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Domain variation and electromagnetic frequencies (電磁波の固有振動数に関するアダマール変分)

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§1. Eigenvalue of Laplacian and Hadamard variation

Let Ω be a bounded domain in the space $\mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$ and consider solutions of a certain boundary value problem of a particular PDE in Ω. When Ω is deformed smoothly, how do they or their structure vary or perturb ? This problem is a fundamental subject to study from the point of view of mathematics and physics for many kinds of PDE (cf. Courant-Hilbert [1], 小澤真 [18])). Because scientists have interested in various phenomena (in real physics or in mathematical model), which differ dependently on the geometric properties of the environments. Hadamard studied the eigenvalues of the Laplacian and bi-Laplacian (with the Dirichlet boundary condition) and their Green function and deduced their certain variational formula under a regular variation of domains. This study is a great leap in PDE history. His formula were justified later in the framework of rigorous analysis and were generalized. After this pioneering work, there have been a lot of studies of variational formula for several quantities of domain under regular domain variation (cf. Fujiwara-Ozawa [3], Garabedian-Schiffer [4], 小澤真 [18], Shimakura [20], Sokolowski-Zolesio [22], Ohsawa [17](nonlinear eigenvalue), Kozlov [10], Grinfeld [5], Kozono-Ushikoshi [11] (Stokes equation), Jimbo [9] for related works. There have been also studies for singular deformation of domains (cf. Jimbo [8], Maz'ya- Nazarov- Plamenevskij [12], 小澤真 [18], Ozawa [19] and their references). In the following, we mention the results for the Hadamard variation of the eigenvalue problem of the Laplacian, because the main motivation of our study is to generalize these earlier works. First we consdier the case of the Dirichlet boundary condition.

$$\Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \quad \Phi = 0 \quad \text{on} \quad \partial\Omega \tag{1.1}$$

We denote the $k-$th eigenvalue and the corresponding eigenfunction by λ_k and Φ_k, respectively. The question is **"How λ_k perturbes if the domain Ω smoothly deforms ?"**. We formulate the deformation of the domain. Let $\rho = \rho(\xi)$ is a smooth function on $\Gamma = \partial\Omega$ and define the set

$$\Gamma(\epsilon) = \{\xi + \epsilon\rho(\xi)\nu(\xi) \in \mathbb{R}^n \mid \xi \in \Gamma\}$$

for small ϵ. There exists a unique bounded domain $\Omega(\zeta)$ which is homeomorphic to Ω, such that the boundary $\partial\Omega(\zeta)$ agrees to $\Gamma(\zeta)$. In this parametrized domain, we consider the eigenvalue problem of Laplacian (with Dirichlet B.C.),

$$\Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \partial\Omega(\epsilon). \tag{1.2}$$

Denote the $k-$th eigenvalue of (1.2) by $\lambda_k(\epsilon)$ and then the followig result holds.

Theorem 1 (Hadamard [6]). Assume that λ_k is simple in (1.1). Then $\lambda_k(\epsilon)$ is differentiable at $\epsilon = 0$ and the following asymptotic formula holds.

$$\lim_{\epsilon \to 0} \frac{\lambda_k(\epsilon) - \lambda_k}{\epsilon} = -\int_\Gamma (\frac{\partial \Phi_k}{\partial \nu}(\xi))^2 \, dS / \|\Phi_k\|^2_{L^2(\Omega)}$$

It is quite natural to ask the same problem for the Neumann or Robin boundary condition. The result can be seen in Ohsawa [17], Grinfeld [5]. Let us consider the following eigenvalue problem.

$$\Delta \Phi + \mu \Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \frac{\partial \Phi}{\partial \nu} + \sigma \, \Phi = 0 \quad \text{on} \quad \partial\Omega(\epsilon) \tag{1.3}$$

where σ is a nonnegative constant and ν is the unit outward normal vector on $\partial\Omega(\epsilon)$. The boundary condition is Robin B.C if $\sigma > 0$ and Neumann B.C. if $\sigma = 0$.

Denote the $k-$th eigenvalue by $\mu_k(\epsilon)$ of (1.3) and then the followig result holds.

