数理解析研究所講究録
第 1878 巻 2014 年 100-111

THE SHARP GROWTH ESTIMATE FOR $\mathcal{U}(\lambda)$

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ABSTRACT. For $0<\lambda\leq 1$ let $\mathcal{U}(\lambda)$ be the class of analytic functions in the unit disk $\mathbb{D}$ with $f(0)=f'(0)-1=0$ satisfying $|f'(z)(z/f(z))^{2}-1|<\lambda$ in $\mathbb{D}$. Then it is known that every $f\in \mathcal{U}(\lambda)$ is univalent in $\mathbb{D}$. In the present article we shall prove the sharp estimates $|f''(0)|\leq 2(1+\lambda)$ and $|z|/\{(1+|z|)(1+\lambda|z|)\}\leq |f(z)|\leq|z|/\{(1-|z|)(1-\lambda|z|)\}$. As an application we shall also give the sharp covering theorems.

1. INTRODUCTION

We denote the complex plane by $\mathbb{C}$ and the extended complex plane by $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. For $c\in \mathbb{C}$ and $r>0$ let $\mathbb{D}(c, r)=\{z\in \mathbb{C}:|z-c|<r\}$ and $\mathbb{D}=\mathbb{D}(0,1)$. Similarly let $\Delta_{r}=\widehat{\mathbb{C}}\backslash \overline{\mathbb{D}}(0, r)=\{z\in\widehat{\mathbb{C}}:r<|z|\leq\infty\}$ and $\Delta=\Delta_{1}$.

Let $\mathcal{A}(\{\mathbb{D}\})$ denote the space of analytic functions in $\mathbb{D}$ and $\mathcal{A}_{0}(\{\mathbb{D}\})=\{f\in \mathcal{A}(\{\mathbb{D}\}):f(0)=f'(0)-1=0\}$. Here we regard $\mathcal{A}(\{\mathbb{D}\})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. A function f is said to be univalent in a domain D if it is one-to-one in D. Let S denote the class of univalent functions in $\mathcal{A}_{0}(\{\mathbb{D}\})$.

For $0<\lambda\leq 1$ let $\mathcal{U}(\lambda)$ be the class of functions $f\in \mathcal{A}_{0}$ satisfying

$$(1.1)\qquad \left|f'(z)\left(\frac{z}{f(z)}\right)^{2}-1\right|<\lambda$$

in $\mathbb{D}$. The boundedness of $f'(z)(z/f(z))^{2}$ forces $f\in \mathcal{U}(\lambda)$ that $f(z)\neq 0$ in $\mathbb{D}\backslash \{0\}$. Hence $f'(z)\neq 0$ holds in $\mathbb{D}$ and f is locally univalent in $\mathbb{D}$. Moreover it is known that $f\in \mathcal{U}(\lambda)$ is univalent in $\mathbb{D}$, i.e., $\mathcal{U}(\lambda)\subset S$.

In the present article we shall prove the sharp inequalities

$$|a_2(f)| = 2^{-1}|f''(0)| \leq 1+\lambda,$$

$$\frac{|z|}{(1-\lambda|z|)(1-|z|)} \leq |f(z)| \leq \frac{|z|}{(1-\lambda|z|)(1-|z|)}$$

for $f \in \mathcal{U}(\lambda)$. To this end we introduce three classes of meromorphic functions in Δ closely related to $\mathcal{U}(\lambda)$. For $0<\lambda\leq 1$ let $\mathcal{M}(\lambda)$ be the class of meromorphic functions g in Δ which has a Laurent expansion of a form

$$g(w) = w + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \quad 1<|w|<\infty \tag{1.2}$$

and satisfying

$$|g'(w) - 1| < \lambda.$$

Now for $f \in \mathcal{U}(\lambda)$ put

$$T(f)(w) = \frac{1}{f(\frac{1}{w})}, \quad w \in \Delta.$$

Then we have

$$\frac{d}{dw}T(f)(w) = \frac{f'(1/w)}{w^2 f^2(\frac{1}{w})} = f'(z)\left(\frac{z}{f(z)}\right)^2, \quad z = \frac{1}{w}$$

and hence $Tf \in \mathcal{M}(\lambda)$. Thus T is a transformation which maps $\mathcal{U}(\lambda)$ injectively into $\mathcal{M}(\lambda)$. The image $T(\mathcal{U}(\lambda))$ is a proper subset of $\mathcal{M}(\lambda)$ and it is easy to see that

$$\mathcal{M}_0(\lambda) = \{g \in \mathcal{M}(\lambda) : g(w) \neq 0 \text{ in } \Delta\} = T(\mathcal{U}(\lambda)).$$

