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Extensions for certain subordination relations

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Abstract

For some complex number γ which has a positive real part, a certain subordination relation concerned with the Bernardi integral operator I_{γ} was proven by D. J. Hallenbeck and St. Ruscheweyh (Proc. Amer. Math. Soc. **52**(1975), 191–195). By considering the analyticity of the functions defined by the Bernardi integral operator I_{γ} for some non-zero complex number γ with $\mathrm{Re}\,\gamma \leqq 0$, an extension for certain subordination relation are discussed.

1 Introduction and definitions

For a positive integer n and a complex number a, let $\mathcal{H}[a,n]$ denote the class of functions $p(z)$ of the form

$$p(z)=a+\sum_{k=n}^{\infty}a_kz^k$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Also, let $\mathcal{A}$ be the class of analytic functions $f(z)$ which are normalized by $f(0)=f'(0)-1=0$.

An analytic function $f(z)$ is said to be convex in $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a convex domain (A domain $\mathbb{D}\subset\mathbb{C}$ is said to be convex if the line segment joining any two points of $\mathbb{D}$ lies entirely in $\mathbb{D}$). It is well-known that the function $f(z)$ is convex in $\mathbb{U}$ if and only if $f'(0)\neq 0$ and

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in\mathbb{U}).$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z)\prec q(z), \tag{1.1}$$

if there exists an analytic function $w(z)$ with $w(0)=0$, $|w(z)|<1$ $(z\in\mathbb{U})$, and such that $p(z)=q\big(w(z)\big)$ $(z\in\mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.1) implies that

$$p(0)=q(0) \quad \text{and} \quad p(\mathbb{U})\subset q(\mathbb{U}). \tag{1.2}$$

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In particular, if $q(z)$ is univalent in $\mathbf{U}$, then the subordination (1.1) is equivalent to the condition (1.2).

For the functions $p(z) \in \mathcal{H}[a,n]$ and $h(z) \in \mathcal{H}[a,1]$, Hallenbeck and Ruscheweyh [3] (also Miller and Mocanu [6]) considered the following first-order differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \tag{1.3}$$

where γ is complex number with $\gamma \neq 0$, and proved the following subordination result.

Lemma 1.1 *Let n be a positive integer, and let γ be a complex number with $\mathrm{Re}\,\gamma > 0$. Also, let $h(z)$ be analytic and convex univalent in $\mathbf{U}$ with $h(0) = a$. If $p(z) \in \mathcal{H}[a,n]$ satisfies the differential subordination (1.3), then $p(z) \prec q(z)$, where*

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}\,dt. \tag{1.4}$$

The function $q(z)$ defined by (1.4) is the best dominant of the subordination (1.3).

Remark 1.2 If $p(z) \prec q(z)$ for all $p(z)$ satisfying the subordination (1.3), then $q(z)$ is called a dominant of the subordination (1.3). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of the subordination (1.3) is said to be the best dominant of the subordination (1.3). Note that the best dominant is unique up to a rotation of $\mathbf{U}$.

For the function $f(z) \in \mathcal{H}[0,n]$, the Bernardi integral operator [2] is defined by

$$\mathrm{I}_\gamma[f](z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1}\,dt, \tag{1.5}$$

where $\gamma = 0, 1, 2, \cdots$. In particular, the integral operator I_0 is well-known as the Alexander integral operator [1]. The integral operator I_γ is well-defined on $\mathcal{H}[0,n]$ and maps $f(z)$ into $\mathcal{H}[0,n]$. Specially, we note that $\mathrm{I}_\gamma[f](z) \in \mathcal{A}$ for $f(z) \in \mathcal{A}$. Next lemma [6, Lemma 1.2c] shows that the Bernardi integral operator can be extended for certain complex values of γ.

Lemma 1.3 *Let m be an integer with $m \geqq 0$, and let γ be a complex number with* $\mathrm{Re}\,\gamma > -m$. *If $f(z) = \sum_{k=m}^{\infty} a_k z^k$ is analytic in $\mathbf{U}$, and $F(z)$ is defined by*

$$F(z) = \frac{1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}\,dt \tag{1.6}$$

then $F(z) = \sum_{k=m}^{\infty} \frac{a_k}{\gamma+k} z^k$ is analytic in $\mathbf{U}$.