Theorem 2 ([5],[17]). Assume that $\mu_k(0)$ is simple in (1.3) for $\epsilon = 0$. Then $\mu_k(\epsilon)$ is differentiable at $\epsilon = 0$ and

$$\lim_{\epsilon \to 0} \frac{\mu_k(\epsilon) - \mu_k}{\epsilon} = \int_\Gamma (|\nabla \Phi_k|^2 + (\sigma h(\xi) - 2\sigma^2 - \mu_k)\Phi_k^2) dS / \|\Phi_k\|^2_{L^2(\Omega)}$$

where $h = h(\xi)$ is the mean curvature of Γ at ξ with respect to the unit outward normal vector $\nu(\xi)$ on Γ.

Remark. T. Ohsawa [17] consider the nonlinear eigenvalue problem of Laplacian with the Robin boundary condition.

$$\Delta v + \mu_k \, |v|^{p-1} v = 0 \quad \text{in} \quad \Omega, \quad \partial v / \partial \nu + \sigma \, v = 0 \quad \text{on} \quad \Gamma(\zeta)$$

and characterized the first (nonlinear) eigenvalue, which agrees to the asmptotic formula as in Theorem 2 (for the case $p = 1, k = 1$). Grinfeld [5] dealt with more general cases.

An interesting point is that the mean curvature term appears for Robin case, but it does not in the case for Dirichlet and Neumann condition.

Remark. The formula in Theorem 2 can be deduced by a similar argument as that for the eigenfrequency of the Maxwell equation.

§2. Eigenvalue problem in Electromagnetism

We deal with the harmonic oscillation in the Maxwell equation in a bounded domain (under a certain boundary condition) and consider the smooth dependency of the eigenfrequency under a smooth domain perturbation. The electric-magnetic phenomena is modelled by the Maxwell equation (with an appropriate boundary condition) in the theory of Electromagnetism. The Maxwell equation is a coupled system of the electric field E and the magnetic field H. Assume $n = 3$ hereafter and consider the Maxwell equation

$$\epsilon_0 \, \partial E / \partial t - \operatorname{rot} H = \mathbf{0}, \quad \mu_0 \, \partial H / \partial t + \operatorname{rot} E = \mathbf{0}, \quad \operatorname{div} E = 0, \quad \operatorname{div} H = 0. \tag{2.1}$$

with some boundary condition (cf. (2.2)). Here $\epsilon_0 > 0$ is the dielectric constant and $\mu_0 > 0$ is the magnetic permeability of the space (cf. 平川 [7]). We impose the boundary condition so that the space is surrounded by a perfect conductor. It gives the following condition

$$E \times \nu = \mathbf{0}, \quad \langle H, \nu \rangle = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

Here ν is the outward unit normal vector on $\partial\Omega$ and $\langle\cdot,\cdot\rangle$ is the standard inner product. The time harmonic solutions are written in the following form

$$E(t,x) = \exp(i\omega t)\widetilde{E}(x), \qquad H(t,x) = \exp(i\omega t)\widetilde{H}(x)$$

where ω is a parameter. Substitute these functions into (2.1) and we get

$$i\,\epsilon_0\,\omega\,\widetilde{E} - \operatorname{rot}\widetilde{H} = \mathbf{0}, \quad i\,\mu_0\,\omega\,\widetilde{H} + \operatorname{rot}\widetilde{E} = \mathbf{0}, \quad \operatorname{div}\widetilde{E} = 0, \quad \operatorname{div}\widetilde{H} = 0.$$

Applying rot on the second equation and use the first equation, we get

$$-\mu_0\epsilon_0\omega^2\,\widetilde{E} + \operatorname{rot}\operatorname{rot}\widetilde{E} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \widetilde{E}\times\nu = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

The eigenfrequency is the value ω, for which (2.3) allows a nontrivial solution $\widetilde{E}$. By a scale transform, we can assume $\mu_0\epsilon_0 = 1$. Denote $\widetilde{E}$ in place of $\mathbf{\Phi}$ and put $\lambda = \omega^2$. Thus we get the mathematical problem.

