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Embedding α-convex functions in the class $\mathcal{U}$

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Abstract

In this paper the relation between classes

$$\mathcal{M}(\alpha,\beta)=\left\{f\in\mathcal{A}:\operatorname{Re}(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta,\ z\in\mathbb{D}\right\}\quad(\alpha\in\mathbb{R},0\leq\beta<1)$$

and

$$\mathcal{U}(\lambda,\mu)=\left\{f\in\mathcal{A}:\frac{f(z)}{z}\neq 0\text{ and }\left|\left(\frac{z}{f(z)}\right)^{1+\mu}\cdot f'(z)-1\right|<\lambda,\ z\in\mathbb{D}\right\}\quad(\mu\in\mathbb{C},\lambda>0)$$

is studied and sharp sufficient conditions that imply

$$\mathcal{M}(\alpha,\beta)\subset\mathcal{U}(\lambda,\mu)$$

are given, together with several examples.

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1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the class of functions that are analytic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and

$$\mathcal{H}[a,n]=\left\{f\in\mathcal{H}(\mathbb{D}):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots\right\},$$

where n is a positive integer and $a\in\mathbb{C}$, with $\mathcal{H}_n\equiv\mathcal{H}[1,n]$. Also, let

$$\mathcal{A}=\{f\in\mathcal{H}(\mathbb{D}):f(z)=z+a_2z^2+a_3z^3+\dots\}.$$

The class of *starlike functions of order* α, $0\leq\alpha<1$, which is a subclass of the class of univalent functions, is defined by

$$\mathcal{S}^*(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf'(z)}{f(z)}>\alpha,\ z\in\mathbb{D}\right\}.$$

The class of *starlike functions* $\mathcal{S}^*\equiv\mathcal{S}^*(0)$ consists of functions f that map the unit disk onto a starlike region, i.e. if $w\in f(\mathbb{D})$, then $tw\in f(\mathbb{D})$ for all $t\in[0,1]$.

Another subclass of univalent functions is the class of *convex functions of order* α, $0\leq\alpha<1$, defined by

$$\mathcal{K}(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\alpha,z\in\mathbb{D}\right\}.$$

Here $\mathcal{K}\equiv\mathcal{K}(0)$ is the class of *convex functions* such that $f\in\mathcal{K}$ if and only if $f(\mathbb{D})$ is a convex region, i.e., if for any $w_1,w_2\in f(\mathbb{D})$ follows $tw_1+(1-t)w_2\in f(\mathbb{D})$ for all $t\in[0,1]$.

Further, using operators

$$J(f,\alpha;z)\equiv(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\quad(\alpha\in\mathbb{R})$$

and

$$U(f,\mu;z) = \left(\frac{z}{f(z)}\right)^{1+\mu} \cdot f'(z) \quad (\mu \in \mathbb{C}),$$

let define classes

$$\mathcal{M}(\alpha,\beta) = \{f \in \mathcal{A} : \operatorname{Re} J(f,\alpha;z) > \beta,\ z \in \mathbb{D}\} \quad (\alpha \in \mathbb{R}, 0 \le \beta < 1),$$

$$\mathcal{M}'(\alpha,\gamma) = \{f \in \mathcal{A} : |J(f,\alpha;z) - 1| < \gamma,\ z \in \mathbb{D}\} \quad (\alpha \in \mathbb{R}, \gamma > 0)$$

and

$$\mathcal{U}(\lambda,\mu) = \{f \in \mathcal{A} : f(z)/z \ne 0 \text{ and } |U(f,\mu;z) - 1| < \lambda,\ z \in \mathbb{D}\} \quad (\mu \in \mathbb{C}, \lambda > 0).$$

Specially, $\mathcal{M}(\alpha) \equiv \mathcal{M}(\alpha, 0)$ is the well known class of *α-convex functions* for which ([4], p.10):

$$\mathcal{M}(\alpha) \subset \mathcal{S}^* \text{ for all } \alpha \in \mathbb{R}$$

and

$$\mathcal{M}(\alpha) \subset \mathcal{K} \subset \mathcal{S}^* \text{ for } \alpha \ge 1.$$

More details on all these classes can be found in [2] and [4].

