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On constructing explicit homomorphisms between generalized Verma modules

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Abstract

In this article we introduce a systematic construction method of homomorphisms between generalized Verma modules, one of which is non-scalar. To obtain concrete results we apply the construction method to the case of maximal parabolic subalgebras of quasi-Heisenberg type.

1 Introduction

Generalized Verma modules are one of the most important objects in representation theory, parabolic geometry, and also in mathematical physics. Because of their significance, there are many interesting problems on such modules in the literature. The classification and construction of the homomorphisms between generalized Verma modules are some of such problems. (For the topic, see, for instance, [2], [3], [4], [5], [8], [10], [11], [12], [17], [18], [19], [24], [25], [26], [27], [28], and the references therein.) In contrast that homomorphisms between scalar generalized Verma modules are intensively investigated, much less is known about homomorphisms between generalized Verma modules that are not necessarily scalar. The aim of this article is to introduce an idea to study homomorphisms between generalized Verma modules, one of which is non-scalar.

To describe our problems precisely, let $\mathfrak{g}$ be a complex simple Lie algebra and fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ with $\mathfrak{h}$ the Cartan subalgebra and $\mathfrak{u}$ the nilpotent radical. Write $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of the roots of $\mathfrak{g}$. We denote by Δ^+ the positive system attached to $\mathfrak{b}$ and by Π the set of the simple roots. Given subspace $U \subset \mathfrak{g}$, we write $\Delta(U) := \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset U\}$, where $\mathfrak{g}_\alpha$ are the root spaces for $\alpha \in \Delta$.

If $\mathfrak{q} \supset \mathfrak{b}$ is a standard parabolic subalgebra then, as usual, we write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ for a Levi decomposition and denote by $\bar{\mathfrak{n}}$ the opposite nilpotent subalgebra. For any subalgebra $\mathfrak{s} \subset \mathfrak{g}$, let $\mathcal{U}(\mathfrak{s})$ denote for the universal enveloping algebra of $\mathfrak{s}$. Given simple $\mathfrak{q}$-module V, we write

$$M_\mathfrak{q}[V] := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V$$

the generalized Verma module induced from V.

Observe that if $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra then there exists $\alpha_\mathfrak{q} \in \Pi$ so that $\Delta(\mathfrak{l}) = \{\alpha \in \Delta \mid \alpha \in \operatorname{span}(\Pi \backslash \{\alpha_\mathfrak{q}\})\}$ and $\Delta(\mathfrak{n}) = \Delta^+ \backslash \Delta(\mathfrak{l})$. Let $\lambda_\mathfrak{q}$ denote the fundamental weight for $\alpha_{\mathfrak{q}}$. We write $\mathbb{C}_{-s} := (-s\lambda_{\mathfrak{q}}, \mathbb{C})$ for a one-dimensional $\mathfrak{q}$-module associated with $\lambda_{\mathfrak{q}}$ with $s \in \mathbb{C}$.

The purpose of this article is to study non-zero $\mathcal{U}(\mathfrak{g})$-homomorphisms

$$\varphi \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_{\mathfrak{q}}[V], M_{\mathfrak{q}}[\mathbb{C}_{-s}]).$$

The first thing to consider is when such φ exists. In this article we consider the existence problem as follows.

Problem A. *Given simple* $\mathfrak{q}$*-module* V*, determine complex parameter* $s_V \in \mathbb{C}$ *so that*

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_{\mathfrak{q}}[V], M_{\mathfrak{q}}[\mathbb{C}_{-s_V}]) \neq \{0\}. \tag{1}$$

It is well-known that associated to the homomorphisms φ between generalized Verma modules are G_0-intertwining differential operators D_φ between the degenerate principle series representations associated to the generalized Verma modules. Here, G_0 is an appropriate real form of a complex Lie group G with Lie algebra $\mathfrak{g}$. (See, for instance, [7] and [9].) To construct representations of G_0 in the kernels of D_φ, it is important to obtain concrete formulas of φ. Keeping the problem in mind, we also consider the following problem.