Remark 1.4 Let us consider the analyticity of the function $F(z)$ defined by (1.6). If $f(z) \in \mathcal{H}[a,1]$ with $a \neq 0$, then by Lemma 1.2 with $m = 0$, the function $F(z)$ is analytic in

$\mathbb{U}$ for some complex number γ such that $\operatorname{Re}\gamma > 0$. On the other hand, considering the case $m = 1$ in Lemma 1.2, we find that the function $F(z)$ with $f(z) \in \mathcal{H}[0,1]$ is analytic in $\mathbb{U}$ for some complex number γ such that $\operatorname{Re}\gamma > -1$.

To prove the subordination relation in Lemma 1.1, Miller and Mocanu [6] discussed the analyticity of the solution $q(z)$ of the following first-order differential equation

$$q(z) + \frac{nzq'(z)}{\gamma} = h(z) \qquad (z \in \mathbb{U}) \tag{1.7}$$

with $q(0) = h(0)$, where γ is complex number with $\gamma \neq 0$. Note that the solution $q(z)$ of the differential equation (1.7) is given by (1.4). Let $h(z) \in \mathcal{H}\big[h(0),1\big]$. According to the first assertion in Remark 1.4, the function $q(z)$ given in (1.4) is analytic in $\mathbb{U}$ for some complex number γ with $\operatorname{Re}\gamma > 0$, because $h(z) \in \mathcal{H}\big[h(0),1\big]$.

On the other hand, let us define the functions $h_0(z)$ and $q_0(z)$ by

$$h_0(z) = h(z) - h(0) \quad \text{and} \quad q_0(z) = q(z) - q(0) \tag{1.8}$$

for $z \in \mathbb{U}$, where $q(0) = h(0)$. Then the differential equation (1.7) is equivalent to

$$q_0(z) + \frac{nzq_0'(z)}{\gamma} = h_0(z) \qquad (z \in \mathbb{U}) \tag{1.9}$$

with $q_0(0) = h_0(0) = 0$. Noting that the solution $q_0(z)$ of the differential equation (1.9) is given by

$$q_0(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h_0(t)t^{\frac{\gamma}{n}-1}\,dt \qquad (z \in \mathbb{U}),$$

it follows from the equalities in (1.8) that the solution $q(z)$ of the differential equation (1.7) can be represented by

$$q(z) = h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z \Big(h(t) - h(0)\Big)t^{\frac{\gamma}{n}-1}\,dt \qquad (z \in \mathbb{U}). \tag{1.10}$$

Then since $h(z)-h(0) \in \mathcal{H}[0,1]$, the second assertion in Remark 1.4 leads us that the function $q(z)$ given by (1.10) is analytic in $\mathbb{U}$ for some complex number γ with $\operatorname{Re}\left(\frac{\gamma}{n}\right) > -1$.

From the above-mentioned, we expect that the subordination relation in Lemma 1.1 can be discussed for some complex number γ with $\gamma \neq 0$ and $\operatorname{Re}\gamma > -n$ by replacing the conclusion in Lemma 1.1 with the following subordination

$$p(z) \prec h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z \Big(h(t) - h(0)\Big)t^{\frac{\gamma}{n}-1}\,dt.$$

In the present paper, by considering some properties for the function $q(z)$ given in (1.10), we will discuss the following subordination relation :

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad \text{implies} \quad p(z) \prec h(0) + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z \Big(h(t) - h(0)\Big)t^{\frac{\gamma}{n}-1}\,dt \tag{1.11}$$

for some complex number γ with $\gamma \neq 0$ and $\operatorname{Re}\gamma > -n$.

2 Preliminary results

Miller and Mocanu [6] developed a lemma which is well-known as the open door lemma. By considering a simpler version of the open door function, Kuroki and Owa [5] provided the better result for the open door lemma.

Definition 2.1 (Simpler version of open door function) Let c be a complex number with $\mathrm{Re}\,c > 0$. Then the open door function $R_c(z)$ is defined by

$$R_c(z) = -\bar{c} - \frac{1}{1-z} + \frac{2\mathrm{Re}\,c + 1}{1 + \frac{c}{\bar{c}}z} \qquad (z \in \mathbf{U}). \tag{2.1}$$

