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# ON THE GROUP OF HOLOMORPHIC AND ANTI-HOLOMORPHIC AUTOMORPHISMS OF A COMPACT HERMITIAN SYMMETRIC SPACE

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ABSTRACT. Let $f$ be a complex function on a domain in the complex plane $\mathbb{C}$. Then $f$ is holomorphic or anti-holomorphic, if and only if $f$ is a conformal map. we are interested in generalizing this to higher dimensional cases. In this paper, for a compact irreducible Hermitian symmetric space $M$, we determine the group $H^{\pm}(M)$ of all holomorphic and anti-holomorphic automorphisms of $M$, and we characterize the group $H^{\pm}(M)$ as the automorphism group of a certain $G$-structure on $M$, called the *generalized conformal structure*. This paper is a short-cut version; the detailed one will appear elsewhere.

## 1. SIMPLE GRADED LIE ALGEBRAS AND COMPACT HERMITIAN SYMMETRIC SPACES

**1.1.**

- Let
$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1. \tag{1.1}$$
be a complex simple graded Lie algebra (abbrev. GLA).
- $Z \in \tilde{\mathfrak{g}}_0$ is the characteristic element of $\tilde{\mathfrak{g}}$, that is, $\operatorname{ad} Z = k1$ on $\tilde{\mathfrak{g}}_k$, $k = 0, \pm 1$.
- $\tau$ is the grade-reversing Cartan involution of $\tilde{\mathfrak{g}}$, that is, $\tau(\tilde{\mathfrak{g}}_k) = \tilde{\mathfrak{g}}_{-k}$ $k = 0, \pm 1$, which is equivalent to $\tau(Z) = -Z$. Note that $\tau$ is a conjugation of $\tilde{\mathfrak{g}}$ with respect to a compact real form $\mathfrak{k}$ of $\tilde{\mathfrak{g}}$.
- $\operatorname{Aut}\tilde{\mathfrak{g}} (\subset \mathrm{GL}(\tilde{\mathfrak{g}}))$ : the automorphism group of the complex Lie algebra $\tilde{\mathfrak{g}}$.
- $\tilde{G}_0 := \operatorname{Aut}_{\mathrm{gr}} \tilde{\mathfrak{g}} := \{g \in \operatorname{Aut}\tilde{\mathfrak{g}} : g(\tilde{\mathfrak{g}}_k) = \tilde{\mathfrak{g}}_k, k = 0, \pm 1\}$: the group of grade-preserving automorphisms of $\tilde{\mathfrak{g}}$.
$\tilde{G}_0$ coincides with the centralizer $C_{\operatorname{Aut}\tilde{\mathfrak{g}}}(Z)$ of $Z$ in $\operatorname{Aut}\tilde{\mathfrak{g}}$.
Note that $\operatorname{Lie}\tilde{G}_0 = \tilde{\mathfrak{g}}_0$.
- $\tilde{U} := \tilde{G}_0 \exp \tilde{\mathfrak{g}}_{-1}$.
- $\tilde{G} := \tilde{G}_0 \operatorname{Int}\tilde{\mathfrak{g}}$ : an open subgroup of $\operatorname{Aut}\tilde{\mathfrak{g}}$.
$\tilde{U}$ is a parabolic subgroup of $\tilde{G}$, and $\tilde{G}_0$ is the Levi subgroup of $\tilde{U}$.
- We have the (complex) flag manifold $M = \tilde{G}/\tilde{U}$. It can be shown that $\tilde{G}$ acts on $M$ effectively.
- The symmetric space expression of $M$.
$\tilde{\tau}$: the Cartan involution of $\tilde{G}$ defined by $\tilde{\tau}(g) = \tau g \tau$, $g \in \tilde{G}$.
Then the set $K$ of all $\tilde{\tau}$-fixed elements in $\tilde{G}$ is a compact real form of $\tilde{G}$. Note that $\operatorname{Lie} K = \mathfrak{k}$. $M$ is expressed as
$$M = \tilde{G}/\tilde{U} = K/K_0,$$
where $K_0 = K \cap \tilde{U}$. Here $K/K_0$ is a compact irreducible Hermitian symmetric space. $K/K_0$ has a $K$-invariant Kähler-Einstein metric (cf. [5]).
- The identity component of $K$ coincides with that of the isometry group $I(M)$.