Further, class $\mathcal{U}(\lambda,\mu)$ and its special cases $\mathcal{U}(\lambda) \equiv \mathcal{U}(\lambda, 1)$ and $\mathcal{U} \equiv \mathcal{U}(1) = \mathcal{U}(1,1)$ are widely studied in the past decades ([1], [3], [8]-[17]). It is known [1, 17] that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \le 1$, but not necessarily univalent if $\lambda > 1$. Also, functions from $\mathcal{U}(\lambda,\mu)$, in general case are not starlike. More precisely, Obradović [7], and Ponnusamy and Singh [18], proved that

$$\mu < 1 \text{ and } 0 \le \lambda \le \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}} \quad \Rightarrow \quad \mathcal{U}(\lambda,\mu) \subset \mathcal{S}^*;$$

extended by Fournier and Ponnusamy [3] as:

$$\operatorname{Re}\mu < 1 \text{ and } 0 \le \lambda \le \frac{|1-\mu|}{\sqrt{|1-\mu|^2 + |\mu|^2}} \quad \Leftrightarrow \quad \mathcal{U}(\lambda,\mu) \subset \mathcal{S}^*.$$

Particulary,

$$\mathcal{U}(1,\mu) \subset \mathcal{S}^* \quad \Leftrightarrow \quad \mu = 0,$$

i.e., $\mathcal{U} \not\subseteq \mathcal{S}^*$, which can be also verified by the function

$$f(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{2}z^3} \in \mathcal{U} \setminus \mathcal{S}^*.$$

Therefore, it is of interest to study the relation between classes $\mathcal{M}(\alpha,\beta)$, $\mathcal{M}'(\alpha,\gamma)$ and $\mathcal{U}(\lambda,\mu)$, which will be done in this paper. More precisely, we will obtain sufficient conditions when

$$\mathcal{M}(\alpha,\beta) \subset \mathcal{U}(\lambda,\mu) \quad \text{or} \quad \mathcal{M}'(\alpha,\gamma) \subset \mathcal{U}(\lambda,\mu).$$

For the investigation we will use methods from the theory of first order differential subordinations and we proceed with some basic definitions. Let $f(z)$ and $g(z)$ be analytic in the unit disk. Then we say that $f(z)$ is subordinate to $g(z)$, and we write $f(z) \prec g(z)$, if $g(z)$ is univalent in $\mathbb{D}$, $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Further on, we use the method of differential subordinations introduced by Miller and Mocanu [5]. In fact, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ ($\mathbb{C}$ is the complex plane) is analytic in domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$, when $z \in \mathbb{D}$, then we say that $p(z)$ satisfies a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z) \tag{1}$$

The univalent function $q(z)$ is dominant of the differential subordination (1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1). If $\tilde{q}(z)$ is a dominant of (1) and $\tilde{q}(z) \prec q(z)$ for all dominants of (1), then we say that $\tilde{q}(z)$ is the best dominant of the differential subordination (1). If $p \in \mathcal{H}[a,n]$, then $q(z)$ is called an (a,n)-dominant and $\tilde{q}(z)$ the best (a,n)-dominant.

From the theory of first-order differential subordinations we will make use of the following lemma.

Lemma 1 (Suffridge [19] or Corollary 3.1d.1 on p.76 from [4]) *Let h be starlike in $\mathbb{D}$, with $h(0)=0$ and $a \neq 0$. If $p \in \mathcal{H}[a,n]$ satisfies*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \cdot \exp\left[\frac{1}{n} \cdot \int_0^z \frac{h(t)}{t}\,dt\right]$$

and q is the best (a,n)-dominant.