Problem B. *If* $s_V \in \mathbb{C}$ *is a complex parameter so that* (1) *holds then, for a homomorphsim* $\varphi \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M_{\mathfrak{q}}[V], M_{\mathfrak{q}}[\mathbb{C}_{-s_V}])$*, give a concrete formula of* $\varphi(1 \otimes v_h)$ *or* $\varphi(1 \otimes v_l)$*, where* v_h *(resp.* v_l*) is a highest (resp. lowest) weight vector for* V*.*

To achieve the problems we introduce a method for constructing homomorphisms between appropriate generalized Verma modules. As proposed in [15], we call it the τ *method.* This method is originally invented by Barchini-Kable-Zierau in [1] to build certain systems of differential operators on degenerate principal series. Later their method was refined by the author ([22]) and, recently, further investigated by Kable ([14], [15]). In this article we introduce a version of the τ method developed in [22]. Here we wish to note that the τ method is closely related to the invariant theory of prehomogeneous vector spaces. For those readers who are interested in the relationship, consult [1], [13], and [14].

The main idea of the method is to construct $\mathcal{U}(\mathfrak{g})$-submodules of $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$ from a graded Lie algebra $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$. More precisely, first, we construct simple $\mathfrak{l}$-submodules V_{-s} of $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$ from $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ via polynomial map $\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$. We then determine $s = s_V$ so that the nilpotent radical $\mathfrak{n}$ for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ acts on $V_{-s_V} \subset M_{\mathfrak{q}}[\mathbb{C}_{-s_V}]$ trivially. The detail will be exhibited in Section 2.

We now briefly describe the rest of this article. There are three sections. As described above, Section 2 is devoted to illustrate the τ method in detail. In Section 3, to show concrete results on Theorems A and B, we apply the τ method to the case that maximal parabolic subalgebra $\mathfrak{q}$ is of "quasi-Heisenberg type". The material of the section is indeed a summary of [20] and [22]. To give results as precisely as possible, we also discuss such parabolic subalgebras $\mathfrak{q}$ and the associated grading on $\mathfrak{g}$. To maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type, Problems A and B are achieved in Theorems 11 and 12, respectively.

2 The τ method

The purpose of this short section is to describe the τ method in detail. We keep the notation introduced in the introduction, unless otherwise specified.

Let $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ be the $\mathbb{Z}$-grading on $\mathfrak{g}$ with $\mathfrak{g}(1) \neq 0$ and $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$, so that $\mathfrak{q} = \bigoplus_{j\geq 0} \mathfrak{g}(j)$ with $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \bigoplus_{j>0} \mathfrak{g}(j)$. (For such a $\mathbb{Z}$-grading, see, for instance, Section X.3 of [16].) Observe that, as $[\mathfrak{g}(0), \mathfrak{g}(j)] \subset \mathfrak{g}(j)$, each graded subspace $\mathfrak{g}(j)$ is an $\mathfrak{l}$-module, and so is $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ for $1 \leq k \leq 2r$. Let X_α be a root vector for $\alpha \in \Delta$ and X_α^* the vector dual to X_α with respect to the Killing form κ, namely, $X_\alpha^*(X_\beta) := \kappa(X_\alpha^*, X_\beta) = \delta_{\alpha,\beta}$ with $\delta_{\alpha,\beta}$ the Kronecker delta.

Now fix $k \in \{1, \ldots, 2r\}$. Our first task is to construct simple $\mathfrak{l}$-submodules of $\mathcal{U}(\bar{\mathfrak{n}})$ from $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$. Toward the end, we define polynomial map

$$\begin{aligned} \tau_k : \mathfrak{g}(1) &\to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) \\ X &\mapsto \big(\mathrm{ad}(X)^k \otimes \mathrm{Id}\big)\omega, \end{aligned}$$

where ω is the element in $\mathfrak{g}(-r) \otimes \mathfrak{g}(r)$ defined by

$$\omega := \sum_{\eta \in \Delta(\mathfrak{g}(r))} X_\eta^* \otimes X_\eta.$$

Observe that if L is a complex analytic group with Lie algebra $\mathfrak{l}$ then τ_k is L-equivariant. Therefore, for each L-irreducible constituent W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$, we can define L-intertwining linear map $\tilde{\tau}_k|_{W^*} \in \mathrm{Hom}_L(W^*, \mathcal{P}^k(\mathfrak{g}))$ by