**Theorem 1.1.** *Let* $\mathrm{Hol}^+(M)$ *be the group of all holomorphic automorphisms of* $M = \tilde{G}/\tilde{U}$. *Then we have*

$$\mathrm{Hol}^+(M) = \tilde{G}.$$

*Proof.* (Sketch)
There are four steps. First of all, $\mathrm{Hol}^+(M)$ is a complex Lie group by a theorem of Bochner-Montgomery ([1, 2]).

(1) As was noted before, $\tilde{G}$ acts on $M$ effectively and holomorphically. Hence $\tilde{G} \subset \mathrm{Hol}^+(M)$.

(2) The existence of the $K$-invariant Kähler-Einstein metric on $M$ implies that

$$\mathrm{Lie}\,\mathrm{Hol}^+(M) = (\mathrm{Lie}\,I(M))^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} = \tilde{\mathfrak{g}},$$

by Matsushima [6]. Thus $\tilde{G}$ is an open subgroup of $\mathrm{Hol}^+(M)$.

(3) One can show that the center of $\mathrm{Hol}^+(M)$ reduces to the identity. Therefore $\mathrm{Hol}^+(M)$ is realized as an open subgroup of $\mathrm{Aut}\,\tilde{\mathfrak{g}}$ by taking the adjoint representation of $\mathrm{Hol}^+(M)$ on $\tilde{\mathfrak{g}}$.

(4) $M$ has the coset space expression in two ways:

$$M = \tilde{G}/\tilde{U} = \mathrm{Hol}^+(M)/\hat{U},$$

where $\hat{U} \supset \tilde{U}$. It is easy to see that $\hat{U} = \tilde{U}$, which shows the coincidence of the numerators. □

**1.2.** Here we consider the scalar restrictions of the objects in 1.1 to $\mathbb{R}$.

- Let

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

  be the real simple GLA, which is the scalar restriction of the complex GLA (1.1) to $\mathbb{R}$.
  Let $I$ be the complex structure on $\mathfrak{g}$ corresponding to the $i$-multiplication on $\tilde{\mathfrak{g}}$.
  $\tilde{\mathfrak{g}}$ can be expressed as the pair $(\mathfrak{g}, I)$.
- $Z \in \mathfrak{g}$ and $\tau$ are the same as those for $\tilde{\mathfrak{g}}$.
- $\mathrm{Aut}\,\mathfrak{g}(\subset \mathrm{GL}(\mathfrak{g}))$ : the automorphism group of the real Lie algebra $\mathfrak{g}$.
  Note that $\mathrm{Aut}\,\tilde{\mathfrak{g}} \subset \mathrm{Aut}\,\mathfrak{g}$.
- $G_0 := \mathrm{Aut}_{\mathrm{gr}}\,\mathfrak{g}$. Note that the inclusion $\tilde{G}_0 \subset G_0$ and $\mathrm{Lie}\,G_0 = \mathfrak{g}_0$ are valid.
- $U := G_0 \exp \mathfrak{g}_{-1} \supset \tilde{U}$.
- The open subgroup $G$ of $\mathrm{Aut}\,\mathfrak{g}$:
  $\mathrm{Aut}\,\mathfrak{g} \supset G := G_0\,\mathrm{Int}\,\mathfrak{g} \supset \tilde{G}$.
  $U$ is a parabolic subgroup of $G$, and $G_0$ is the Levi subgroup of $U$.
- As a real manifold, $M$ is expressed as a (real) flag manifold $G/U$.
  This is non-trivial, and will be proved in Corollary 2.4.

The following theorem will be proved in the section 3.

**Theorem 1.2.** *Let* $\mathrm{Hol}^{\pm}(M)$ *be the group of all holomorphic or anti-holomorphic automorphisms of* $M$. *Then we have*

$$\mathrm{Hol}^{\pm}(M) = G.$$

## 2. The relation between the groups $\tilde{G}$ and $G$

**Lemma 2.1.** *Let* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1$ *be a complex simple GLA and let* $Z$ *and* $\tau$ *be as before. Then there exists a unique normal real form* $\mathfrak{g}^N$ *of* $\tilde{\mathfrak{g}}$ *such that* $Z \in \mathfrak{g}^N$ *and that* $\tau(\mathfrak{g}^N) \subset \mathfrak{g}^N$.

