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Packing A-paths in Group-Labelled Graphs via Linear Matroid Parity*

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Abstract

Mader's disjoint $\mathcal{S}$-paths problem is a common generalization of non-bipartite matching and Menger's disjoint paths problems. Lovász (1980) suggested a polynomial-time algorithm for this problem through a reduction to matroid matching. A more direct reduction to the linear matroid parity problem was given later by Schrijver (2003), which leads to faster algorithms.

As a generalization of Mader's problem, Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour (2006) introduced a framework of packing non-zero A-paths in group-labelled graphs, and proved a min-max theorem. Chudnovsky, Cunningham, and Geelen (2008) provided an efficient combinatorial algorithm for this generalized problem. On the other hand, Pap (2007) introduced a framework of packing non-returning A-paths as a further genaralization.

In this paper, we discuss a possible extension of Schrijver's reduction technique to another framework introduced by Pap (2006), under the name of the subgroup model, which apparently generalizes but in fact is equivalent to packing non-returning A-paths. We provide a necessary and sufficient condition for the groups in question to admit a reduction to the linear matroid parity problem. As a consequence, we give faster algorithms for important special cases of packing non-zero A-paths such as odd-length A-paths. In addition, it turns out that packing non-returning A-paths admits a reduction to the linear matroid parity problem, which leads to its efficient solvability, if and only if the size of the input label set is at most four.

1 Introduction

Let Γ be a group. A Γ-*labelled graph* (G,ψ) is a pair of an undirected graph $G=(V,E)$ and a *label function* ψ on the edge set to Γ, which is defined below. For a directed graph $\vec{G}=(V,\vec{E})$ obtained from G by replacing each edge with a pair of arcs of opposite directions, a function $\psi:\vec{E}\to\Gamma$ is called a label function if $\psi(\bar{e})=\psi(e)^{-1}$ holds for each $e\in\vec{E}$, where $\bar{e}$ denotes the reverse arc of e. In this paper, for $e=uv=vu\in E$ replaced with $e'=uv\in\vec{E}$ and $\bar{e'}=vu\in\vec{E}$, we will use the notation of $\psi(e,v):=\psi(e')=\psi(\bar{e'})^{-1}=:\psi(e,u)^{-1}$. For each undirected path $P=(v_0,e_1,v_1,\ldots,e_k,v_k)$ in G, where $e_i=v_{i-1}v_i\in E$ for every $1\le i\le k$, we define the *label* of P as $\psi(P):=\psi(e_k,v_k)\cdots\psi(e_2,v_2)\cdot\psi(e_1,v_1)$.

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For a prescribed terminal set $A \subseteq V$, an *A-path* is an undirected path between distinct terminals in A which does not intersect with A in between. In this paper, we consider the *subgroup model* of packing A-paths in group-labelled graphs introduced by Pap [10]. In this model, for a given proper subgroup Γ' of Γ, an A-path P is called *admissible* if $\psi(P) \notin \Gamma'$, and is called *non-admissible* otherwise. Our objective is to find a maximum family of (fully) vertex-disjoint admissible A-paths in a given Γ-labelled graph with terminal set A. Note that it is not necessary that G is simple.

The subgroup model was introduced at the end of a sequence of extensions of Mader's disjoint $\mathcal{S}$-paths problem, which is known to be solvable by a reduction to matroid matching due to Lovász [5]. A more direct reduction to linear matroid parity has been presented by Schrijver [13]. In this paper, we extract a structure of Schrijver's reduction and introduce the concept of *coherent representation.* For an instance of the subgroup model, we call a matrix a coherent representation if it satisfies Properties 3.1 and 3.2 described in Section 3.1. The main result of this paper is a characterization of the subgroup model that admits a coherent representation.