$$\tilde{\tau}_k|_{W^*}(Y^*)(X) := Y^*(\tau_k(X)).$$

Here, W^* is the dual space for W with respect to the Killing form κ and $\mathcal{P}^k(\mathfrak{g}(1))$ is the space of homogeneous polynomials on $\mathfrak{g}(1)$ of degree k. Now, from irreducible constituents W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ with $\tilde{\tau}_k|_{W^*} \not\equiv 0$, we obtain simple $\mathfrak{l}$-submodules of $\mathcal{U}(\bar{\mathfrak{n}})$ via the following algebraic procedure:

$$W^* \stackrel{\tilde{\tau}_k|_{W^*}}{\to} \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1)) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}), \tag{2}$$

where $\sigma : \mathrm{Sym}^k(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization operator.

As observed above the irreducible constituents $W \subset \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ with $\tilde{\tau}_k|_{W^*} \not\equiv 0$ play a key role. We then give a name to such constituents.

Definition 3. [20] *We call L-irreducible constituents W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$* ***special*** *if the associated L-linear map $\tilde{\tau}_k|_{W^*}$ is not identically zero.*

Let $\omega_k|_{W^*} := \sigma \circ \tilde{\tau}|_{W^*}$ be the composition of the L-equivariant maps. Observe that, as $\omega_k|_{W^*} : W^* \to \mathcal{U}(\bar{\mathfrak{n}})$ is L-equivariant, if W is a special constituent then $\omega_k(W^*) := \omega_k|_{W^*}(W^*)$ is a (non-zero) simple $\mathfrak{l}$-submodule of $\mathcal{U}(\bar{\mathfrak{n}})$. Now, by tensoring $\mathbb{C}_{-s}$ to $\omega_k(W^*)$, we obtain (non-zero) simple $\mathfrak{l}$-submodule $\omega_k(W^*) \otimes \mathbb{C}_{-s}$ of $M_\mathfrak{q}[\mathbb{C}_{-s}] \cong \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-s}$. (See the diagram below.)

$$\begin{tikzcd} & W^* \arrow[d, "\omega_k|_{W^*}"] \\ M_\mathfrak{q}[\mathbb{C}_{-s}] & \mathcal{U}(\bar{\mathfrak{n}}) \arrow[l, "\cdot\otimes\mathbb{C}_{-s}"'] \\ \omega_k(W^*) \otimes \mathbb{C}_{-s} & \omega_k(W^*) \arrow[l, mapsto] \end{tikzcd}$$

Now, naively speaking, via the inclusion map $\iota_k \in \mathrm{Hom}_L\big(\omega_k(W^*)\otimes\mathbb{C}_{-s}, M_{\mathfrak{q}}[\mathbb{C}_{-s}]\big)$, we wish to construct a $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_k \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s}], M_{\mathfrak{q}}[\mathbb{C}_{-s}]\big)$, namely,

$$
\begin{aligned}
M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s}] &\overset{\varphi_k}{\to} M_{\mathfrak{q}}[\mathbb{C}_{-s}] \\
u\otimes\big(\omega_k(Y^*)\otimes 1\big) &\mapsto u\cdot\iota_k\big(\omega_k(Y^*)\otimes 1\big).
\end{aligned}
\tag{4}
$$

Observe that, since $\omega_k(W^*)\otimes\mathbb{C}_{-s}$ is only an $\mathfrak{l}$-module for arbitrary $s\in\mathbb{C}$, in general, $M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s}]$ is not well-defined. To make it indeed a generalized Verma module, one has to make the nilpotent radical $\mathfrak{n}$ for $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{n}$ act on $\omega_k(W^*)\otimes\mathbb{C}_{-s}$ trivially; equivalently saying, one should have

$$
\iota_k \in \mathrm{Hom}_L\big(\omega_k(W^*)\otimes\mathbb{C}_{-s}, M_{\mathfrak{q}}[\mathbb{C}_{-s}]^{\mathfrak{n}}\big), \tag{5}
$$

where $M_{\mathfrak{q}}[\mathbb{C}_{-s}]^{\mathfrak{n}} := \{u\otimes 1 \in M_{\mathfrak{q}}[\mathbb{C}_{-s}] \mid X\cdot u\otimes 1 = 0 \text{ for all } X\in\mathfrak{n}\}$. This problem is achieved by finding a suitable complex parameter $s\in\mathbb{C}$ so that (5) holds. It follows from infinitesimal characters of $M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s}]$ and $M_{\mathfrak{q}}[\mathbb{C}_{-s}]$ that if such a complex parameter exists then it is unique. (See, for instance, Section 7 of [2] and Section 3 of [23].) Because of the importance we also give a name to such a complex parameter.