The function $R_c(z)$ is analytic and univalent in $\mathbf{U}$ with $R_c(0) = c$. In addition, $R_c(z)$ maps $\mathbf{U}$ onto the complex plane with slits along the half-lines ℓ_c^+ and ℓ_c^-, where

$$\ell_c^+ = \left\{ w \in \mathbf{C} : \mathrm{Re}\,w = 0 \text{ and } \mathrm{Im}\,w \geqq \frac{1}{\mathrm{Re}\,c}\left(|c|\sqrt{2\mathrm{Re}\,c+1} - \mathrm{Im}\,c\right)\right\}$$

and

$$\ell_c^- = \left\{ w \in \mathbf{C} : \mathrm{Re}\,w = 0 \text{ and } \mathrm{Im}\,w \leqq -\frac{1}{\mathrm{Re}\,c}\left(|c|\sqrt{2\mathrm{Re}\,c+1} + \mathrm{Im}\,c\right)\right\}.$$

Note that the slit domain $\mathbf{C} \setminus \{\ell_c^+ \cup \ell_c^-\}$ is not symmetric with respect to the real axis (see [5]).

Lemma 2.2 (Open door lemma) *Let c be a complex number with $\mathrm{Re}\,c > 0$, and let $R(z) \in \mathcal{H}[c, 1]$ satisfy the subordination*

$$R(z) \prec R_c(z),$$

where $R_c(z)$ is defined by (2.1). If $p(z) \in \mathcal{H}[\frac{1}{c}, 1]$ satisfies the differential equation

$$zp'(z) + R(z)p(z) = 1 \qquad (z \in \mathbf{U}),$$

then $\mathrm{Re}\,p(z) > 0 \quad (z \in \mathbf{U})$.

More general form of this lemma for $p(z) \in \mathcal{H}[\frac{1}{c}, n]$ was given in the work [5].

3 Some properties for certain integral operator

To considering the subordination relation (1.11), we need to develop some property for certain integral operator by using the open door lemma.

Theorem 3.1 *Let γ be a complex number with $\mathrm{Re}\,\gamma > -1$, and let $f(z) \in \mathcal{A}$ satisfy*

$$1 + \frac{zf''(z)}{f'(z)} + \gamma \prec R_{\gamma+1}(z), \tag{3.1}$$

where $R_{\gamma+1}(z)$ ***is the open door function defined by***

$$R_{\gamma+1}(z) = -(\overline{\gamma}+1) - \frac{1}{1-z} + \frac{2\mathrm{Re}\,\gamma+3}{1+\frac{\gamma+1}{\overline{\gamma}+1}z} \qquad (z \in \mathbb{U}). \tag{3.2}$$

If $F = \mathrm{L}_\gamma[f]$ *is defined by* (1.5), *then* $F(z) \in \mathcal{A}$, $F'(z) \neq 0 \quad (z \in \mathbb{U})$ *and*

$$\mathrm{Re}\left(1 + \frac{zF''(z)}{F'(z)} + \gamma\right) > 0 \qquad (z \in \mathbb{U}). \tag{3.3}$$

Proof. From the subordination (3.1), we note that $f'(z) \neq 0 \quad (z \in \mathbb{U})$. By Lemma 1.3, it is easy to see that

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)t^{\gamma-1}\,dt \in \mathcal{A}$$

for $\mathrm{Re}\,\gamma > -1$. If we define the function $p(z)$ by

$$p(z) = \frac{1}{z^{\gamma+1}f'(z)} \int_0^z f'(t)t^{(\gamma+1)-1}\,dt \qquad (z \in \mathbb{U}), \tag{3.4}$$

then since $f'(z) \in \mathcal{H}[1,1]$, it follows from Lemma 1.3 that $p(z) \in \mathcal{H}\left[\frac{1}{\gamma+1}, 1\right]$. By differentiating the equality (3.4), we find that $p(z)$ satisfies the differential equation

$$zp'(z) + R(z)p(z) = 1 \qquad (z \in \mathbb{U}), \tag{3.5}$$

where $R(z)$ is defined by

$$R(z) = 1 + \frac{zf''(z)}{f'(z)} + \gamma \qquad (z \in \mathbb{U}). \tag{3.6}$$

