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TANGENTIAL REPRESENTATIONS ON A SPHERE

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1. Introduction

Let G be a finite group. The Smith problem is as follows. Let Σ be a homotopy sphere with smooth G-action such that Σ has just two fixed points, say a and b. Are tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic as real G-modules? Two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real G-modules. We know infinitely many Oliver groups possessing non-isomorphic Smith equivalent real modules. We consider about the subset $\mathrm{Sm}(G)$ of the real representation ring $\mathrm{RO}(G)$ of G consisting of all differences $U - V$ of Smith equivalent real G-modules. Recently we have several results corresponding to the Smith set. In this note, we study a sufficient condition for the Smith set to be an additive subgroup of the real representation ring $\mathrm{RO}(G)$. This work is a continuous study from [24].

2. Smith Problem

The Smith problem asks whether the Smith set $\mathrm{Sm}(G)$ is zero or not. There are many results corresponding to the Smith problem.

Atiyah and Bott [1] or Milnor [7] showed that for a homotopy sphere Σ with semi-free smooth compact Lie group with just two fixed points, the tangential representations are isomorphic. Thus, any Smith equivalent real modules over an abelian simple group are isomorphic, that is, $\mathrm{Sm}(C) = 0$ for a prime order cyclic group C. Sanchez [18] generalized the result as follows by computing G-signature and using Franz-Bass's theorem. For a cyclic group P of odd prime power order, Smith equivalent real P-modules are isomorphic. Therefore $\mathrm{Sm}(P) = 0$ for any group P of odd prime power order by combining the Smith theory.

On the other hand, Cappell and Shaneson [2] showed that there exists non-isomorphic, Smith equivalent real module over a cyclic group C_{4n} of order $4n$ for $n \geq 2$, that is, $\mathrm{Sm}(C_{4n}) \neq \{0\}$. Petrie [17] showed that the Smith set of an abelian group of odd order which has at least four non-cyclic subgroups is nontrivial, eg. $\mathrm{Sm}(C_{pqrs} \times C_{pqrs}) \neq 0$,

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where p, q, r, s are distinct odd primes. And in 1980's, Dovermann, Suh, Masuda, etc. studied the Smith equivalent real modules.

Oliver [13] showed that G acts smoothly on a disk without fixed points if and only if there are no subgroups P and H such that P is a p-group, H/P is cyclic, G/H is a q-group for some primes p and q, possibly $p = q$. A group acting on a disk without fixed points is called an Oliver group. Laitinen and Morimoto [5] showed that G is an Oliver group if and only if there exists a one fixed point G-action on sphere. Laitinen and Pawałowski [6] showed that there exists Smith equivalent, non-isomorphic real G-modules for a perfect group G with $r_G \geq 2$ by connecting sum with a sphere with just one fixed point, where r_G is the number of real conjugacy classes of elements of G not of prime power order. After that, Pawałowski and Solomon [14] extended to that $\mathrm{Sm}(G) \neq 0$ if G is a gap Oliver group with $r_G \geq 2$ except $\mathrm{Aut}(A_6)$ and $\mathrm{P\Sigma L}(2, 27)$. A group G is a gap group if there exists a real G-module V such that

- $\dim V^L = 0$ for any prime power index subgroup L of G and
- for any subgroups P of prime power order and H with $H > P$,

$$\dim V^P \geq 2 \dim V^H.$$

In particular, a perfect group G with $r_G \geq 2$ is a gap Oliver group. A study for gap groups is seen in [12, 19, 20, 22, 23].

Now we need some notations. A real conjugacy class $(x)^{\pm}$ of an element x of G is the union of the conjugacy class

$$(x) = \{g^{-1}xg \mid g \in G\}$$

of x and one of its inverse x^{-1}. We denote by $\mathrm{NPP}(G)$ the set of elements of G not of prime power order, by $\overline{\mathrm{NPP}}(G)$ the set of elements of the real conjugacy classes of elements of $\mathrm{NPP}(G)$. Then r_G is the cardinality of the set $\overline{\mathrm{NPP}}(G)$. For a prime p, let $\mathcal{N}_p(G)$ be the set of normal subgroups N of G with $[G : N] \leq p$. We denote by $\mathrm{RO}(G)$ the real representation ring, by $\mathcal{P}(G)$ the set of all subgroups of G of prime power, possibly 1, order, by $O^p(G)$ the Dress subgroup of type p for a prime p defined as