Definition 6. *Given special constituent W of $\mathfrak{g}(-r+k)\otimes\mathfrak{g}(r)$, we say that a complex parameter $s_k\in\mathbb{C}$ is the* ***special value*** *for W if*

$$
\mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}(M_{\mathfrak{q}}[\omega_k(W^*)\otimes\mathbb{C}_{-s_k}], M_{\mathfrak{q}}[\mathbb{C}_{-s_k}]) \neq \{0\}.
$$

In the next section, to a particular class of maximal parabolic subalgebras $\mathfrak{q}$ and a particular τ_k map, we illustrate how one proceeds the τ method.

3 Quasi-Heisenberg case

In this section we briefly exhibit how one actually applies the τ method. (For the detail see [20] and [22].) The notation introduced in the previous sections are left in force, unless otherwise specified.

To illustrate the results, we start with discussing some observation and definitions given in [20] and [22]. First, recall from [22] that we call a parabolic subalgebra $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{n}$ **quasi-Heisenberg type**[1] if its nilpotent radical $\mathfrak{n}$ satisfies the conditions that $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]=0$ and $\dim([\mathfrak{n},\mathfrak{n}])>1$. Let $\alpha_{\mathfrak{q}}$ be the simple root, so that the maximal parabolic subalgebra $\mathfrak{q}=\mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}=\mathfrak{l}\oplus\mathfrak{n}$ determined by $\alpha_{\mathfrak{q}}$ is of quasi-Heisenberg type. Given Dynkin type $\mathcal{T}$ of $\mathfrak{g}$, if we write $\mathcal{T}(i)$ for the Lie algebra together with the choice of maximal parabolic subalgebra $\mathfrak{q}=\mathfrak{q}_{\{\alpha_i\}}$ determined by α_i then the maximal parabolic subalgebras $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{n}$ of quasi-Heisenberg type are classified as follows:

$$
B_n(i)\ (3\le i\le n),\quad C_n(i)\ (2\le i\le n-1),\quad D_n(i)\ (3\le i\le n-2),
$$

and

$$
E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ F_4(4).
$$

[1]Note that the condition is $\dim([\mathfrak{n},\mathfrak{n}])>1$, not $\dim([\mathfrak{n},\mathfrak{n}])\ge 1$.

Here, the Bourbaki conventions [6] are used for the labels of the simple roots. Note that, in type A_n, any maximal parabolic subalgebra has abelian nilpotent radical, and also that, in type G_2, the nilpotent radicals of two maximal parabolic subalgebras are a 3-step nilpotent or Heisenberg algebra.

We next observe that a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type induces a 2-grading on $\mathfrak{g}$. As $\mathfrak{q}$ has two-step nilpotent radical, if $\lambda_{\mathfrak{q}}$ is the fundamental weight for $\alpha_{\mathfrak{q}}$ then, for all $\beta \in \Delta$, the quotient $2\langle \lambda_{\mathfrak{q}}, \beta \rangle / ||\alpha_{\mathfrak{q}}||^2$ takes the values of 0, ± 1, or ± 2. (See, for example, Section 4.1 of [22].) Therefore, if $H_{\mathfrak{q}}$ is the element in $\mathfrak{h}$ so that $\beta(H_{\mathfrak{q}}) = 2\langle \lambda_{\mathfrak{q}}, \beta \rangle / ||\alpha_{\mathfrak{q}}||^2$ for all $\beta \in \Delta$, and if $\mathfrak{g}(j)$ is the j-eigenspace of $\mathrm{ad}(H_{\mathfrak{q}})$ on $\mathfrak{g}$ then the adjoint action of $H_{\mathfrak{q}}$ induces a 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ on $\mathfrak{g}$ with parabolic subalgebra $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. The nilpotent radical $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}$ is given by $\bar{\mathfrak{n}} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$. Observe that, as $\mathfrak{g}_{\alpha_{\mathfrak{q}}} \subset \mathfrak{g}(1)$, we have $\mathfrak{g}(1) \neq \{0\}$. Here we have $\mathfrak{g}(2) = \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{g}(-2) = \mathfrak{z}(\bar{\mathfrak{n}})$, where $\mathfrak{z}(\mathfrak{n})$ (resp. $\mathfrak{z}(\bar{\mathfrak{n}})$) is the center of $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$). Thus we denote the resulted graded Lie algebra by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{g}(-1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$$

with parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}).$$

Now, for $1 \leq k \leq 4$, the L-equivariant polynomial maps τ_k are given by

$$\begin{aligned} \tau_k : \mathfrak{g}(1) &\to \mathfrak{g}(-2+k) \otimes \mathfrak{z}(\mathfrak{n}) \\ X &\mapsto \frac{1}{k!}\big(\mathrm{ad}(X)^k \otimes \mathrm{Id}\big)\omega \end{aligned}$$

with $\omega = \sum_{\eta \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_\eta^* \otimes X_\eta$. When $k = 2$, we have

$$\begin{aligned} \tau_2 : \mathfrak{g}(1) &\to \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) \\ X &\mapsto \frac{1}{2}\big(\mathrm{ad}(X)^2 \otimes \mathrm{Id}\big)\omega. \end{aligned}$$

For the rest of this section we proceed the τ method with the τ_2 map.

To apply the τ method one first needs to find special constituents W of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$; namely, L-irreducible constituents W of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ so that the linear map

$$\tilde{\tau}_2|_{W^*} \in \mathrm{Hom}_L(W^*, \mathcal{P}^2(\mathfrak{g}(1)))$$

is not identically zero. To the end we observe the structure of $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. First, as $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is the maximal parabolic subalgebra determined by $\alpha_{\mathfrak{q}}$, we have $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\mathfrak{q}}$. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ is either simple or the direct sum of two or three simple ideals. (See, for instance, Appendix A of [22].) To characterize the simple ideals, let γ be the highest root for $\mathfrak{g}$. If $\mathfrak{g}$ is not of type A_n then there is exactly one simple root that is not orthogonal to γ. It is well known that if α_γ is the unique simple root then the nilpotent radical $\mathfrak{n}'$ of the parabolic subalgebra $\mathfrak{q}' = \mathfrak{q}_{\{\alpha_\gamma\}}$ satisfies

$\dim([\mathfrak{n}', \mathfrak{n}']) = 1$. By definition parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is of quasi-Heisenberg type if and only if $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. Hence, if $\mathfrak{q} = \mathfrak{q}_{\{\alpha_\mathfrak{q}\}}$ is such a parabolic subalgebra then α_γ is in $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_\mathfrak{q}\}$, where $\Pi(\mathfrak{l}) := \{\alpha \in \Pi \mid \mathfrak{g}_\alpha \subset \mathfrak{l}\}$. In particular, there is a unique simple ideal of $[\mathfrak{l}, \mathfrak{l}]$ containing the root space $\mathfrak{g}_{\alpha_\gamma}$ for α_γ. We denote by $\mathfrak{l}_\gamma$ the unique simple ideal containing $\mathfrak{g}_{\alpha_\gamma}$. Similarly, when $[\mathfrak{l}, \mathfrak{l}]$ consists of two (resp. three) simple ideals, we denote the other simple ideal(s) by $\mathfrak{l}_{n\gamma}$ (resp. $\mathfrak{l}^+_{n\gamma}$ and $\mathfrak{l}^-_{n\gamma}$). The three simple factors occur only when $\mathfrak{q}$ is of type $D_n(n-2)$. So, when $\mathfrak{q}$ is not of type $D_n(n-2)$, the Levi subalgebra $\mathfrak{l}$ may decompose into

$$\mathfrak{l} = \mathbb{C}H_\mathfrak{q} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}. \tag{7}$$

Similarly, when $\mathfrak{q}$ is of type $D_n(n-2)$, one may write

$$\mathfrak{l} = \mathbb{C}H_\mathfrak{q} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}^+_{n\gamma} \oplus \mathfrak{l}^-_{n\gamma}. \tag{8}$$