From the subordination (3.1), we see that $R(z) \in \mathcal{H}[\gamma+1, 1]$, and $R(z)$ satisfies the subordination

$$R(z) \prec R_{\gamma+1}(z),$$

where $R_{\gamma+1}(z)$ is defined by (3.2). Thus, the function $p(z)$ satisfies the conditions of Lemma 2.2 with $c = \gamma+1$, and so we deduce that

$$\mathrm{Re}\,p(z) > 0 \quad \text{and} \quad p(z) \neq 0 \tag{3.7}$$

for $z \in \mathbb{U}$. Moreover, the function $p(z)$ defined by (3.4) can be represented by

$$p(z) = \frac{1}{(\gamma+1)zf'(z)} \frac{\gamma+1}{z^\gamma} \int_0^z \left(tf'(t)\right) t^{\gamma-1}\,dt$$

$$= \frac{1}{(\gamma+1)zf'(z)} \left\{ z\left(\mathrm{L}_\gamma[f](z)\right)' \right\} = \frac{F'(z)}{(\gamma+1)f'(z)} \qquad (z \in \mathbb{U}),$$

which implies that

$$F'(z) = (\gamma+1)f'(z)p(z) \qquad (z \in \mathbb{U}). \tag{3.8}$$

Then since $p(z) \neq 0 \quad (z \in \mathbb{U})$, it is clear that $F'(z) \neq 0 \quad (z \in \mathbb{U})$. Making differentiation (3.8) logarithmically, we have

$$1 + \frac{zF''(z)}{F'(z)} + \gamma = \left(1 + \frac{zf''(z)}{f'(z)} + \gamma\right) + \frac{zp'(z)}{p(z)} = \frac{1}{p(z)} \qquad (z \in \mathbb{U})$$

from the equations (3.5) and (3.6). Hence, the condition (3.7) shows that the function $F(z)$ satisfies the inequality (3.3). This completes the proof of Theorem 3.1. □

4 Main results

Kuroki and Owa [4] proved the following lemma concerned with a special first-order differential subordination for certain complex values of λ which has a negative real part.

Lemma 4.1 *Let n be a positive integer, and let λ be a complex number with*

$$\text{(4.1)} \qquad \operatorname{Re}\lambda \leqq 0 \quad \textit{and} \quad \left|\lambda + \frac{1}{2n}\right| > \frac{1}{2n}.$$

Also, let $q(z)$ be analytic in $\mathbb{U}$ with $q(0) = a$, $q'(0) \neq 0$ and

$$\text{(4.2)} \qquad \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > -\frac{1}{n}\operatorname{Re}\left(\frac{1}{\lambda}\right) \qquad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{H}[a, n]$ satisfies the subordination

$$\text{(4.3)} \qquad p(z) + \lambda zp'(z) \prec q(z) + \lambda nzq'(z),$$

then $p(z) \prec q(z)$.

If we take $\lambda = \dfrac{1}{\gamma}$ for some non-zero complex number γ, then the condition (4.1) is equivalent to the inequality $-n < \operatorname{Re}\gamma \leqq 0$. In addition, the subordination (4.3) can be written as follows:

$$p(z) + \frac{zp'(z)}{\gamma} \prec q(z) + \frac{nzq'(z)}{\gamma}.$$

By making use of Lemma 4.1 with $\lambda = \dfrac{1}{\gamma}$, and applying Theorem 3.1, we derive the following result concerned with the subordination (1.11).

Theorem 4.2 *Let n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and*

$$-n < \operatorname{Re}\gamma \leqq 0.$$

Also, let $h(z) \in \mathcal{H}[a, 1]$ satisfy the subordination

$$\text{(4.4)} \qquad 1 + \frac{zh''(z)}{h'(z)} + \frac{\gamma}{n} \prec R_{\frac{\gamma}{n}+1}(z),$$

where $R_{\frac{\gamma}{n}+1}(z)$ *is the open door function defined by*

(4.5) $$R_{\frac{\gamma}{n}+1}(z) = -\left(\frac{\overline{\gamma}}{n}+1\right) - \frac{1}{1-z} + \frac{2\text{Re}\left(\frac{\gamma}{n}\right)+3}{1+\frac{\frac{\gamma}{n}+1}{\frac{\overline{\gamma}}{n}+1}z} \qquad (z \in \mathbb{U}).$$

If $p(z) \in \mathcal{H}[a, n]$ *satisfies the differential subordination* (1.3), *then* $p(z) \prec q(z)$, *where*

(4.6) $$q(z) = a + \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z (h(t)-a)t^{\frac{\gamma}{n}-1}\,dt.$$

The function $q(z)$ *defined by* (4.6) *is the best dominant of the subordination* (1.3).