Notice that $a_2(f) = -c_0(T(f))$ hold for $f \in \mathcal{U}(\lambda)$. Moreover let

$$\widetilde{\mathcal{M}}(\lambda) = \{g \in \mathcal{M}(\lambda) : c_0(g) = 0\}.$$

In the next section we shall show that every $g \in \widetilde{\mathcal{M}}(\lambda)$ satisfies $g(w) \neq 0$ in Δ. Thus the relation

$$T(\widetilde{\mathcal{U}}(\lambda)) = \widetilde{\mathcal{M}}(\lambda) \subset T(\mathcal{U}(\lambda)) = \mathcal{M}_0(\lambda) \subset \mathcal{M}(\lambda)$$

holds.

In Section 2 we shall derive an integral representation of $g \in \mathcal{M}(\lambda)$ and prove existence of the boundary limit $g(\eta) = \lim_{\Delta\ni w\to\eta} g(w)$ for each $\eta \in \partial\Delta$. Further we shall precisely study the boundary values of $g \in \widetilde{\mathcal{M}}(\lambda)$ and obtain the sharp estimate $|g(\eta)| \leq 1+\lambda$ for $g \in \widetilde{\mathcal{M}}(\lambda)$ and $\eta \in \partial\Delta$, which is equivalent to the sharp upper bound $|a_2(f)| \leq 1+\lambda$ for $f \in \mathcal{U}(\lambda)$.

In Section 3 for each fixed $w_0 \in \Delta\backslash\{\infty\}$ we shall treat the sharp estimate on $|g(w_0)|$ for $g \in \mathcal{M}_0(\lambda)$. The result has an immediate counterpart in $\mathcal{U}(\lambda)$ and we shall derive the sharp growth estimate and the sharp covering theorem for $\mathcal{U}(\lambda)$.

2. Integral representation

For $g \in \mathcal{M}(\lambda)$ let

$$b_g(w) = \frac{w^2}{\lambda}(1 - g'(w)), \quad w \in \Delta, \tag{2.1}$$

$$\beta_g(z) = b_g(1/z), \quad z \in \mathbb{D}. \tag{2.2}$$

Note that $g'(w) - 1 = O(w^{-2})$ as $w \to \infty$ and that $g'(1/z)$ is analytic in $\mathbb{D}$. Applying the maximum modulus principle to $g'(1/z) - 1$ we have

$$|g'(w) - 1| \le \frac{\lambda}{|w|^2}, \quad 1 < |w| < \infty.$$

Hence $\beta_g \in H_1^\infty(\mathbb{D})$ and for any $w, w_0 \in \Delta\backslash\{\infty\}$ by integrating $g'(w) - 1 = -\lambda b_g(w) w^{-2}$ we obtain

$$g(w) - w = g(w_0) - w_0 - \lambda \int_{w_0}^{w} \frac{b_g(\zeta)}{\zeta^2}\, d\zeta.$$

Since $\lim_{w_0 \to \infty}(g(w_0) - w_0) = c_0$, we have

$$g(w) = w + c_0 - \lambda \int_{\infty}^{w} \frac{b_g(\zeta)}{\zeta^2}\, d\zeta = w + c_0 + \lambda \int_0^{1/w} \beta_g(\zeta)\, d\zeta.$$

Converse is also true and we have the following.

Theorem 2.1. *For a meromorphic function g in Δ, $g \in \mathcal{M}(\lambda)$ if and only if there exist $\beta \in H_1^\infty(\mathbb{D})$ and $c \in \mathbb{C}$ such that*

$$g(w) = w + c + \lambda \int_0^{1/w} \beta(\zeta)\, d\zeta.$$

Corollary 2.2. *Each $g \in \mathcal{M}(\lambda)$ is Lipschitz continuous and satisfies*

$$\left(1 - \frac{\lambda}{|w_0 w_1|}\right)|w_1 - w_0| \le |g(w_1) - g(w_0)| \le \left(1 + \frac{\lambda}{|w_0 w_1|}\right)|w_1 - w_0| \tag{2.3}$$

for $w_0, w_1 \in \Delta$. Particularly

(i) *The limit $g(\eta) = \lim_{\Delta \ni w \to \eta} g(w)$ exists for every $\eta \in \partial\Delta$.*
(ii) *g is univalent in Δ. Furthermore if $0 < \lambda < 1$, then g is univalent on $\overline{\Delta}$.*

Proof. Inequality (2.3) easily follows from Theorem 2.1 and

$$\left|\int_{1/w_0}^{1/w_1} \beta(\zeta)\, d\zeta\right| \le \left|\frac{1}{w_1} - \frac{1}{w_0}\right| = \frac{|w_1 - w_0|}{|w_0 w_1|}.$$