$$O^p(G) = \bigcap_{L \trianglelefteq G; [G:L] = p^a} L,$$

and by $\mathcal{L}(G)$ the set of all prime power, possible 1, index subgroups of G. Then for $L \in \mathcal{L}(G)$, L contains $O^p(G)$ for some prime p. We put

$$\cap p(G) = \cap_{N \in \mathcal{N}_p(G)} N$$

which quotient is an elementary abelian p-group and denote by G^{nil} the smallest normal subgroup of G by which quotient is nilpotent:

$$G^{\mathrm{nil}} = \bigcap_p O^p(G).$$

Note that

$$G \trianglerighteq \cap p(G) \trianglerighteq O^p(G) \trianglerighteq G^{\mathrm{nil}}.$$

For families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subgroups of G and a subset A of $\mathrm{RO}(G)$, we put

$$A_{\mathcal{F}_1} = \bigcap_{P \in \mathcal{F}_1} \ker(\mathrm{Res}^G_P \colon \mathrm{RO}(G) \to \mathrm{RO}(P)) \cap A,$$

$$A^{\mathcal{F}_2} = \bigcap_{L \in \mathcal{F}_2} \ker(\mathrm{Fix}^L \colon \mathrm{RO}(G) \to \mathrm{RO}(N_G(L)/L)) \cap A,$$

and

$$A^{\mathcal{F}_2}_{\mathcal{F}_1} = A_{\mathcal{F}_1} \cap A^{\mathcal{F}_2} = \bigcap_{P \in \mathcal{F}_1} \ker \mathrm{Res}^G_P \cap \bigcap_{L \in \mathcal{F}_2} \ker \mathrm{Fix}^L \cap A.$$

The automorphism group $\mathrm{Aut}(A_6)$ of the alternating group A_6 is not a gap group, $r_{\mathrm{Aut}(A_6)} = 2$, and $\mathrm{Sm}(G) = 0$ [8]. Morimoto [8] gave a condition

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)} = \mathrm{RO}(G)^{\cap 2(G)}$$

for Smith equivalent real modules. The rank of $\mathrm{RO}(G)^{\mathcal{N}_2(G)}$ is equal to

$$r_G - r_{G,\cap 2(G)},$$

where $r_{G,\cap 2(G)}$ is the cardinality of the set $\pi(\overline{\mathrm{NPP}}(G))$ for a canonical projection $\pi \colon G \to G/\cap 2(G)$ (cf. [14]). This condition implies that there are Oliver solvable groups G such that $r_G \geq 2$ and $\mathrm{Sm}(G) = 0$ [15]. The group $\mathrm{P\Sigma L}(2, 27)$ is an extension of $\mathrm{PSL}(2, 27)$ by a field automorphism group of order 3 which is a gap non-solvable group, $r_{\mathrm{P\Sigma L}(2,27)} = 2$ and $\mathrm{Sm}(\mathrm{P\Sigma L}(2, 27)) \neq 0$ [9]. Moreover, putting together with [16], for an Oliver non-solvable group G with $r_G \geq 2$, $\mathrm{Sm}(G) = 0$ if and only if G is isomorphic to $\mathrm{Aut}(A_6)$.

3. Subsets of the Smith set

Sanchez's criterion and Petrie's observation says that

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}_o(G)},$$

where $\mathcal{P}_o(G)$ is the set of subgroups of G of order $1, 2, 4$, or odd prime power. Thus we have

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}_o(G)}.$$

Note that if G has no element of order 8 then $\mathcal{P}_o(G) = \mathcal{P}(G)$. Recall that two real G-modules U and V are Smith equivalent if there exists a smooth action of G on a sphere Σ such that $S^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$ as real G-modules and put

$$\mathrm{Sm}(G) = \{[U] - [V] \mid U \text{ and } V \text{ are Smith equivalent}\}.$$

Similarly we consider the sets $\mathrm{PSm}^c(G)$ (resp. $\mathrm{LSm}(G)$) of all differences $[U] - [V]$ such that U and V are Smith equivalent and in addition the homotopy sphere Σ satisfies that Σ^P is connected for any prime power order subgroups P of G (resp. for any 2-groups of G).