[Mathematical formulation for the eigenvalues]

We consider the eigenvalue problem.

$$\operatorname{rot}\operatorname{rot}\mathbf{\Phi} - \lambda\,\mathbf{\Phi} = \mathbf{0}, \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{\Phi}\times\nu = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{2.4}$$

Any eigenvalue is nonnegative. Actually, take the inner product of (2.4) and $\mathbf{\Phi}$, and integrate in Ω, we have

$$\int_\Omega |\operatorname{rot}\mathbf{\Phi}(x)|^2 dx = \lambda\int_\Omega |\mathbf{\Phi}(x)|^2\,dx. \tag{2.5}$$

This implies that λ is nonnegative if $\mathbf{\Phi} \not\equiv \mathbf{0}$ in Ω.

[Zero eigenspace] If $\lambda = 0$ in (2.4), we deduce $\operatorname{rot}\mathbf{\Phi} = 0$ in Ω from (2.5). This condition and div-free property imply that $\mathbf{\Phi}$ has an expression $\mathbf{\Phi} = \nabla\eta$ by a harmonic function η. Here η may be multi-valued scalar function. From the boundary condition, $\nabla\eta$ is parallel to ν on $\partial\Omega$. This implies that η is constant in any connected component of $\partial\Omega$. So η is necessarily single-valued function. On the other hand, take any function $\eta \in H^1(\Omega)$ which is constant on any connected component of $\partial\Omega$ and put $\Phi = \nabla\eta$ and then it becomes an eigenfunction for $\lambda = 0$. Thus we conclude that the zero eigenspace is the following.

$$X_0 = \{\nabla\eta \mid \eta \in C^2(\overline{\Omega}),\ \Delta\eta = 0 \text{ in } \Omega,\ \eta \text{ is constant in each component of } \partial\Omega\}$$

To prove the existence of positive eigenvalues, we prepare a certain basic function space.

$$X = \{\mathbf{\Phi} \in H^1(\Omega;\mathbb{R}^3) \mid \operatorname{div}\mathbf{\Phi} = 0 \text{ in } \Omega,\ \mathbf{\Phi}\times\nu = \mathbf{0} \text{ on } \partial\Omega\}.$$

It is easy to see that $\dim X_0 = \sharp(\text{components of}\,\partial\Omega) - 1$. It is known that X is a closed subspace of $H^1(\Omega;\mathbb{R}^3)$ and X is also closed in the sense of weak convergence in $H^1(\Omega;\mathbb{R}^3)$ (because X is linear). Hereafter we deal with the positive eigenvalues from now.

Proposition 2.1. The eigenvalue problem (2.4) has a set of positive eigenvalues $\{\Lambda_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty}\Lambda_k = \infty$.

This is proved by a completely similar argument as the Laplacian (cf. Courant-Hilbert [1], Edmunds-Evans [2]) with the Rayleigh quotient in $X \cap X_0^\perp$

$$\mathcal{R}(\phi) = \int_\Omega |\operatorname{rot}\phi|^2 dx \Big/ \int_\Omega |\phi|^2 dx.$$

(Proof of Proposition 2.1) To prove the existence of the eigenvalues, we can carry out a completely similar argument as the case of the Laplacian and the Schrödinger operator (cf. Edmunds-Evans [2]). So we only give a sketch of the argument. Hereafter the symbol $\perp$ means the orthogonality in $L^2(\Omega;\mathbb{R}^3)$. Put

$$\Lambda_1 = \inf\{\mathcal{R}(\phi) \mid \phi \in X,\ \phi \perp X_0\}, \quad \text{where} \quad \mathcal{R}(\phi) = \int_\Omega |\mathrm{rot}\,\phi|^2 / \int_\Omega |\phi|^2 dx.$$

$\mathcal{R}$ attains the minimum Λ_1 with a minimizer $\mathbf{\Phi}^{(1)} \in X$ which is an eigenfunction corresponding to the eigenvalue Λ_1. This is proved as follows. Take a minimizing sequence $\{\phi_\ell\}_{\ell=1}^\infty$ with $\|\phi_\ell\|_{L^2(\Omega;\mathbb{R}^3)} = 1$. It is bounded also in $H^1(\Omega;\mathbb{R}^3)$ due to Lemma 2.2 and Lemma 2.3 below. This sequence contains a weakly converent subsequence in $H^1(\Omega;\mathbb{R}^3)$ which is also strongly convergent in $L^2(\Omega;\mathbb{R}^3)$. Since X is closed, the limit $\mathbf{\Phi}^{(1)}$ of the subsequence belongs to X and satisfies $\mathbf{\Phi}^{(1)} \perp X_0$. From the lower semicontinuity of $\mathcal{R}$ in X, $\mathbf{\Phi}^{(1)}$ becomes a minimizer in $X_0^\perp \cap X$. Taking the variation of $\mathcal{R}$ at $\mathbf{\Phi}^{(1)}$ (minimizer), we get