$\mathfrak{g}^N$ can be expressed as a GLA

$$\mathfrak{g}^N = \mathfrak{g}^N_{-1} + \mathfrak{g}^N_0 + \mathfrak{g}^N_1,$$

where $\mathfrak{g}^N_k = \mathfrak{g}^N \cap \tilde{\mathfrak{g}}_k$ $(k = 0, \pm 1)$.

Now let $\nu$ be the conjugation of $(\mathfrak{g}, I)$ with respect to $\mathfrak{g}^N$. Then $\nu$ satisfies the following equalities:

$$\nu^2 = 1, \ \nu I = -I\nu.$$

Since $\nu(Z) = Z$, $\nu$ is grade-preserving on $\mathfrak{g}$. Hence we have

$$\nu \in G_0 \setminus \tilde{G}_0, \quad \nu \in \mathrm{Aut}\,\mathfrak{g} \setminus \mathrm{Aut}\,\tilde{\mathfrak{g}}.$$

Let $\bar{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. We extend $\nu$ $\mathbb{C}$-linearly to $\bar{\mathfrak{g}}$.

**Proposition 2.2.**

$$\mathrm{Aut}\,\mathfrak{g} = (\mathrm{Aut}\,\tilde{\mathfrak{g}}) \cdot < \nu > . \tag{2.1}$$

*Proof.* Let $\Pi$ be the Dynkin diagram of the complex simple Lie algebra $\tilde{\mathfrak{g}}$. Then it is well-known that

$$\mathrm{Aut}\,\tilde{\mathfrak{g}} / \mathrm{Int}\,\tilde{\mathfrak{g}} = \mathrm{Aut}(\Pi). \tag{2.2}$$

The Satake diagram of the real simple Lie algebra $\mathfrak{g}$ is given by the pair $(\bar{\Pi}, \nu)$, where $\bar{\Pi}$ is the Dynkin diagram of $\bar{\mathfrak{g}}$ which is the pair of two copies of $\Pi$. $\nu$ acts on $\bar{\Pi}$ as the Satake involution. Now let us denote by $(\mathrm{Aut}\,\mathfrak{g})^z$ the Zariski connected component of $\mathrm{Aut}\,\mathfrak{g}$. Then we see that $(\mathrm{Aut}\,\mathfrak{g})^z = \mathrm{Int}\,\tilde{\mathfrak{g}}$. Applying a result of H. Matsumoto ([7]) we conclude that

$$\mathrm{Aut}\,\mathfrak{g} / \mathrm{Int}\,\tilde{\mathfrak{g}} = \mathrm{Aut}\,\mathfrak{g} / (\mathrm{Aut}\,\mathfrak{g})^z = \mathrm{Aut}(\bar{\Pi}, \nu) = < \nu > (\mathrm{Aut}(\Pi)). \tag{2.3}$$

(2.1) follows from (2.2) and (2.3). □

From Proposition 2.2 we have

**Theorem 2.3.** (1) $G_0 = \tilde{G}_0 \cdot < \nu >$,
(2) $U = \tilde{U} \cdot < \nu >$,
(3) $G = \tilde{G} \cdot < \nu >$. *In particular,* $\tilde{G}$ *is a normal subgroup of* $G$.

**Corollary 2.4.** *The complex flag manifold* $M$ *is expressed as the real flag manifold*

$$M = \tilde{G}/\tilde{U} = G/U.$$

*Proof.* By Theorem 2.3, we have $G = \tilde{G}U$. Consequently we get

$$G/U = \tilde{G}U/U = \tilde{G}/\tilde{G} \cap U = \tilde{G}/\tilde{U} = M.$$

□

## 3. The proof of Theorem 1.2

**Definition 3.1.** Let $X$ be a smooth manifold, $I$ a complex structure on $X$ and let $\sigma : X \to X$ be a diffeomorphism. Then $\sigma$ is said to be an *anti-holomorphic involution*, if the following conditions are satisfied on $X$

$$\sigma^2 = 1, \ \sigma_* I = -I\sigma_*,$$

where $\sigma_*$ is the differential of $\sigma$. The pair $(\sigma, I)$ is called an *anti-holomorphic pair* (shortly, AHP).