For a positive integer $n \in \mathbb{N}$ and a field $\mathbb{F}$, $\mathrm{GL}(n, \mathbb{F})$ denotes the set of the nonsingular $n \times n$ matrices over $\mathbb{F}$, and let $\mathrm{PGL}(n, \mathbb{F}) := \mathrm{GL}(n, \mathbb{F})/\{kI_n \mid k \in \mathbb{F}\}$, where I_n is the $n \times n$ identity matrix. In this paper, each element of PGL is denoted by its representative in GL.

Theorem 1.1. *Let Γ be a group, Γ' be its proper subgroup, and $\mathbb{F}$ be a field. Then the following two statements are equivalent.*

- **(i)** *For any Γ-labelled graph $(G = (V, E), \psi)$ with any terminal set $A \subseteq V$, the subgroup model with respect to Γ' can be reduced to the linear matroid parity problem with a coherent representation over $\mathbb{F}$.*
- **(ii)** *There exist a homomorphism $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{F})$ and a 1-dimensional linear subspace Y of $\mathbb{F}^2$ such that $\Gamma' = \{\alpha \in \Gamma \mid \rho(\alpha)Y = Y\}$.*

Theorem 1.1 clarifies a necessary and sufficient condition for the groups in question to admit a reduction with a coherent representation, which leads to fast algorithms. A recent work of Tanigawa and the author [14] showed that Lovász's reduction idea to matroid matching, which implies the polynomial-time solvability by Lovász's matroid matching algorithm [6], is always extendable even when there is no coherent representation.

2 Preliminaries

2.1 Packing A-paths

Finding a maximum family of (fully) vertex-disjoint A-paths is a path-packing problem which includes non-bipartite matching as a special case with $A = V$. Mader [8] suggested a more generalized problem, called Mader's disjoint $\mathcal{S}$-paths problem, and showed a min-max relation. Here $\mathcal{S}$ is a partition of A and an *$\mathcal{S}$-path* is an A-path between terminals in distinct subsets in $\mathcal{S}$. Hence, for any disjoint $S, T \subseteq V$, the concept of $\mathcal{S}$-path includes that of S–T path as a special case with $\mathcal{S} = \{S, T\}$.

As a generalization of Mader's problem, Chudnovsky et al. [3] introduced a framework of packing A-paths in group-labelled graphs, called the *non-zero model* in this paper, and proved a min-max theorem which generalizes Mader's theorem. Later, Pap [11] introduced a slightly more generalized model, called the *non-returning model* in this paper, and gave a simpler proof of a further generalized min-max theorem for his model. Chudnovsky et al. [2] gave an efficient combinatorial algorithm for the non-zero model, and Pap [12] suggested one for the non-returning model, whose running time bound is not known.

Theorem 2.1 (Chudnovsky, Cunningham, and Geelen [2]). *A maximum family of vertex-disjoint non-zero A-paths can be found in $O(|V|^5)$ time.*

These frameworks include interesting special cases besides Mader's problem such as packing odd-length A-paths, and packing A-paths on surfaces under various constraints according to the homotopy class of the curve associated with each A-path. The non-returning model generalizes the non-zero model and is in fact equivalent to the subgroup model (see [10, §3.6] for a detail argument), so we mainly discuss the subgroup model in this paper.

2.2 Linear matroid parity

Given a matrix $Z \in \mathbb{F}^{n\times 2m}$ over a field $\mathbb{F}$, the *linear matroid patiry problem* is to find a maximum subset X of $[m] = \{1, 2, \ldots, m\}$ such that the corresponding submatrix $Z_X := (z_{2i-1}, z_{2i} \mid i \in X)$ is column-full-rank, where z_j denotes the j-th column of Z. Let ω be the matrix multiplication exponent, which is at most 2.373.