The set $\mathrm{PSm}^c(G)$ (resp. $\mathrm{LSm}(G)$) is empty if and only if G is of order prime power (resp. 2-power). It holds the inclusions

$$\mathrm{PSm}^c(G) \subset \mathrm{LSm}(G) \subset \mathrm{Sm}(G)$$

and

$$\mathrm{LSm}(G) \subset \mathrm{RO}(G)_{\mathcal{P}(G)}.$$

$$\begin{array}{ccccc}
\mathrm{PSm}^c(G) & & & & \\
\cap\big\downarrow & & & & \\
\mathrm{LSm}(G) & & \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} & & \\
\cap\big\downarrow & & \cap\big\downarrow\exists\neq & & \\
\mathrm{Sm}(G)_{\mathcal{P}(G)} & \xrightarrow{\subset} & \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{\cap 2(G)\}}_{\mathcal{P}(G)} & \xrightarrow{\subset} & \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} \\
\cap\big\downarrow\exists\neq & & \cap\big\downarrow\exists\neq & & =\big\downarrow \\
\mathrm{Sm}(G) & \xrightarrow{\subset} & \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}_o(G)} & \xrightarrow{\subset} & \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}
\end{array}$$

TABLE 1. Diagram of inclusions

Theorem 3.1. *Let G be an Oliver group whose nil-quotient G/G^{nil} is not a 2-group. Then*

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{PSm}^c(G).$$

Moreover, we have

Theorem 3.2. *Let G be an Oliver non-gap group with $[G : O^2(G)] = 2$. Suppose that all elements x of $G \smallsetminus O^2(G)$ of order 2 such that $C_G(x)$ is not a 2-group. Then*

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subseteq \mathrm{PSm}^c(G).$$

We denote by $\mathrm{SG}(m, n)$ the small group of order m and type n which is obtained as SmallGroup(m, n) in the software GAP [3]. Morimoto studied (or is studying) the set $\mathrm{Sm}(G)_{\mathcal{P}(G)} \smallsetminus \mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. He [9] showed that for $G = \mathrm{P\Sigma L}(2, 27)$, $\mathrm{SG}(864, 2666)$, $\mathrm{SG}(864, 4666)$, $\mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = 0$ but $\mathrm{Sm}(G)_{\mathcal{P}(G)} = \mathrm{Sm}(G) \cong \mathbb{Z}$. Also he and his colleagues [4] showed that if a Sylow 2-subgroup is normal, then

$$\mathrm{Sm}(G) \subset \mathrm{RO}(G)^{\mathcal{N}_3(G)}$$

and in particular $\mathrm{Sm}(G) = 0$ holds for $G = \mathrm{SG}(1176, 220), \mathrm{SG}(1176, 221)$.

For an Oliver group, we see $\mathrm{PSm}^c(G) \neq 0$ to show $\mathrm{Sm}(G)_{\mathcal{P}(G)} \neq 0$. We have no rich examples so that $\mathrm{Sm}(G)_{\mathcal{P}(G)} \neq \mathrm{Sm}(G)$, whole $\mathrm{Sm}(G) \smallsetminus \mathrm{Sm}(G)_{\mathcal{P}(G)}$ is a finite set. We do not have an example for an Oliver group G such that $\mathrm{PSm}^c(G) \neq \mathrm{Sm}(G)_{\mathcal{P}(G)}$. It remains the problem whether $\mathrm{PSm}^c(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)}$ for an Oliver group.

4. Criterion for the Smith set to be a group

We discuss for Oliver groups G such that $\mathrm{PSm}^c(G)$ is a subgroup of $\mathrm{RO}(G)$. We introduce two condition. One is a part of a sufficient condition to show $\mathrm{Sm}(G)_{\mathcal{P}(G)} \setminus \mathrm{Sm}(G)^{\mathcal{L}(G)} \neq 0$ and the other is a sufficient condition so that $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is a group.

Let $Q = \bigcap_{p \neq 2} O^p(G)$ be a normal subgroup of G with odd index and let N be a normal subgroup of G with $G^{\mathrm{nil}} \leq N \leq \cap 2(G) \cap Q$. Then

$$Q \geq \cap 2(G) \cap Q \geq N \geq G^{\mathrm{nil}} \geq O^2(Q).$$

Definition 4.1. We say that G satisfies the *quasi-N-$\mathcal{P}$-condition* if there are real Q-modules U and V such that

- $\dim U^{\cap 2(G) \cap Q} = \dim V^N = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \mathrm{RO}(Q)_{\mathcal{P}(Q)}$.