By (2.3) the function g is Lipschitz continuous in Δ and from this the boundary limit $g(\eta) = \lim_{\Delta \ni w \to \eta} g(w)$ exists for each $\eta \in \partial\Delta$. Inequality (2.3) also shows that g is univalent in Δ. Since (2.3) still holds on $\overline{\Delta}$ by continuity, g is univalent on $\overline{\Delta}$, when $0 < \lambda < 1$. □

Each $f \in \mathcal{U}(\lambda)$ is univalent in $\mathbb{D}$, since Tf is univalent in Δ by Corollary 2.2.

Corollary 2.3. *For each* $0 < \lambda \le 1$ *the inclusion relation* $\widetilde{\mathcal{M}}(\lambda) \subset \mathcal{M}_0(\lambda)$ *holds.*

Proof. For $g \in \widetilde{\mathcal{M}}(\lambda) (\subset \mathcal{M}(\lambda))$ we have by Theorem 2.1

$$|g(w)| = \left|w + \lambda \int_0^{1/w} \beta(\zeta)\, d\zeta\right| \ge |w| - \frac{\lambda}{|w|} > 0, \quad |w| > 1.$$

Thus g has no zeros in Δ and hence $g \in \mathcal{M}_0(\lambda)$. □

For $g \in \mathcal{M}(\lambda)$ let $E(g)$ be the omitted set of g, i.e.,

$$E(g) = \widehat{\mathbb{C}} \backslash g(\Delta).$$

For $R > 1$ the image $g(\partial\Delta_R)$ is an analytic Jordan curve and $g(\Delta_R)$ is the domain outside $g(\partial\Delta_R)$. Let D_R be the domain bounded by $g(\partial\Delta_R)$. Then $\{D_R : 1 < R\}$ is a 1-parameter family of increasing domains in $\mathbb{C}$ and

$$E(g) = \bigcap_{R>1} D_R.$$

In particular when $0 < \lambda < 1$, by Corollary 2.2 $E(g)$ is a closed Jordan domain bounded by $g(\partial\Delta)$. Notice that for any $g \in \mathcal{M}(\lambda)$ with $0 < \lambda \le 1$ and $a \in \partial E(g)$ there exists $\eta \in \partial\Delta$ such that $a = g(\eta)$.

Theorem 2.4. *Let* $g \in \mathcal{M}(\lambda)$ *and* $g(w) = w + c_0 + \int_0^{1/w} \beta(\zeta)\, d\zeta$ *with* $\beta \in H_1^\infty(\mathbb{D})$ *and* $c_0 \in \mathbb{C}$. *Then* $g \in \mathcal{M}_0(\lambda)$ *if and only if*

$$-c_0 \in E(\widetilde{g}), \tag{2.4}$$

where

$$\widetilde{g}(w) = w + \int_0^{1/w} \beta(\zeta)\, d\zeta \in \widetilde{\mathcal{M}}(\lambda).$$

Proof. For $g \in \mathcal{M}(\lambda)$, by definition, $g \in \mathcal{M}_0(\lambda)$ if and only if $g(w) = \tilde{g}(w) + c_0 \neq 0$ for all $w \in \Delta$. This is equivalent to (2.4). □

For $g \in \mathcal{M}_0(\lambda)$ the coefficient $c_0 = c_0(g)$ in the expansion (1.2) is called the conformal center of the set $E(g)$. For more details on conformal center we refer to [8].

Notice that $g + c \in \mathcal{M}(\lambda)$ holds for any $g \in \mathcal{M}(\lambda)$ and $c \in \mathbb{C}$. This implies that there are no upper bound on $|c_0(g)|$ for $g \in \mathcal{M}(\lambda)$. However concerning with the class $\mathcal{M}_0(\lambda)$, it is not difficult to get the sharp estimate.

Theorem 2.5. *Let* $\lambda \in (0,1]$. *Then,*

(a) *For* $g \in \widetilde{\mathcal{M}}(\lambda)$ *the sharp estimate* $1 - \lambda \leq |g(\eta)| \leq 1 + \lambda$ *holds on* $\partial\Delta$. *Furthermore equality* $|g(\eta)| = 1 - \lambda$ *at some* $\eta \in \partial\Delta$ *if and only if* $g(w) \equiv w - \lambda\eta^2/w$, *and* $|g(\eta)| = 1 + \lambda$ *if and only if* $g(w) \equiv w + \lambda\eta^2/w$.