Theorem 2.2 (Gabow and Stallmann [4], Orlin [9]). *The linear matroid parity problem can be solved in $O(mn^3)$ time. If fast matrix multiplication is used, then the running time is improved to $O(mn^\omega)$.*

Theorem 2.3 (Cheung, Lau, and Leung [1]). *The linear matroid parity problem can be solved with high probability in $O(mn^2)$ time. If fast matrix multiplication is used, then the running time is improved to $O(mn^{\omega-1})$.*

A direct application of these theorems to Schrijver's reduction of Mader's problem implies that a maximum family of vertex-disjoint $\mathcal{S}$-paths in an undirected graph $G = (V, E)$ can be found in $O(|E| \cdot |V|^\omega)$ time, and moreover with high probability in $O(|E| \cdot |V|^{\omega-1})$ time. Cheung et al. [1] improved the latter running time bound to $O(|V|^\omega)$ under the assumption, without loss of generality, that the input graph is simple.

3 Reduction to Linear Matroid Parity

3.1 Coherent Representation

We introduce two natural properties satisfied by Schrijver's reduction of Mader's problem to the linear matroid parity problem. Let Γ be a group and $\mathbb{F}$ be a field. For a Γ-labelled graph $(G = (V, E), \psi)$ with terminal set $A \subseteq V$, we consider constructing a representation matrix $Z \in \mathbb{F}^{2n\times 2m}$ which defines an instance of the linear matroid parity problem, where $|V| = n$, $|E| = m$. For the simplicity of description, we assume $V = [n] := \{1, 2, \ldots, n\}$ and $E = [m]$. The representation is desired to be based on the incidence matrix of G.

Property 3.1. *For each $e = uv \in E$, there exists exactly one pair of two corresponding columns z_{2e-1}, z_{2e} of Z, each of which has at most four nonzero entries at $2u-1, 2u, 2v-1, 2v$-th rows. In other words,*

$$z_{2e-1}, z_{2e} \in \{a_{2u-1}\vec{e}_{2u-1} + a_{2u}\vec{e}_{2u} + a_{2v-1}\vec{e}_{2v-1} + a_{2v}\vec{e}_{2v} \mid a_i \in \mathbb{F}\ (i = 2u-1, 2u, 2v-1, 2v)\},$$

where $\vec{e}_i \in \mathbb{F}^{2n}$ denote i-th unit vectors for $i \in [n]$.

Let us call an edge set $F \subseteq E$ *feasible* if the set $\{z_{2e-1}, z_{2e} \mid e \in F\}$ of all correspong-ing vectors is linearly independent. The following property guarantees natural relation between the subgroup model and the linear matroid parity problem.

Property 3.2. *For each A-path P in G, its edge set $E(P)$ is feasible if and only if P is admissible.*

3.2 Sufficiency

In this section, we show how to construct a coherent representation under the condition **(ii)** in Theorem 1.1. Fix a field $\mathbb{F}$, a projective representation $\rho : \Gamma \to \mathrm{PGL}(2, \mathbb{F})$, and a 1-dimensional subspace Y of $\mathbb{F}^2$ which satisfy **(ii)**. Furthermore, fix an arbitrary Γ-labelled graph $(G = (V, E), \psi)$ and an arbitrary terminal set $A \subseteq V$.

- Let $L_e := \{x \in (\mathbb{F}^2)^V \mid \rho(\psi(e, w))x(u) + x(w) = \mathbf{0},\ x(v) = \mathbf{0}\ (v \in V \setminus \{u, w\})\}$ for each edge $e = uw \in E$.
- Let $Q_v := \{x \in (\mathbb{F}^2)^V \mid x(v) \in Y,\ x(u) = \mathbf{0}\ (u \in V \setminus \{v\})\}$ for each terminal $v \in A$.
- Let $Q_U := \sum_{v \in U} Q_v$ for each $U \subseteq A$, and let $Q := Q_A$.
- Let $\mathcal{E} := \{L_e/Q \mid e \in E\}$ (we may assume $\dim(L_e/Q) = 2$ for every edge $e \in E$).