### 3.1. The AHP $(\tilde{\nu}, \tilde{I})$ on $\tilde{G}$

We identify the Lie algebra $(\mathfrak{g}, I)$ with the Lie algebra of left-invariant vector fields on $\tilde{G}$. The complex structure $I$ on $\mathfrak{g}$ and the left-invariant complex structure $\tilde{I}$ on $\tilde{G}$ are related with each other by the equality

$$\tilde{I}_p X_p = (IX)_p, \ p \in \tilde{G}, X \in \mathfrak{g},$$

which is also expressed as

$$\tilde{I}X = IX, \tag{3.1}$$

where both sides are vector fields on $\tilde{G}$.

Next, noting that $\nu\tilde{G}\nu^{-1} \subset \tilde{G}$, we define the automorphism $\tilde{\nu} : \tilde{G} \to \tilde{G}$ as

$$\tilde{\nu}(a) = \nu a \nu^{-1}, \ a \in \tilde{G}. \tag{3.2}$$

Then $\tilde{\nu}$ is naturally extended to the whole $G$.

**Lemma 3.2.** $(\tilde{\nu}, \tilde{I})$ *is an* AHP on $\tilde{G}$

*Proof.* Note that $\tilde{\nu}_* = \nu$. By using this equality, (3.1) and the anti-linearity of $\nu$, we can conclude the equality $\tilde{\nu}_* \tilde{I} = -\tilde{I}\tilde{\nu}_*$. □

### 3.2. The AHP $(\nu_M, J)$ on $M$

First of all, note that

$$\tilde{\nu}(\tilde{U}) = \nu\tilde{U}\nu^{-1} = \tilde{U}. \tag{3.3}$$

The left action of $\nu$ on $G/U$ at a point $gU$ $(g \in G)$ can be expressed as

$$\nu(gU) = \nu gU = \nu g\nu^{-1}\nu U\nu^{-1} = \nu g\nu^{-1}U = \tilde{\nu}(g)U.$$

Restricting this equality to $\tilde{G}/\tilde{U}$, we have the following action of $\nu$ on $\tilde{G}/\tilde{U}$:

$$\nu(a\tilde{U}) = \tilde{\nu}(a)\tilde{U}, \ a \in \tilde{G}. \tag{3.4}$$

In the following, the $\nu$ acting on $\tilde{G}/\tilde{U}$ will be denoted by $\nu_M$.

Let $\pi : \tilde{G} \to M = \tilde{G}/\tilde{U}$ be the natural projection. Then the following commutativity follows from (3.4):

$$\pi\tilde{\nu} = \nu_M\pi. \tag{3.5}$$

Next we will define the invariant complex structure $J$ on $M = \tilde{G}/\tilde{U}$, which is $\pi$-related to $\tilde{I}$. We consider the following identification for the complex tangent space of $M$ at the origin $o$:

$$T_o(M) = \operatorname{Lie}\tilde{G}/\operatorname{Lie}\tilde{U} = \tilde{\mathfrak{g}}_1 = \mathfrak{g}_1^N + I\mathfrak{g}_1^N.$$

The complex structure $J_o$ on $\tilde{\mathfrak{g}}_1$ is given by

$$J_o = I|_{\tilde{\mathfrak{g}}_1} = \operatorname{ad}_{\tilde{\mathfrak{g}}_1}(iZ).$$

$J_o$ commutes with the linear isotropy representation of $\tilde{U}$, that is,

$$[\operatorname{Ad}_{\tilde{\mathfrak{g}}_1}\tilde{G}_0, J_o] = 0.$$

Therefore $J_o$ extends uniquely to a $\tilde{G}$-invariant almost complex structure $J$ on $M$. It can be seen from the construction that $\tilde{I}$ and $J$ are $\pi$-related, that is,

$$\pi_*\tilde{I} = J\pi_*. \tag{3.6}$$

It follows from (3.6) that the almost complex structure $J$ is integrable.