Proof. Note that $h'(0) \neq 0$ from $h(z) \in \mathcal{H}[a, 1]$. If we define the function $q(z)$ by

$$q(z) = a + \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z (h(t)-a)t^{\frac{\gamma}{n}-1}\,dt \qquad (z \in \mathbb{U}),$$

then by Lemma 1.3, we find that $q(z) \in \mathcal{H}[a, 1]$ with $q'(0) \neq 0$. Also, it is easy to see that

$$h(z) = q(z) + \frac{nzq'(z)}{\gamma} \qquad (z \in \mathbb{U}).$$

It sufficies to show that the function $q(z)$ satisfies the inequality (4.2) with $\lambda = \frac{1}{\gamma}$ according to Lemma 4.1. If we let

$$f(z) = \frac{h(z)-a}{h'(0)} \qquad (z \in \mathbb{U}),$$

then the subordination (4.4) can be written by

$$1 + \frac{zf''(z)}{f'(z)} + \frac{\gamma}{n} \prec R_{\frac{\gamma}{n}+1}(z).$$

We also set

$$F(z) = \mathrm{I}_{\frac{\gamma}{n}}[f](z),$$

where $\mathrm{I}_{\frac{\gamma}{n}}$ is defined by (1.5). Since $f(z) \in \mathcal{A}$ and $\text{Re}\left(\frac{\gamma}{n}\right) > -1$, we deduce that $F(z)$ satisfies the inequality

(4.7) $$\text{Re}\left(1 + \frac{zF''(z)}{F'(z)} + \frac{\gamma}{n}\right) > 0 \qquad (z \in \mathbb{U})$$

by applying Theorem 3.1. A simple check gives us that $q(z)$ can be represented by

$$q(z) = a + \frac{\frac{\gamma}{n}h'(0)}{1+\frac{\gamma}{n}}F(z) \qquad (z \in \mathbb{U}).$$

Therefore, the inequality (4.7) shows that $q(z)$ satisfies the inequality

$$\text{Re}\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\gamma}{n}\right) > 0 \qquad (z \in \mathbb{U}).$$

Since all conditions of Lemma 4.1 with $\lambda = \frac{1}{\gamma}$ are satisfied, we conclude that $p(z) \prec q(z)$, which completes the proof of Theorem 4.2. □

Remark 4.3 For $h(z) \in \mathcal{H}[a,1]$, if we define the function $H(z)$ by

$$H(z) = 1 + \frac{zh''(z)}{h'(z)} \qquad (z \in \mathbb{U}), \tag{4.8}$$

then the assumption (4.4) in Theorem 4.2 can be written by

$$H(z) \prec R_{\frac{\gamma}{n}+1}(z) - \frac{\gamma}{n}, \tag{4.9}$$

where $R_{\frac{\gamma}{n}+1}(z)$ is defined by (4.5). The subordination (4.9) means that $H(z)$ maps $\mathbb{U}$ onto inside of the slit domain $\mathbb{C} \setminus \{\ell^+ \cup \ell^-\}$, where

$$\ell^+ = \left\{ w \in \mathbb{C} : \operatorname{Re} w = -\frac{\operatorname{Re}\gamma}{n} \text{ and } \operatorname{Im} w \geqq \frac{|\gamma + n|\sqrt{\frac{2\operatorname{Re}\gamma}{n} + 3} - (\operatorname{Im}\gamma)\left(\frac{\operatorname{Re}\gamma}{n} + 2\right)}{\operatorname{Re}\gamma + n} \right\}$$

and

$$\ell^- = \left\{ w \in \mathbb{C} : \operatorname{Re} w = -\frac{\operatorname{Re}\gamma}{n} \text{ and } \operatorname{Im} w \leqq \frac{-|\gamma + n|\sqrt{\frac{2\operatorname{Re}\gamma}{n} + 3} - (\operatorname{Im}\gamma)\left(\frac{\operatorname{Re}\gamma}{n} + 2\right)}{\operatorname{Re}\gamma + n} \right\}.$$