Let us construct a representation matrix $Z \in \mathbb{F}^{2|V| \times 2|E|}$ associated with $\mathcal{E}$ by enumerating the bases of L_e/Q for all $e \in E$. Then each edge set $F \subseteq E$ is feasible if and only if $\dim(L_F/Q) = 2|F|$, where $L_F := \sum_{e \in F} L_e$.

Lemma 3.3. *Let $\nu(\mathcal{E})$ denote the cardinality of a maximum feasible edge set, and let $\mu(G, \psi, A)$ denote the maximum number of vertex-disjoint admissible A-paths in G with respect to $\psi : E \to \Gamma$. If G is connected and $A \neq \emptyset$, then $\nu(\mathcal{E}) = |V| - |A| + \mu(G, \psi, A)$.*

This lemma is derived from the following properties: (1) each maximum feasible edge set includes an edge set forming a maximum family of vertex-disjoint admissible A-paths, (2) the edge set of each maximum family of vertex-disjoint admissible A-paths is included a maximum feasible edge set, and (3) each connected component formed by a feasible edge set contains at most one A-path, which is admissible. From (1) and (3), in particular, one can construct a maximum family of vertex-disjoint admissible A-paths from a maximum feasible edge set by the depth first search from each terminal, in linear time. Thus the subgroup model reduces to the linear matroid parity problem under the condition **(ii)** in Theorem 1.1.

3.3 Necessity

We construct a special Γ-labelled graph that has a coherent representation, and show that the coherent representation leads to a projective representation of Γ that satisfies the condition **(ii)** in Theorem 1.1. Here we just show the construction of such a special graph, and the proof is left for the full paper.

Let Γ/Γ' denote the left cosets $\{\alpha\Gamma' \mid \alpha \in \Gamma\}$, and consider a Γ-labelled graph $(G = (V, E), \psi)$ with terminal set A defined as follows.

- For each $i \in \{1, 2, 3\}$, let $G_i = (V_i, E_i)$ be a star with the center vertex $v_i \in V_i$.
- For each $i \in \{1, 2, 3\}$ and each representative α of Γ/Γ', there are exactly two edges $e = uv_i \in E_i$ such that $\psi(e, v_i) = \alpha$.
- For each $1 \leq i < j \leq 3$ and each $\alpha \in \Gamma$, there is exactly one edge $e = v_i v_j$ such that $\psi(e, v_j) = \alpha$. Let E' denote the set of such parallel edges and $E := E' \cup E_1 \cup E_2 \cup E_3$.
- Let $V := V_1 \cup V_2 \cup V_3$ and let $A \subseteq V$ be the set of all leaves of G.

4 Applications

By our reduction, linear matroid parity algorithms can be used to solve a number of special cases of the subgroup model. Naive applications of Theorems 2.2 and 2.3 lead to deterministic $O(|E| \cdot |V|^{\omega})$ and randomized $O(|E| \cdot |V|^{\omega-1})$ algorithms. One can improve the latter bound to $O(|E| + |V|^{\omega})$ by the same argument as [1, §5.1.3]. Since $|\Gamma'| + 1$ edges are enough between each pair of vertices, we may assume $|E| \leq (|\Gamma'| + 1) \cdot |V|^2$. Therefore, the new bound is better than $O(|V|^5)$, which can be achieved by an extension of the algorithm of Chudnovsky et al. [2], unless G has a large number of dense parallel edges (i.e., there are $\Omega(|\Gamma'|)$ parallel edges between each of $\Omega(|V|^2)$ pairs of two distinct vertices) and Γ' is so large that $|\Gamma'| = \Omega(|V|^3)$.

In this section, we present simple but important special cases of the subgroup model. All following cases that admit coherent representations satisfy $|\Gamma'| = O(1)$, and hence the running time of the linear matroid parity algorithm of Cheung et al. [1] is bounded by $O(|V|^{\omega})$, which is much better than $O(|V|^5)$. Without loss of generality, Y is fixed to $\langle \vec{e}_1 \rangle$ where $\vec{e}_1 \in \mathbb{F}^2$ denotes the first unit vector over each field $\mathbb{F}$.