In particular, the quasi-G^{nil}-$\mathcal{P}$-condition is simply called quasi-Nil-$\mathcal{P}$-condition.

Definition 4.2. We say that G satisfies the *weak-Nil-$\mathcal{P}$-condition* if there are real G-modules U and V such that

- $\dim U^{\cap 2(G)} = \dim V^{G^{\mathrm{nil}}} = 0$ and
- $[\mathbb{R} \oplus U] - [V] \in \mathrm{RO}(G)_{\mathcal{P}(G)}$.

Lemma 4.3. *If G satisfies the quasi-*Nil-*$\mathcal{P}$-condition, then G satisfies the weak* Nil-*$\mathcal{P}$-condition.*

Proposition 4.4 (cf. [10, Lemma 15]). *Let G be a finite group with $O^2(G) = G$. The following statements are equivalent.*

(1) *G^{nil} has a sub-quotient isomorphic to D_{2pq} for distinct primes p, q.*
(2) *G satisfies the quasi-*Nil-*$\mathcal{P}$-condition.*

Morimoto and Qi [11] obtained a sufficient condition for an Oliver group G to hold that $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is not equal to $\mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$. This result supplies that $\mathrm{Sm}(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)} \cong \mathbb{Z}$ for $G = \mathrm{SG}(864, 2666)$ or $\mathrm{SG}(864, 4666)$. For $G = \mathrm{SG}(864, 2666)$ or $\mathrm{SG}(864, 4666)$, G/G^{nil} is a cyclic group of order 3 and $\mathrm{RO}(G)_{\mathcal{P}(G)}$ is generated by two element $\mathbb{R}[G/G^{\mathrm{nil}}] + X_1$ and $3(\mathbb{R}[G/G^{\mathrm{nil}}] - \mathbb{R}) + X_2$ for some elements $X_1, X_2 \in \mathrm{RO}(G)^{\{G^{\mathrm{nil}}\}}$ and thus, G satisfies the weak-Nil-$\mathcal{P}$-condition since G/G^{nil} is a cyclic group of order 3. We see it in the next section. Indeed, G has a sub-quotient isomorphic to D_{12} and G satisfies the quasi-Nil-$\mathcal{P}$-condition.

Definition 4.5. For a normal subgroup N of G, we say that G satisfies the *N-$\mathcal{P}$-condition* if there are real G-modules U and V such that $U^N = V^N = 0$ and $[\mathbb{R} \oplus U] - [V] \in \mathrm{RO}(G)_{\mathcal{P}(G)}$. If $N = G^{\mathrm{nil}}$ we say that G satisfies the Nil-*$\mathcal{P}$-condition*.

Lemma 4.6 or Theorem 4.8 in [9] essentially yields us the following two theorems.

Theorem 4.6. *If a gap Oliver group G satisfies the weak-*Nil-$\mathcal{P}$*-condition with* $\mathrm{NPP}(G) \cap G^{\mathrm{nil}} \neq \emptyset$ *and has an element of* $\mathrm{NPP}(G)$ *outside* $O^p(G)$ *for some prime p, then*

$$\mathrm{PSm}^c(G) \smallsetminus \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \neq 0.$$

Note that under the assumption that $\mathrm{NPP}(G) \cap G^{\mathrm{nil}} \neq \emptyset$ the inequality $\mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} \neq \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ if and only if $\mathrm{NPP}(G) \smallsetminus O^p(G)$ is not empty for some prime p. By using the multiplication of $\mathrm{RO}(G)$, we get the following theorem.

Theorem 4.7. *Let G be a gap Oliver group satisfying the* Nil-$\mathcal{P}$*-condition. Then*

$$\mathrm{PSm}^c(G) = \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} = \mathrm{Sm}(G)_{\mathcal{P}(G)}$$

and in particular $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ *is an additive group.*

If a Sylow 2-subgroup of G is normal, G does not satisfy the Nil-$\mathcal{P}$-condition. Although the Nil-$\mathcal{P}$-condition is a sufficient one for an Oliver group G such that $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is a additive group, it is not a necessary condition. For example, $A_5 \times C_4$ does not satisfy the Nil-$\mathcal{P}$-condition but the following result holds.

Proposition 4.8. $\mathrm{PSm}^c(A_5 \times C_4) = \mathrm{Sm}(A_5 \times C_4) = \mathrm{RO}(A_5 \times C_4)^{\{A_5\}}$.