2 Main results and consequences

Let note that for $p(z) = U(f,-1/\alpha;z)$ we have

$$J(f,\alpha;z) = 1 + \alpha \cdot \frac{zU'(f,-1/\alpha;z)}{U(f,-1/\alpha;z)}. \tag{2}$$

and $a = p(0) = 1$, i.e. $p \in \mathcal{H}[1,n]$. So, directly from Lemma 1, having in mind relation (2), we receive the following result.

Theorem 1 *Let $f \in \mathcal{A}$ and $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Also, let h be starlike in $\mathbb{D}$, $h(0)=0$ and $\alpha \neq 0$. If*

$$\frac{1}{\alpha} \cdot [J(f,\alpha;z) - 1] \prec h(z) \tag{3}$$

or equivalently

$$J(f,\alpha;z) \prec 1 + \alpha h(z),$$

then

$$U(f,-1/\alpha;z) \prec \exp\left[\frac{1}{n} \cdot \int_0^z \frac{h(t)}{t}\,dt\right] \equiv q(z),$$

and q is the best $(1,n)$-dominant of (3). Even more, if $f''(0) \neq 0$, then $n=1$.

Using the definition of subordination we receive the next corollary that gives information about the relation between the class $\mathcal{U}(\lambda,\mu)$ and the classes $\mathcal{M}(\alpha,\beta)$ and $\mathcal{M}'(\alpha,\gamma)$.

Corollary 1 *Let $f \in \mathcal{A}$ and $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Also, let $\alpha \neq 0$, $0 \leq \beta < 1$ and $\gamma > 0$.*

(i) If $\alpha < 0$ and $\frac{1-\beta}{n\alpha} \geq -\frac{1}{2}$, then

$$\mathcal{M}(\alpha,\beta) \subset \mathcal{U}(\lambda_1, -1/\alpha),$$

where $\lambda_1 = 2^{c_1} - 1$ and $c_1 = -\frac{2(1-\beta)}{n\alpha}$.

(ii) If $\gamma \leq n|\alpha|$, then

$$\mathcal{M}'(\alpha,\gamma) \subset \mathcal{U}(\lambda_2, -1/\alpha),$$

where $\lambda_2 = e^{|c_2|} - 1$ and $c_2 = \frac{\gamma}{n\alpha}$.

Even more, if $f''(0) \neq 0$, then $n=1$ in the previous results. These results are sharp, i.e. given values of λ_1 and λ_2 are the smallest ones so that the corresponding inclusion holds.

Proof.

(i) Let $f \in \mathcal{M}(\alpha, \beta)$. Then, by the definition of subordination we conclude that (3) holds for

$$1 + \alpha h(z) = \frac{1 + (1 - 2\beta)z}{1 - z}, \quad \text{i.e.,} \quad h(z) = \frac{1}{\alpha} \cdot \frac{2z(1 - \beta)}{1 - z}.$$

Now, from Theorem 1, we receive

$$p(z) = U(f, -1/\alpha; z) \prec \exp\left[\frac{1}{n} \cdot \int_0^z \frac{h(t)}{t}\, dt\right] = (1 - z)^{c_1} = q(z) \tag{4}$$

and q is the best $(1, n)$-dominant of (3). Further, having in mind that $0 < c_1 \le 1$ we conclude that $q(\mathbb{D})$ is a convex region since $q_1(z) = \frac{1 - q(z)}{c_1} \in \mathcal{K}$ because of

$$\mathrm{Re}\left[1 + \frac{zq_1''(z)}{q_1'(z)}\right] > 0 \ \ (z \in \mathbb{D}) \quad \Leftrightarrow \quad \mathrm{Re}\frac{1 - c_1 z}{1 - z} > 0 \ \ (z \in \mathbb{D}) \quad \Leftrightarrow \quad |c_1| \le 1.$$

Also, $q(\mathbb{D})$ is symmetric with respect to the real axes ($q(\overline{z}) = \overline{q(z)}$). Therefore,

$$|q(z) - 1| < \lambda \quad (z \in \mathbb{D}),$$

where

$$\lambda = \sup\{|q(z) - 1| : z \in \mathbb{D}\} = \max\{q(-1) - 1, q(1) - 1\} = 2^{c_1} - 1 = q(-1),$$

for $c_1 > 0$. So, by the definition of subordination and subordination (4) we have $f \in \mathcal{U}(\lambda, -1/\alpha)$. In the case when $c_1 < 0$ we receive $\lambda \to +\infty$.