(b) *Inequality* $|c_0(g)| \leq 1 + \lambda$ *holds for* $g \in \mathcal{M}_0(\lambda)$ *with equality if and only if*

$$g(w) = w\left(1 + \frac{\lambda e^{i\theta}}{w}\right)\left(1 + \frac{e^{i\theta}}{w}\right), \quad w \in \Delta.$$

for some real θ.

(c) *Inequality* $|a_2(f)| \leq 1 + \lambda$ *holds for* $f \in \mathcal{U}(\lambda)$ *with equality if and only if*

$$f(z) = \frac{z}{(1 + \lambda e^{i\theta} z)(1 + e^{i\theta} z)}, \quad z \in \mathbb{D}$$

for some real θ.

Proof. Let $g \in \widetilde{\mathcal{M}}(\lambda)$ and put $\beta \in H_1^\infty(\mathbb{D})$ as in (2.2). Then

$$g(w) = w + \lambda \int_0^{1/w} \beta(\zeta)\, d\zeta. \tag{2.5}$$

For $\eta \in \partial\Delta$ we consider $\int_0^{1/\eta} \beta(\zeta)\, d\zeta$ as the Lebesgue integral along a C^1-path connecting 0 to $1/\eta$ and contained in $\mathbb{D}$ except for the end point $1/\eta$. Then the integral does not depend on choice of path. Thus (2.5) still holds for $\eta \in \partial\Delta$ and we have

$$|g(\eta)| \leq |\eta| + \lambda \left| \int_0^{1/\eta} \beta(\zeta)\, d\zeta \right| \leq 1 + \lambda \int_{[0,1/\eta]} |\beta(\zeta)|\, |d\zeta| \leq 1 + \lambda,$$

where $[0, 1/\eta]$ is the radial line segment connecting 0 and $1/\eta$. If $|g(\eta_0)| = 1 + \lambda$ at some $\eta_0 \in \partial\Delta$, then by the maximum modulus theorem $\beta = \varepsilon$ for some $\varepsilon \in \partial\mathbb{D}$ and

$$\frac{\lambda \int_0^{1/\eta} \beta(\zeta)\, d\zeta}{\eta} = \frac{\lambda\varepsilon}{\eta^2} > 0.$$

Therefore $\varepsilon = \eta^2$ and $g(w) \equiv w + \lambda\eta^2/w$. Similarly

$$|g(\eta)| \geq |\eta| - \lambda \left| \int_0^{1/\eta} \beta(\zeta)\, d\zeta \right| \geq 1 - \lambda \int_{[0,1/\eta]} |\beta(\zeta)|\, |d\zeta| \geq 1 - \lambda$$

with equality if and only if $\beta = \varepsilon \in \partial\mathbb{D}$ and $\lambda\varepsilon/\eta^2 < 0$, i.e., $\varepsilon = -\eta^2$ and therefore $g(w) \equiv w - \lambda\eta^2/w$. This completes the proof of (a).

To show (b) let $g \in \mathcal{M}_0(\lambda)$. Then g can be expressed as $g = \tilde{g} + c_0$ with $\tilde{g} \in \widetilde{\mathcal{M}}(\lambda)$ and $-c_0 \in E(\tilde{g})$. By (a) and $E(\tilde{g}) = \cap_{R>1} D(R)$, where $D(R)$ is a domain bounded by the Jordan curve $\tilde{g}(\partial\Delta_R)$. Hence we have $E(\tilde{g}) \subset \overline{\mathbb{D}}(0, 1+\lambda)$ and $|c_0(g)| \leq 1 + \lambda$.

Suppose now that $|c_0(g)| = 1 + \lambda$. Combining $-c_0(g) \in E(\tilde{g}) \cap \partial\mathbb{D}(0, 1+\lambda)$ and $E(\tilde{g}) \subset \overline{\mathbb{D}}(0, 1+\lambda)$, we have $-c_0(g) \in \partial E(\tilde{g})$. By Lipschitz continuity of $\tilde{g}$ there exists $\eta \in \partial\Delta$ with $-c_0(g) = \tilde{g}(\eta)$. Since $|\tilde{g}(\eta)| = |-c_0(g)| = 1 + \lambda$, it follows from (a) that $\tilde{g}(w) = w + \lambda\eta^2/w$ and hence

$$g(w) = \tilde{g}(w) + c_0 = \tilde{g}(w) - \tilde{g}(\eta) = w\left(1 + \frac{\lambda\eta}{w}\right)\left(1 + \frac{\eta}{w}\right).$$

By letting $e^{i\theta} = -\eta$ we obtain (b).