Problem. $\mathrm{PSm}^c(A_5 \times (C_4)^n) = \mathrm{Sm}(A_5 \times (C_4)^n)$ *holds. Is it true that* $\mathrm{PSm}^c(A_5 \times (C_4)^n) = \mathrm{RO}(A_5 \times (C_4)^n)^{\{A_5 \times (C_2)^n\}}$ *for* $n \geq 2$*?*

5. Quasi-Nil-$\mathcal{P}$-condition

In this section we study properties for the weak-Nil-$\mathcal{P}$-condition. Remark that there is an Oliver group which satisfies the weak-Nil-$\mathcal{P}$-condition but does not satisfy the Nil-$\mathcal{P}$-condition (eg. SG(864, 2666), SG(864, 4666)).

Proposition 5.1. *Let K be a subgroup of G such that* $\cap 2(G) \cdot K = G$*. If K satisfies the weak-*$(G^{\mathrm{nil}} \cap K)$*-*$\mathcal{P}$*-condition, then G satisfies the weak-*Nil-$\mathcal{P}$*-condition.*

Theorem 5.2. *Let G be a gap Oliver group. Suppose that* $\mathrm{NPP}(G) \cap G^{\mathrm{nil}}$ *is not empty and that there is an element* $\mathrm{NPP}(G)$ *outside of* $O^p(G)$ *for some prime p. If an odd index subgroup K of G satisfies the weak-*$(G^{\mathrm{nil}} \cap K)$*-*$\mathcal{P}$*-condition, then*

$$\mathrm{PSm}^c(G) \smallsetminus \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \neq 0.$$

Morimoto and Qi [10, Lemma 21 and Theorem 22] showed that $\mathrm{Sm}(G)_{\mathcal{P}(G)} \neq \mathrm{Sm}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ for an odd integer $n > 1$, an odd prime p, and $G = D_{2n} \int C_p$, the wreath product of the dihedral group D_{2n} of order $2n$ by a cyclic group C_p of order p. The group G satisfies the assumption of Proposition 5.1 as follows. The group G has a presentation

$$\left\langle a_1, b_1, \dots, a_p, b_p, c \;\middle|\; \begin{array}{l} a_i^n = b_i^2 = (a_ib_i)^2 = 1,\ (\forall i), \\ a_ia_j = a_ja_i, a_ib_j = b_ja_i, b_ib_j = b_jb_i,\ (i \neq j), \\ c^p = 1, c^{-1}a_ic = a_{i+1}, c^{-1}b_ic = b_{i+1},\ (\forall i) \end{array} \right\rangle,$$

where $a_{p+1} = a_1$ and $b_{p+1} = b_1$. The group G^{nil} is a subgroup of G generated by elements $a_1, \dots, a_p$ and b_ib_j $(i < j)$, and then $G/G^{\mathrm{nil}} \cong C_{2p}$. Thus G is a gap Oliver group. Put $K = O^p(G)$. Let $f\colon D_{2n}^p \to D_{2n}$ be the first projection and let $\hat{U}$ and $\hat{V}$ be $\mathcal{P}(D_{2n})$-matched real D_{2n}-modules such that $\hat{U}^{D_{2n}} = \mathbb{R}$ and $\hat{V}^{D_{2n}} = 0$. The real K-modules $f^*\hat{U}$ and $f^*\hat{V}$ implies that K satisfies the assumption of Proposition 5.1 since $f(G^{\mathrm{nil}}) = D_{2n}$. (Or directly, two real G-modules $\mathrm{Ind}_K^G f^*\hat{U}$ and $\mathrm{Ind}_K^G f^*\hat{V}$ implies that G satisfies the weak-Nil-$\mathcal{P}$-condition.)