The result is sharp because for $\lambda_* < \lambda_1$ and $\mathcal{M}(\alpha, \beta) \subset \mathcal{U}(\lambda_*, -1/\alpha)$ we have $p(z) \prec 1 + \lambda_* z$ and $q(z) \not\prec 1 + \lambda_* z$ which contradicts the fact that q is the best $(1, n)$-dominant of (3).

(ii) First, let note that $|c_2| \le 1$. In a similar way as in the proof of part (i), using $h(z) = \frac{1}{\alpha} z$ in Theorem 1 we receive

$$q(z) = e^{c_2 \cdot z}.$$

Again, $q(\mathbb{D})$ is a convex region since $q_2(z) = \frac{q(z) - 1}{c_2} \in \mathcal{K}$ due to

$$\mathrm{Re}\left[1 + \frac{zq_2''(z)}{q_2'(z)}\right] > 0 \ \ (z \in \mathbb{D}) \quad \Leftrightarrow \quad \mathrm{Re}(1 - c_2 z) > 0 \ \ (z \in \mathbb{D}) \quad \Leftrightarrow \quad |c_2| \le 1.$$

Region $q(\mathbb{D})$ is symmetric with respect to the real axes ($q(\overline{z}) = \overline{q(z)}$), and we realize that

$$\sup\{|q(z) - 1| : z \in \mathbb{D}\} = \max\{q(-1) - 1, q(1) - 1\} = e^{|c_2|} - 1.$$

Therefore,

$$|U(f, -1/\alpha; z) - 1| < e^{|c_2|} - 1 \quad (z \in \mathbb{D}),$$

i.e., $f \in \mathcal{U}(\lambda_2, -1/\alpha)$. Proof of the sharpness goes in a similar way as in part (i). □

3 Examples

By specifying some concrete values for α, γ and/or β we have next examples.

Example 1 *Let $f \in \mathcal{A}$, $f''(0) \neq 0$ and $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Also, let $\alpha \neq 0$, $\gamma \geq 0$ and $0 \leq \beta < 1$. The following results are sharp.*

(i) If $\alpha \le -2$ *then* $\mathcal{M}(\alpha) \subset \mathcal{U}(1, -1/\alpha)$, *i.e.*

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} > 0 \;\; (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\left[\frac{z}{f(z)}\right]^{1-1/\alpha} \cdot f'(z) - 1\right| < 1 \;\; (z \in \mathbb{D}).$$

($\beta = 0$ *in Corollary 1(i));*

(ii) If $\alpha = -2(1-\beta)$, *then* $\mathcal{M}(\alpha, \beta) \subset \mathcal{U}(1, -1/\alpha)$. ($c_1 = 1$ *in Corollary 1(i));*

(iii) $\mathcal{M}'(1,1) \subset \mathcal{U}(e-1, -1)$, *i.e.*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \;\; (z \in \mathbb{D}) \quad \Rightarrow \quad |f'(z) - 1| < e - 1 \;\; (z \in \mathbb{D}).$$

($\alpha = \gamma = 1$ *in Corollary 1(ii));*

(iv) $\mathcal{M}'(1/2, 1/2) \subset \mathcal{U}(e-1, -2)$, *i.e.*

$$\left|\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - 1\right| < 1 \;\; (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\frac{f(z)f'(z)}{z} - 1\right| < e - 1 \;\; (z \in \mathbb{D}).$$

($\alpha = \gamma = 1/2$ *in Corollary 1(ii));*

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