Since $a_2(f) = -c_0(T(f))$ holds for $f \in \mathcal{U}(\lambda)$, (c) follows directly from (b). □

3. Growth Estimates

Let $0 < \lambda \leq 1$ and $1 < |w_0| < \infty$. Combining Theorem 2.1 and Theorem 2.5 (b) it is easily seen that if $g \in \mathcal{M}_0(\lambda)$ then

$$\begin{aligned} (3.1) \qquad |g(w_0)| &\leq |w_0| + \lambda \int_{[0,1/w_0]} |\beta(\zeta)|\, |d\zeta| + |c_0| \\ &\leq |w_0| + \frac{\lambda}{|w_0|} + 1 + \lambda = |w_0|\left(1 + \frac{\lambda}{|w_0|}\right)\left(1 + \frac{1}{|w_0|}\right) \end{aligned}$$

with equality at $w_0 = Re^{i\theta}$ if and only if $g(w) \equiv w(1+\lambda e^{i\theta}w^{-1})(1+e^{i\theta}w^{-1})$. Similarly the lower estimate

$$|g(w_0)| \geq |w_0| - \frac{\lambda}{|w_0|} - (1+\lambda)$$

holds for $g \in \mathcal{M}(\lambda)$. Clearly it is not sharp, since the right hand side is negative for all r sufficiently close to 1.

In this section we deal with the region of variability $V_\lambda(w_0)$ of $g(w_0)$, when g varies on $\mathcal{M}_0(\lambda)$, i.e.,

$$V_\lambda(w_0) = \{g(w_0) : g \in \mathcal{M}_0(\lambda)\}.$$

We shall show that $V_\lambda(w_0)$ is a closed Jordan domain bounded by a simple closed curve and give a parameterization of the boundary curve. Using these ideas we will obtain the sharp lower estimate on $|g(w_0)|$ when $g \in \mathcal{M}_0(\lambda)$.

First we notice that $V_\lambda(w_0)$ is a compact subset of $\mathbb{C}$. Indeed by (3.1) it is clear that $\mathcal{M}_0(\lambda)$ is a family of analytic functions in $\Delta\backslash\{\infty\}$ which is locally uniformly bounded and hence normal. Moreover if a sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{M}_0(\lambda)$ converge to g locally uniformly in $\Delta\backslash\{\infty\}$, then it is not difficult to see that $g \in \mathcal{M}_0(\lambda)$. Thus $\mathcal{M}_0(\lambda)$ is a compact family with respect to the topology of locally uniform convergence. Since $V_\lambda(w_0)$ is the image of $\mathcal{M}_0(\lambda)$ with respect to the continuous mapping $\mathcal{M}_0(\lambda) \ni g \mapsto g(w_0) \in \mathbb{C}$, $V_\lambda(w_0)$ is also an compact subset of $\mathbb{C}$.

Next for $g \in \mathcal{M}_0(\lambda)$ let $g_\theta(w) = e^{-i\theta}g(e^{i\theta}w)$. Then $g_\theta \in \mathcal{M}_0(\lambda)$ for any $\theta \in \mathbb{R}$. From this it follows that

$$V_\lambda(Re^{i\theta}) = e^{i\theta}V_\lambda(R)$$

and it suffices to determine $V_\lambda(R)$ for $1 < R < \infty$. Similarly for $g \in \mathcal{M}_0(\lambda)$ let $\overline{g}(w) = \overline{g(\overline{w})}$. Then $\overline{g} \in \mathcal{M}_0(\lambda)$ and hence $V_\lambda(R)$ is symmetric with respect to $\mathbb{R}$.

Theorem 3.1. *Let* $0 < \lambda \leq 1$. *Then*

(i) *For* $g \in \mathcal{M}_0(\lambda)$

$$|w|\left(1-\frac{\lambda}{|w|}\right)\left(1-\frac{1}{|w|}\right) \leq |g(w)| \leq |w|\left(1+\frac{\lambda}{|w|}\right)\left(1+\frac{1}{|w|}\right),$$

for $1 < |w| < \infty$ *with equality* $w_0 = R_0e^{i\theta_0}$ *if and only if*

$$g(w) = w\left(1-\frac{\lambda e^{i\theta_0}}{w}\right)\left(1-\frac{e^{i\theta_0}}{w}\right) \text{ or } g(w) = w\left(1+\frac{\lambda e^{i\theta_0}}{w}\right)\left(1+\frac{e^{i\theta_0}}{w}\right),$$

respectively.