4.1 Mader's $\mathcal{S}$-paths

Let $\mathcal{S} = \{A_1, \ldots, A_k\}$ be a partition of the terminal set A. Then Mader's problem is a special case of the subgroup model: $\Gamma = (\mathbb{Z}, +)$, $\Gamma' = \{0\}$ and $\psi(e) = i - j$ for each $e = uv \in \vec{E}$ with $u \in A_i$ and $v \in A_j$, where $A_0 := V \setminus A$. In this case, ρ defined as follows leads to the same coherent representation over $\mathbb{Q}$ as Schrijver's one with appropriate base transformations:

$$\rho(i) := \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \quad (i \in \mathbb{Z}).$$

4.2 Odd-length A-paths

To find a maximum family of vertex-disjoint odd-length A-paths is a special case of the subgroup model: $\Gamma = (\{1,-1\}, \times) \simeq \mathbb{Z}/2\mathbb{Z}$, $\Gamma' = \{1\}$, and $\psi(e) = -1$ for each $e \in \vec{E}$. In this case, ρ defined as follows leads to a coherent representation over an arbitrary field:

$$\rho(1) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(-1) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

4.3 Non-returning model

First we describe the definition of the non-returning model dealt with in [11, 12]. Let Π be a finite set, $\omega : A \to \Pi$ be a map on the terminal set, and $\pi : \vec{E} \to S(\Pi)$ be a map on the edge set to the permutations on Π with reference orientation. In this model, an A-path $(v_0, e_1, v_1, \ldots, e_k, v_k)$ is admissible if and only if $\omega(v_k) \neq \pi(e_k, v_k) \circ \cdots \circ \pi(e_1, v_1)(\omega(v_0))$ holds. Let $d := |\Omega| \geq 2$. This model is equivalent to the subgroup model, and in particular it reduces to the following setting. Let Γ be the symmetric group S_d of degree d, and $\Gamma' := \{\sigma \in \Gamma \mid \sigma(d) = d\} = S_{d-1}$.

Theorem 4.1. *The subgroup model reduced from the non-returning model with the label set Ω admits a coherent representation if and only if $|\Omega| \leq 4$.*

In particular, if $|\Omega| = 4$, we have desired ρ over $\mathbb{F} := \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ as follows:

$$\begin{aligned}
\rho(\mathrm{id}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho((1\ 2\ 3)) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \rho((1\ 3\ 2)) &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
\rho((1\ 2)) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, & \rho((2\ 3)) &= \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, & \rho((1\ 3)) &= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \\
\rho((1\ 4)) &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, & \rho((2\ 4)) &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, & \rho((3\ 4)) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\rho((1\ 2)(3\ 4)) &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, & \rho((1\ 3)(2\ 4)) &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, & \rho((1\ 4)(2\ 3)) &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \\
\rho((1\ 2\ 4)) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \rho((1\ 4\ 2)) &= \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, & \rho((1\ 3\ 4)) &= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \\
\rho((1\ 4\ 3)) &= \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, & \rho((2\ 3\ 4)) &= \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, & \rho((2\ 4\ 3)) &= \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \\
\rho((1\ 3\ 2\ 4)) &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, & \rho((1\ 4\ 2\ 3)) &= \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, & \rho((1\ 2\ 3\ 4)) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \\
\rho((1\ 4\ 3\ 2)) &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, & \rho((1\ 3\ 4\ 2)) &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, & \rho((1\ 2\ 4\ 3)) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}$$

This is an isomorphism from S_4 to $\mathrm{PGL}(2, \mathbb{F}_3)$. The correctness can be easily confirmed by checking $A^2 = B^2 = C^2 = I_2$, $AC = CA$, $ABA = BAB$, $BCB = CBC$ where $A := \rho((1\ 2))$, $B := \rho((2\ 3))$, and $C := \rho((3\ 4))$.

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