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{\Phi}^{(1)} - \Lambda_1 \mathbf{\Phi}^{(1)} = \mathbf{0} \quad \text{in} \quad \Omega.$$

Carry out this argument in the space $X \cap (X_0 \oplus L.H.[\mathbf{\Phi}^{(1)}])^\perp$, we get the second positive eigenvalue Λ_2 as the minimum of $\mathcal{R}$ with the eigenfunction (minimizer) $\mathbf{\Phi}^{(2)} \in X$ with $\mathbf{\Phi}^{(2)} \in (X_0 \oplus L.H.[\mathbf{\Phi}^{(1)}])^\perp$. We can repeat this argument and get the sequence $0 < \Lambda_1 \leqq \Lambda_2 \leqq \Lambda_3 \leqq \cdots$. We also note that this sequence is unbounded. The eigenfunctions obtained above are sufficiently regular if $\partial\Omega$ is regular. This can be proved by the arguments in the chapter 7 in Morrey [14], where the harmonic forms in the smooth manifold with a boundary, are studied. The regularity of $\Phi^{(k)}$ inside Ω is proved by the argument in Mizohata [13] for each component. For the regularity near the boundary, the technique in [14] is applied. The higher regularity estimates of the eigenfunction are also obtain in this process. □

Lemma 2.2 (Trace inequality). For any $\eta > 0$, there exists $c(\eta) > 0$ such that

$$\int_{\partial\Omega} \phi(x)^2 dS \leqq \eta \int_\Omega |\nabla\phi(x)|^2 dx + c(\eta) \int_\Omega \phi(x)^2 dx \quad (\phi \in H^1(\Omega)).$$

See Mizohata [13; Chap.3] for the proof.

Lemma 2.3. If $\mathbf{\Psi} \in H^1(\Omega;\mathbb{R}^3)$ and $\mathbf{\Psi} \times \nu = \mathbf{0}$ on $\partial\Omega$, then

$$\int_\Omega |\mathrm{rot}\,\mathbf{\Psi}|^2 dx + \int_\Omega |\mathrm{div}\,\mathbf{\Psi}|^2\, dx = \int_\Omega |\nabla\mathbf{\Psi}|^2\, dx + \int_{\partial\Omega} H(x)|\mathbf{\Psi}(x)|^2\, dS.$$

Here $H(x)$ is the mean curvature at $x \in \partial\Omega$ with respect to the unit outward normal vector ν.

(Proof of Lemma 2.3) The proof is carried out through the straightforward calculation. □

Proposition 2.4 (Max-Min principle). The k-th positive eigenvalue Λ_k is characterized by the following formula.

$$\Lambda_k = \sup_{E \subset X_0^\perp, \dim E \leqq k-1} \inf\{\mathcal{R}(\mathbf{\Phi}) \mid \Phi \in X,\ \mathbf{\Phi} \perp X_0,\ \mathbf{\Phi} \perp E\}$$

Here E is a subspace of $L^2(\Omega;\mathbb{R}^3)$. For Max-Min principle for more general frame work of selfadjoint elliptic operators in Hilbert spaces, see Reed-Simon's book.

We begin the domain variation problem for electromagnetic eigenvalue problem. For the domain $\Omega(\epsilon)$, we consider the following eigenvalue problem,

$$\begin{cases} \operatorname{rot}\operatorname{rot}\mathbf{\Phi} - \lambda\,\mathbf{\Phi} = \mathbf{0}, \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \mathbf{\Phi}\times\nu = \mathbf{0} \quad \text{on} \quad \partial\Omega(\epsilon). \end{cases} \tag{2.6}$$

From the formula $\operatorname{rot}\operatorname{rot} = \nabla\operatorname{div} - \Delta$, the eigenvalue problem (2.6) is also written as

$$\begin{cases} \Delta\mathbf{\Phi} + \lambda\,\mathbf{\Phi} = \mathbf{0}, \ \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \mathbf{\Phi}\times\nu = \mathbf{0} \qquad \text{on} \quad \partial\Omega(\epsilon). \end{cases} \tag{2.7}$$

Definition. Let $\{\lambda_k(\epsilon)\}_{k=1}^{\infty}$ be the set of positive eigenvalues (of (2.6)) which are arranged in increasing order with counting multiplicity.