(ii) *For* $f \in \mathcal{U}(\lambda)$

$$\frac{|z|}{(1+|z|)(1+\lambda|z|)} \le |f(z)| \le \frac{|z|}{(1-|z|)(1-\lambda|z|)}, \quad 0 < |z| < 1$$

with equality at $z = r_0 e^{i\theta_0}$ *if and only if*

$$f(z) = \frac{z}{(1+\lambda e^{i\theta_0}z)(1+e^{i\theta_0}z)} \quad or \quad f(z) = \frac{z}{(1-\lambda e^{i\theta_0}z)(1-e^{i\theta_0}z)}$$

respectively.

Theorem 3.2. *Let* $f \in \mathcal{U}(\lambda)$ *with* $0 < \lambda \le 1$. *Then*

$$\mathbb{D}\left(0, \frac{1}{2(1+\lambda)}\right) \subset f(\mathbb{D}).$$

Furthermore $\frac{e^{i\theta_0}}{2(1+\lambda)} \notin f(\mathbb{D})$ *holds if and only if*

$$f(z) = \frac{z}{(1+\lambda c^{-i\theta_0}z)(1+e^{-i\theta_0}z)}.$$

Now we define auxiliary functions. For $\varepsilon \in \overline{\mathbb{D}}$ let

$$\widetilde{G}_{\lambda,\varepsilon}(w) = w + \frac{\lambda\varepsilon}{w}$$

and

$$E_\lambda = \begin{cases} \{u+iv : (u/(1+\lambda))^2 + (v/(1-\lambda))^2 \le 1\}, & 0 < \lambda < 1 \\ [-2,2], & \lambda = 1. \end{cases}$$

Notice that $E(\widetilde{G}_{\lambda,e^{i\theta}}) = e^{i\theta/2}E_\lambda$ for $\theta \in \mathbb{R}$.

Proposition 3.3. *Let* $g \in \mathcal{M}_0(\lambda)$. *If* $g(R) \in \partial V_\lambda(R)$, *then there exists* ε, η *with* $|\varepsilon| = |\eta| = 1$, *such that* $g = \widetilde{G}_{\lambda,\varepsilon} - \widetilde{G}_{\lambda,\varepsilon}(\eta)$.

Proof. By Theorem 2.4 g can be decomposed as $g = \tilde{g} + c_0$, where $\tilde{g}(w) = w + \lambda\int_0^{1/w} \beta_g(\zeta)\,d\zeta \in \widetilde{M}(\lambda)$ and $-c_0 \in E(\tilde{g})$. Again by Theorem 2.4

$$g(R) = \tilde{g}(R) + c_0 \in \tilde{g}(R) - E(\tilde{g}) \subset V_\lambda(R).$$

Thus $-c_0$ cannot be an interior point of $E(\tilde{g})$, otherwise $g(R)$ is an interior point of $V_\lambda(R)$. Hence $-c_0 \in \partial E(\tilde{g})$. By Lipschitz continuity of $\tilde{g}$ there exists $\eta \in \partial\Delta$ such that $-c_0 = \tilde{g}(\eta)$. Therefore

$$g(R) = \tilde{g}(R) - \tilde{g}(\eta) = R - \eta + \lambda\int_{1/\eta}^{1/R} \beta_g(\zeta)\,d\zeta.$$

Notice that $R \neq \eta$, since $g(R) \neq 0$. Then we have

$$\left| \int_{1/\eta}^{1/R} \beta_g(\zeta)\, d\zeta \right| \leq \left| \frac{1}{R} - \frac{1}{\eta} \right|$$

with equality if and only if $\beta_g = \varepsilon$ for some ε with $|\varepsilon| = 1$.

Suppose that

$$\left| \int_{1/\eta}^{1/R} \beta_g(\zeta)\, d\zeta \right| < \left| \frac{1}{R} - \frac{1}{\eta} \right| = \left| \frac{\eta - R}{R\eta} \right|.$$

Put

$$\varepsilon_0 = \frac{R\eta}{\eta - R} \int_{1/\eta}^{1/R} \beta_g(\zeta)\, d\zeta \, (\in \mathbb{D}).$$

Then

$$\begin{aligned} \widetilde{G}_{\lambda,\varepsilon_0}(R) - \widetilde{G}_{\lambda,\varepsilon_0}(\eta) &= R - \eta + \lambda \varepsilon_0 \left(\frac{1}{R} - \frac{1}{\eta} \right) \\ &= R - \eta + \lambda \int_{1/\eta}^{1/R} \beta_g(\zeta)\, d\zeta = \tilde{g}(R) - \tilde{g}(\eta) = g(R). \end{aligned}$$