5 An extension of subordination relation for certain complex values of γ

Let ζ be a smooth arc in $\mathbb{U}$ connecting 0 to z, and assign a value to $\lim_{t\to 0} \arg t \quad (t \in \zeta)$. We define $t^\gamma = e^{\gamma \log t} \quad (t \in \zeta)$ by continuation. Noting that

$$\lim_{t\to 0}|t^\gamma| = \lim_{t\to 0} |t|^{\operatorname{Re}\gamma} e^{-(\operatorname{Im}\gamma)\arg t} = 0 \qquad (t \in \zeta)$$

when $\operatorname{Re}\gamma > 0$, we define $t^\gamma = 0$ at $t = 0 \quad (\operatorname{Re}\gamma > 0)$. Thus, a simple calculation gives that

$$\int_0^z t^{\gamma-1}\,dt = \int_\zeta t^{\gamma-1}\,dt = \left[\frac{t^\gamma}{\gamma}\right]_0^z = \frac{z^\gamma}{\gamma} \qquad (\operatorname{Re}\gamma > 0).$$

Therefore, it follows from the above fact that

$$\frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1}\,dt = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z a t^{\frac{\gamma}{n}-1}\,dt + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a) t^{\frac{\gamma}{n}-1}\,dt$$

$$= a + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a) t^{\frac{\gamma}{n}-1}\,dt \qquad (\operatorname{Re}\gamma > 0),$$

where $h(z) \in \mathcal{H}[a,1]$. This leads that the equality (1.4) in Lemma 1.1 can be replaced with the equality (4.6). Hence by combining the assertions in Lemma 1.1 and Theorem 4.2, we derive the subordination result for some non-zero complex number γ with $\operatorname{Re}\gamma > -n$.

Theorem 5.1 *Let n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and $\operatorname{Re}\gamma > -n$. Also, let $h(z) \in \mathcal{H}[a,1]$, and suppose that $H(z)$ defined by (4.8) satisfy one of the following :*

$$(i) \quad \operatorname{Re} H(z) > 0 \quad (z \in \mathbb{U}) \quad \textit{when } \operatorname{Re}\gamma > 0,$$

$$(ii) \quad H(z) \prec R_{\frac{\gamma}{n}+1}(z) - \frac{\gamma}{n} \quad \textit{when } -n < \operatorname{Re}\gamma \leqq 0,$$

where $R_{\frac{\gamma}{n}+1}(z)$ is the open door function defined by (4.5). Then $p(z) \in \mathcal{H}[a,n]$ satisfies the following subordination relation :

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad \textit{implies} \quad p(z) \prec q(z) = a + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - a)t^{\frac{\gamma}{n}-1}\,dt,$$

and the function $q(z)$ is the best dominant of the subordination (1.3).

If we consider the function $h(z)$ given by

$$h(z) = 1 + z \in \mathcal{H}[1,1],$$

then, it is easy to see that $h(z)$ satisfies all assumptions in Theorem 5.1. Also, it follows that

$$q(z) = 1 + \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z (h(t) - 1)t^{\frac{\gamma}{n}-1}\,dt = 1 + \frac{\gamma}{n+\gamma} z \qquad (\operatorname{Re}\gamma > -n).$$

Hence by Theorem 5.1, we find the following corollary.

Corollary 5.2 *Let n be a positive integer, and let γ be a complex number with $\gamma \neq 0$ and $\operatorname{Re}\gamma > -n$. Then $p(z) \in \mathcal{H}[1,n]$ satisfies the following subordination relation :*

$$p(z) + \frac{zp'(z)}{\gamma} \prec 1 + z \quad \textit{implies} \quad p(z) \prec 1 + \frac{\gamma}{n+\gamma} z. \tag{5.1}$$

Example 5.3 Let us consider the function $p(z)$ given by

$$p(z) = 1 + \frac{i}{2} z^2 \in \mathcal{H}[1,1] \tag{5.2}$$

in Corollary 5.2 with $n = 1$ and $\gamma = -\frac{1}{2}(1 - i)$. A simple calculation gives that

$$p(z) + \frac{zp'(z)}{-\frac{1}{2}(1-i)} = 1 + \frac{1}{2} z^2 \qquad (z \in \mathbb{U}).$$

Then, we see that $p(z)$ defined by (5.2) satisfies the subordination relation (5.1) with $n = 1$ and $\gamma = -\frac{1}{2}(1 - i)$.

Remark 5.4 From Corollary 5.2, we find that $p(z) \in \mathcal{H}[1,n]$ satisfies the following relation

$$\left|p(z)+\frac{zp'(z)}{\gamma}-1\right|<1 \quad (z\in\mathbb{U}) \quad \textbf{implies} \quad |p(z)-1|<\frac{|\gamma|}{|n+\gamma|} \quad (z\in\mathbb{U})$$

for some non-zero complex number γ with $\operatorname{Re}\gamma > -n$.

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