Note that when $[\mathfrak{l}, \mathfrak{l}]$ is a simple ideal, we have $\mathfrak{l}_{n\gamma} = \{0\}$ ($\mathfrak{l}^\pm_{n\gamma} = \{0\}$). It follows from the decompositions (7) and (8) that the tensor product $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ may be written as

$$\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = \begin{cases} \big(\mathbb{C}H_\mathfrak{q} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) & \text{if } \mathfrak{q} \text{ is not of type } D_n(n-2) \\ \big(\mathbb{C}H_\mathfrak{q} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}^+_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}^-_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big) & \text{if } \mathfrak{q} \text{ is of type } D_n(n-2). \end{cases}$$

Now we give a necessary condition for $\mathfrak{l}$-irreducible constituents W to be special. To do so, first observe that, as $\mathfrak{q}$ is a maximal parabolic, the subspace $\mathfrak{g}(1)$ is a simple $\mathfrak{l}$-module. (See, for instance, Proposition 3.2.2 of [21].) We write μ for the highest weight for $\mathfrak{g}(1)$. For $\nu \in \mathfrak{h}^*$ with $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\Pi(\mathfrak{l})$, we denote by $V(\nu)$ the simple $\mathfrak{l}$-module with highest weight $\nu|_{\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]}$.

Lemma 9. [22, Section 6.2] *If $V(\nu)$ is an irreducible constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ with $\tilde{\tau}_2|_{V(\nu)^*} \not\equiv 0$ then ν satisfies the following two conditions:* [2]

(C1) $\nu = \mu + \epsilon$ for some $\epsilon \in \Delta(\mathfrak{g}(1))$.

(C2) $\nu \neq \gamma$.

If $\mathfrak{q}$ is not of type $D_n(n-2)$, there are exactly one or two special constituents $V(\nu) = V(\mu + \epsilon)$ of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$; one is an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and the other is equal to $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$ ([22, Section 6.2]). We denote by $V(\mu + \epsilon_\gamma)$ and $V(\mu + \epsilon_{n\gamma})$ the special constituents so that $V(\mu + \epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. If $\mathfrak{q}$ is of type $D_n(n-2)$ then there are three special constituents, namely, $V(\mu + \epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu + \epsilon^\pm_{n\gamma}) = \mathfrak{l}^\pm_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$ ([20, Section 3]).

With respect to a condition of the highest weight $\mu+\epsilon$, the special constituents $V(\mu+\epsilon)$ are classified as type 1a, type 1b, type 2, or type 3 as follows.

[2] We wish to note that, in [22], the terminology "special constituents" is used for irreducible constituents satisfying the conditions (C1) and (C2). (So it is used in a sense weaker than Definition 3.) For some remark on the discrepancy of the phrase see the introduction of [20].

Definition 10. [22, Definition 6.20] *We say that a special constituent $V(\mu+\epsilon)$ is of*

(1) ***type 1a*** *if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and both μ and ϵ are long roots,*

(2) ***type 1b*** *if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and either μ or ϵ is a short root,*

(3) ***type 2*** *if $\mu+\epsilon = 2\mu$ is not a root, or*

(4) ***type 3*** *if $\mu+\epsilon$ is a root.*

Table 1 summarizes the types of special constituents. A dash in the column for $V(\mu+\epsilon_{n\gamma})$ indicates that $\mathfrak{l}_{n\gamma} = \{0\}$ for the case. (So there is no special constituent $V(\mu+\epsilon_{n\gamma})$.)

Table 1: Types of special constituents

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i)$, $3 \leq i \leq n-2$	Type 1a	Type 1a
$B_n(n-1)$	Type 1a	Type 1b
$B_n(n)$	Type 2	–
$C_n(i)$, $2 \leq i \leq n-1$	Type 3	Type 2
$D_n(i)$, $3 \leq i \leq n-3$	Type 1a	Type 1a
$E_6(3)$	Type 1a	Type 1a
$E_6(5)$	Type 1a	Type 1a
$E_7(2)$	Type 1a	–
$E_7(6)$	Type 1a	Type 1a
$E_8(1)$	Type 1a	–
$F_4(4)$	Type 2	–

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon^+_{n\gamma})$	$V(\mu+\epsilon^-_{n\gamma})$
$D_n(n-2)$	Type 1a	Type 1a	Type 1a

Now we give the special values. Given subspace $U \subset \mathfrak{g}$ and weight $\lambda \in \mathfrak{h}^*$, we write $\Delta_\lambda(U) := \{\alpha \in \Delta(U) \mid \lambda - \alpha \in \Delta\}$.