Before closing this section, we should say the strongness of the weak-Nil-$\mathcal{P}$-condition. Let G be a finite group such that G/G^{nil} is a nilpotent group of odd order and there are an element of G^{nil} not of prime power order and an element of G outside G^{nil} not of prime power order. Then

$$\mathrm{RO}(G)_{\mathcal{P}(G)}^{\{G^{\mathrm{nil}}\}} \neq \mathrm{RO}(G)_{\mathcal{P}(G)}^{\{G\}}.$$

Note that if a Sylow 2-subgroup of G is normal then $\mathrm{Sm}(G) \subset \mathrm{RO}(G)^{\{N_s(G)|s\}}$ (cf. [4]) and G does not satisfy the weak-Nil-$\mathcal{P}$-condition. Otherwise, if G has a sub-quotient isomorphic to D_{2qr} for some distinct primes q and r, there are real G-modules U and V such that the equalities $U^{G^{\mathrm{nil}}} = 0 = V^{G^{\mathrm{nil}}}$ hold and that $\mathbb{R}[G/G^{\mathrm{nil}}] \oplus U$ and V are $\mathcal{P}(G)$-matched:

$$\mathbb{R} + [(\mathbb{R}[G/G^{\mathrm{nil}}] - \mathbb{R}) \oplus U] - [V] = \mathbb{R}[G/G^{\mathrm{nil}}] + [U] - [V] \in \mathrm{RO}(G)_{\mathcal{P}(G)}.$$

Thus, G satisfies the weak-Nil-$\mathcal{P}$-condition and in addition if G is a gap Oliver group then

$$\mathrm{PSm}^c(G)^{\{G^{\mathrm{nil}}\}} \neq \mathrm{PSm}^c(G).$$

6. Nil-$\mathcal{P}$-condition

In this section we study properties for the Nil-$\mathcal{P}$-condition.

Proposition 6.1. *If G satisfies the* Nil-$\mathcal{P}$*-condition, then G satisfies the weak-*Nil-$\mathcal{P}$*-condition.*

Proposition 6.2. *If a quotient group of G satisfies the* Nil-$\mathcal{P}$*-condition, then G also satisfies the* Nil-$\mathcal{P}$*-condition.*

Proposition 6.3. *Let N be a normal subgroup of G. If there are a subgroup K of G and an epimorphism $f\colon K \to H$ such that $f(K \cap N) = H$, $KN = G$ and H has sub-quotient isomorphic to D_{2pq}, where p and q are distinct primes, then G satisfies the N-$\mathcal{P}$-condition.*

For a perfect group G, the weak-Nil-$\mathcal{P}$-condition and Nil-$\mathcal{P}$-conditionare equivalent and moreover equivalent to that G has a sub-quotient isomorphic to a dihedral group D_{2pq} for distinct primes p and q.

Proposition 6.4 (cf. [21]). *Simple groups except the following groups satisfy the* Nil-$\mathcal{P}$-*condition.*

(1) *Cyclic group*
(2) *Projective special linear groups:* $\mathrm{PSL}(2,4) = \mathrm{PSL}(2,5) = A_5$, $\mathrm{PSL}(2,7) = \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,9) = A_6$, $\mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(3,8)$
(3) *Suzuki groups* $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$
(4) *Projective unitary groups:* $\mathrm{PSU}(3,3)$, $\mathrm{PSU}(3,4)$, $\mathrm{PSU}(3,8)$

Theorem 6.5. *Let $q > 1$ be a prime power. The following groups are gap groups satisfying the* Nil-$\mathcal{P}$-*condition.*

(1) *Symmetric groups S_n, $n \geq 7$*
(2) *Projective general linear groups* $\mathrm{PGL}(2,q)$, $q \neq 2,3,4,5,7,8,9,17$
(3) *Projective general linear groups* $\mathrm{PGL}(3,q)$, $q \neq 2,4,8$
(4) *Projective general linear groups* $\mathrm{PGL}(n,q)$, $n \geq 4$
(5) *General linear groups* $\mathrm{GL}(2,q)$, $q \neq 2,3,4,5,7,8,9,17$
(6) *General linear groups* $\mathrm{GL}(3,q)$, $q \neq 2,4,8$
(7) *General linear groups* $\mathrm{GL}(n,q)$, $n \geq 4$
(8) *The automorphism group of sporadic groups*

The Smith sets of $\mathrm{PGL}(2,q)$ and $\mathrm{PGL}(3,q)$ have been already obtained in [24]. This can be proved by finding subgroups as in Proposition 6.3. The groups listed up in Theorem 6.5 are non-solvable gap group. Then we have the following theorem.

Theorem 6.6. *Let G be a group which has quotient isomorphic to a group in Theorem 6.5. Then*

$$\mathrm{PSm}^c(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)}.$$

Corollary 6.7. *Let K be a group in Theorem 6.5 and F any finite group. Then for $G = K \times F$,*

$$\mathrm{PSm}^c(G) = \mathrm{Sm}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)}.$$

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