On the other hand since $\widetilde{G}_{\lambda,c} - \widetilde{G}_{\lambda,c}(\eta) \in \mathcal{M}_0(\lambda)$ for $c \in \overline{\mathbb{D}}$, we have $\widetilde{G}_{\lambda,c}(R) - \widetilde{G}_{\lambda,c}(\eta) \in V_\lambda(R)$ for $c \in \mathbb{D}$. The mapping $\mathbb{D} \ni c \mapsto \widetilde{G}_{\lambda,c}(R) - \widetilde{G}_{\lambda,c}(\eta) \in V_\lambda(R)$ is an analytic function of $c \in \mathbb{D}$. Since $R \neq \eta$, the mapping is not constant and hence it is an open mapping. Thus $g(R) = \widetilde{G}_{\lambda,\varepsilon_0}(R) - \widetilde{G}_{\lambda,\varepsilon_0}(\eta)$ is an interior point of $V_\lambda(R)$, which contradict the assumption that $g(R) \in \partial V_\lambda(R)$. Therefore $\beta_g = \varepsilon$ for some $\varepsilon \in \partial\mathbb{D}$ and $g = \widetilde{G}_{\lambda,\varepsilon} - \widetilde{G}_{\lambda,\varepsilon}(\eta)$. □

Proof of Theorem 3.1. Since (ii) follows directly from (i), it suffices to show (i). From compactness of $\mathcal{M}_0(\lambda)$ it follows that there exist $g_1, g_2 \in \mathcal{M}_0(\lambda)$ such that

$$|g_1(R)| = \min_{g \in \mathcal{M}_0(\lambda)} |g(R)| \quad \text{and} \quad |g_2(R)| = \max_{g \in \mathcal{M}_0(\lambda)} |g(R)|.$$

Then clearly $g_2(R) \in \partial V_\lambda(R)$. Also $g_1(R) \in \partial V_\lambda(R)$ follows from the fact that $0 \notin V_\lambda(R)$. Thus by Proposition 3.3 there exist ε_j, η_j with $|\varepsilon_j| = |\eta_j| = 1$ such that $g_j = \widetilde{G}_{\lambda,\varepsilon_j} - \widetilde{G}_{\lambda,\varepsilon_j}(\eta_j)$ for $j = 1, 2$. Since

$$g_j(R) = \widetilde{G}_{\lambda,\varepsilon_j}(R) - \widetilde{G}_{\lambda,\varepsilon_j}(\eta_j) = (R - \eta_j)\left(1 - \frac{\lambda \varepsilon_j}{R\eta_j}\right),$$

we have

$$
\begin{aligned}
|g_1(R)| = \min_{g\in\mathcal{M}_0(\lambda)} |g(R)| \leq & \widetilde{G}_{\lambda,1}(R) - \widetilde{G}_{\lambda,1}(1) \\
= & (R-1)\left(1-\frac{\lambda}{R}\right) \\
\leq & \left|(R-\eta_1)\left(1-\frac{\lambda\varepsilon_1}{R\eta_1}\right)\right| = |g_1(R)|.
\end{aligned}
$$

Thus $\eta_1 = \varepsilon_1 = 1$ and hence

$$g_1(w) \equiv \widetilde{G}_{\lambda,1}(w) - \widetilde{G}_{\lambda,1}(1) = w + \frac{\lambda}{w} - (1+\lambda).$$

We have shown that for $g \in \mathcal{M}(\lambda)$

$$(R-1)\left(1-\frac{\lambda}{R}\right) \leq |g(R_1)|$$

with equality if and only if $g(w) = w + \lambda w^{-1} - (1+\lambda)$. Applying this to $g_\theta(w) = e^{-i\theta}g(e^{i\theta}w)$ we have for $w = Re^{i\theta}$ and $g \in \mathcal{M}_0(\lambda)$

$$(|w|-1)\left(1-\frac{\lambda}{|w|}\right) = (R-1)\left(1-\frac{\lambda}{R}\right) \leq |g_\theta(R)| = |g(w)|$$

with equality $g_\theta(w) = w + \lambda w^{-1} - (1+\lambda)$, i.e.,

$$g(w) = w + \lambda e^{2i\theta}w^{-1} - (1+\lambda)e^{i\theta} = w\left(1-\frac{\lambda e^{i\theta}}{w}\right)\left(1-\frac{e^{i\theta}}{w}\right).$$

In the same manner we can treat the rest of the proof of (i). □

Proof of Theorem 3.2. For $f \in \mathcal{U}(\lambda)$ the relation $\mathbb{D}(0,(2(1+\lambda))^{-1}) \subset f(\mathbb{D})$ directly follows from 3.1 (ii).