**Proposition 3.3.** *$(\nu_M, J)$ is an* AHP on $M$.

*Proof.* In view of (3.5), (3.6) and Lemma 3.2, we have

$$\nu_{M*}J\pi_* = \nu_{M*}\pi_*\tilde{I} = \pi_*\tilde{\nu}_*\tilde{I} = -\pi_*\tilde{I}\tilde{\nu}_* = -J\pi_*\tilde{\nu}_* = -J\nu_{M*}\pi_*.$$

Therefore we have the equality $\nu_{M*}J = -J\nu_{M*}$. □

### Proof of Theorem 1.2

We denote by $\mathrm{Hol}^-(M)$ the totality of anti-holomorphic automorphisms of $M$. Since $\nu_M$ interchanges $\mathrm{Hol}^+(M)$ with $\mathrm{Hol}^-(M)$, we have the expression

$$\mathrm{Hol}^\pm(M) = \mathrm{Hol}^+(M) \amalg \nu_M \mathrm{Hol}^+(M). \tag{3.7}$$

As is seen in the proof of Theorem 1.1, $\mathrm{Hol}^+(M)$, realized as a subgroup of $\operatorname{Aut}\tilde{\mathfrak{g}}$, coincides with $\tilde{G}$. Also $\nu$ is the realization of $\nu_M$ as an element of $G$. Therefore, considering (3.7) and Theorem 1.1, we have

$$\mathrm{Hol}^\pm(M) = \tilde{G} \amalg \nu\tilde{G} = \tilde{G}\cdot < \nu >= G.$$

## 4. Relation to the generalized conformal structure on $M$

First of all, let us remind the basic facts on the generalized conformal structure (simply, GCS) on the real flag manifold $M = G/U$ (cf. [3]). Let $r$ be the rank of the symmetric space $M$, and let $o$ be the origin of the coset space $M = G/U$. As for the case of the complex tangent space $T_o(M)$, the real tangent space at the origin $o \in M$ can be identified with $\mathfrak{g}_1$. Let $\rho$ be the linear isotropy representation of $U$ on $\mathfrak{g}_1$. Then we have $\rho(U) = G_0$. The $G_0$-orbit decomposition of $\mathfrak{g}_1$ is given by

$$\mathfrak{g}_1 = V_r \amalg V_{r-1} \amalg .... \amalg V_0,$$

where $V_r$ is a single open orbit and $V_0 = (0)$. Since $G_0$ contains $\mathbb{C}^*$, all orbits are cones. The union of singular orbits, denoted by $C_o$, is an algebraic cone. The automorphism group $\operatorname{Aut} C_o$ is defined as the subgroup of $GL(\mathfrak{g}_1)$ consisting of all elements leaving $C_0$ stable.

**Lemma 4.1.** *([3]) Suppose that $r \geq 2$. Then we have*

$$\operatorname{Aut} C_o = G_0.$$

By this lemma, one can translate the cone $C_o$ to each point of $M$ by the action of $G$. Thus we have the cone field $\mathcal{C} = \{C_p\}_{p\in M}$ on $M$, which is called the generalized conformal structure (simply GCS) on $M$. Now we are going to define the automorphism group $\operatorname{Aut}(M, \mathcal{C})$ of the GCS $\mathcal{C}$. $\operatorname{Aut}(M, \mathcal{C})$ is defined to be the group of all smooth diffeomorphisms $f$ of $M$ leaving $\mathcal{C}$ invariant, in other words, for $\mathcal{C} = \{C_p\}_{p\in M}$, $f$ satisfies

$$f_{*p}C_p = C_{f(p)}, \ p \in M.$$

We can characterize the group $G$ as the automorphism group of the GCS, namely,

**Theorem 4.2.** *([3])Let $G$ be as above. Suppose that $r \geq 2$. Then*

$$\operatorname{Aut}(M, \mathcal{C}) = G.$$

Combining the above theorem with Theorem 1.2, we have

**Theorem 4.3.** *Let $M$ be a compact irreducible Hermitian symmetric space of* rank $\geq 2$. Then we have

$$\operatorname{Hol}^{\pm}(M) = \operatorname{Aut}(M, \mathcal{C}).$$

The following theorem gives a necessary and sufficient condition for the global extension of a local holomorphic or local anti-holomorphic transformation on $M$. The proof is similar to the case of the causal structure (cf. [4]).

**Theorem 4.4.** *Let $D$ be a domain in $M$ and let $f$ be a local holomorphic or local anti-holomorphic transformation of $M$ defined on $D$. Suppose that* rank M $\geq 2$. *Then $f$ extends uniquely to an element of* $\operatorname{Hol}^{\pm}(M)$ *if and only if $f$ is a local $\mathcal{C}$-conformal transformation on $D$.*

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