Theorem 11. [20, 22] *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. The special value s_2 of the special constituent $V(\mu+\epsilon)$ is*

$$s_2 = \begin{cases} \dfrac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} - 1 & \text{if } V(\mu+\epsilon) \text{ is of type 1,} \\ -1 & \text{if } V(\mu+\epsilon) \text{ is of type 2, and} \\ n-i+1 & \text{if } V(\mu+\epsilon) \text{ is of type 3,} \end{cases}$$

where $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ is the number of elements of $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$.

Table 2: Special values for $V(\mu+\epsilon)$

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i), 3 \le i \le n-2$	$n-i-\frac{1}{2}$	1
$B_n(n-1)$	$\frac{1}{2}$	1
$B_n(n)$	-1	–
$C_n(i), 2 \le i \le n-1$	$n-i+1$	-1
$D_n(i), 3 \le i \le n-3$	$n-i-1$	1
$E_6(3)$	1	2
$E_6(5)$	1	2
$E_7(2)$	2	–
$E_7(6)$	1	3
$E_8(1)$	3	–
$F_4(4)$	-1	–

Parabolic subalgebra	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon^+_{n\gamma})$	$V(\mu+\epsilon^-_{n\gamma})$
$D_n(n-2)$	1	1	1

Note that, as it turns out that the special values for type 1a case and type 1b case can be expressed as the same formula, we combine these cases and just refer to as type 1. Table 2 exhibits the special values for $V(\mu+\epsilon)$ under consideration.

To find the special values we compute a concrete formula for $\omega_2(Y_l^*)$, where Y_l^* is a lowest weight vector for $V(\mu+\epsilon)^*$. (For ω_2, see the discussion below Definition 3.) It follows from (4) that $\omega_2(Y_l^*) \otimes 1$ gives an explicit formula for φ_2. Then, to achieve Problem B in the introduction, we give a concrete formula for a lowest weight vector for $\omega_2(V(\mu+\epsilon)^*)$. Our strategy is first to explicitly find a lowest weight vector Y_l^* for $V(\mu+\epsilon)^*$ either by using the τ_2 map or by observing $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. We then compute a formula for $\omega_2(Y_l^*)$ via (2). For the detail see Section 7.2 of [22] and Sections 4 and 5 of [20]. Here, we give formulas for Y_l^* and $\omega_2(Y_l^*)$ for $V(\mu+\epsilon)$ of type 1a or type 2. For type 1b case and type 3 cases, see Sections 4 and 5 of [20], respectively.

Theorem 12. [22, Section 7.2] *If $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2 then lowest weight vectors Y_l^* and $\omega_2(Y_l^*)$ for $V(\mu+\epsilon)$ and $\omega_2(V(\mu+\epsilon)^*)$, respectively, may be given by*

$$Y_l^* = \sum_{\eta \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))} N_{-\mu,\eta-\epsilon} N_{-\epsilon,\eta}\, X_{-\theta(\eta)} \otimes X_{-\eta}.$$

and

$$\omega_2(Y_l^*) = \sum_{\substack{\alpha \in \Delta_{\mu+\epsilon}(\mathfrak{g}(1)) \\ \eta \in \Delta_{\mu+\epsilon}(\mathfrak{z}(\mathfrak{n}))}} N_{-\mu,\eta-\epsilon} N_{-\epsilon,\eta} N_{\alpha,-\eta} N_{-\theta(\eta),\theta(\alpha)}\, X_{-\alpha} X_{-\theta(\alpha)}.$$

Here $\theta(\alpha) := (\mu+\epsilon)-\alpha$ and $N_{\alpha,\beta}$ are the structure constants of root vectors X_α and X_β, normalized as $\kappa(X_\alpha, X_{-\alpha}) = 1$.

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