Suppose that $e^{i\theta_0}\{2(1+\lambda)\}^{-1} \notin f(\mathbb{D})$. Then $2(1+\lambda)e^{-i\theta_0} \in E(g) = E(\tilde{g}) + c_0(g)$, where $g = Tf = \tilde{g} + c_0(g)$ with $\tilde{g} \in \widetilde{\mathcal{M}}(\lambda)$. Since $2(1+\lambda)e^{-i\theta_0} - c_0(g) \in E(\tilde{g}) \subset \overline{\mathbb{D}}(0, 1+\lambda)$ and $|c_0(g)| \leq 1+\lambda$ by Theorem 2.5, we have

$$1+\lambda \leq 2(1+\lambda) - |c_0(g)| \leq |2(1+\lambda)e^{-i\theta_0} - c_0(g)| \leq 1+\lambda.$$

Thus $c_0(g) = (1+\lambda)e^{-i\theta_0}$. By Theorem 2.5 (b) $g(w) = w(1+\lambda e^{i\theta}w^{-1})(1+e^{i\theta}w^{-1}) = w+(1+\lambda)e^{i\theta}+\lambda e^{2i\theta}w^{-1}$ for some $\theta \in \mathbb{R}$. Therefore $e^{i\theta} = e^{-i\theta_0}$ and $g(w) = w(1+\lambda e^{-i\theta_0}w^{-1})(1+e^{-i\theta_0}w^{-1})$. This implies

$$f(z) = \frac{z}{(1+\lambda e^{-i\theta_0}z)(1+e^{-i\theta_0}z)}.$$

□

Proposition 3.3 implies that $\partial\partial V_\lambda(R)$ is contained in

$$V^*_\lambda(R) = \left\{(R-\eta)\left(1-\frac{\lambda\varepsilon}{R\eta}\right) : |\varepsilon| = |\eta| = 1\right\}.$$

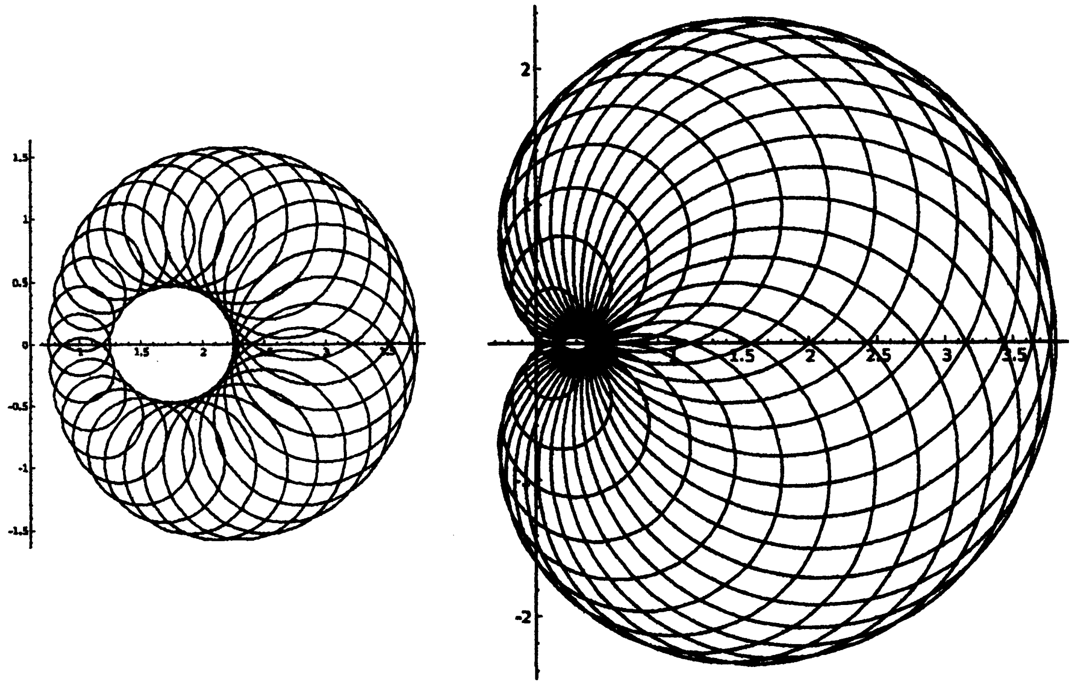

FIGURE 1. $V^*_{0.5}(2)$ and $V^*_{0.9}(1.1)$

One can prove $\partial V^*_\lambda(R)$ consists of two Jordan curves $J_e(R)$ and $J_i(R)$ which are starlike with respect to R and $J_i(R)$ is contained inside of $J_e(R)$, and that $V_\lambda(R)$ is a closed Jordan domain surrounded by $J_e(R)$. For details see forthcoming paper [11].

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