Definition. Let $\{\mathbf{\Phi}_\epsilon^{(k)}\}_{k=1}^{\infty}$ be the corresponding system of the eigenfunctions, which is orthonormal as

$$(\mathbf{\Phi}_\epsilon^{(p)}, \mathbf{\Phi}_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathbb{R}^3)} = \delta(p,q) \quad (|\epsilon| : \text{small}, \ p, q \geqq 1).$$

We have the following result.

Theorem 3. Assume that the $k-$th eigenvalue $\lambda_k(0)$ is simple. Then $\lambda_k(\epsilon)$ is differentiable at $\epsilon = 0$ and its derivative is given by the following formula.

$$\frac{d\lambda_k(\epsilon)}{d\epsilon}_{|\epsilon=0} = \int_{\partial\Omega}\left(|\nabla\mathbf{\Phi}_0^{(k)}|^2 - 2|\frac{\partial\mathbf{\Phi}_0^{(k)}}{\partial\nu}|^2 + (2K - \lambda_k(0))|\mathbf{\Phi}_0^{(k)}|^2\right)\rho\, dS$$

$$+2\int_{\partial\Omega}\langle\mathbf{\Phi}_0^{(k)}, \nu\rangle\,\langle\operatorname{rot}\mathbf{\Phi}_0^{(k)} \times \nabla\rho, \nu\rangle\, dS$$

Here $K(x)$ is the Gaussian curvature of $\partial\Omega$ at x. $\nabla\rho$ is the gradient field in the tangent space of $\partial\Omega$.

Remark. In the case of multiple eigenvalue $\lambda_k(0)$, $\lambda_k(\epsilon)$ is right-differentiable and and also left-differentiable. However they may not agree with each other in general.

In the later sections, we draw a rough sketch of the deduction of the asymptotic formula. But we do not give the justification of the formula. See Jimbo [9] for the complete proof.

§3. Transformation of the problem

The method of the proof is to make a transformation (diffeomorphism) $\gamma_\epsilon : \Omega \longrightarrow \Omega(\epsilon)$ and to transform the problem to fix the domain (through the change of the variable $x = \gamma_\epsilon(y)$). So the problem on the $\epsilon-$dependent variable domain reduces to the equations which includes ϵ in coefficients. So we prepare the transformation map and calculate the equation in a fixed domain Ω.

Lemma 3.1. There exists $\delta_0 > 0$ and a smooth diffeomorphism map

$$\gamma_\epsilon : \overline{\Omega} \longrightarrow \overline{\Omega(\epsilon)}$$

such that γ_ϵ depends smoothly on $\epsilon \in (-\delta_0, \delta_0)$ and

$$\gamma_\epsilon(\xi + t\nu(\xi)) = \xi + (t+\epsilon)\rho(\xi)\nu(\xi) \quad \text{for} \quad \xi \in \partial\Omega, \ |t| < \delta_0, \ |\epsilon| < \delta_0. \tag{3.1}$$

(Proof) Prepare a coordinate near the boundary $\partial\Omega$ and consider the map which moves a point of the δ_0−neighborhood of $\partial\Omega$ as in (3.1). We can construct a smooth map γ_ϵ with this property using a smooth cut-off and extension up to the whole Ω. □

We prepare some notation. The variation of the map γ_ϵ under perturbation by ϵ is given by a vector field g as follows,

$$g(y) = (g_1(y), g_2(y), g_3(y))^t = \frac{\partial\gamma_\epsilon(y)}{\partial\epsilon}_{|\epsilon=0} \quad (y \in \Omega).$$

From the condition (3.1), we have

$$\frac{d\gamma_\epsilon}{d\epsilon}(\xi + t\nu(\xi))_{\epsilon=0} = \rho(\xi)\nu(\xi) \quad \text{for} \quad \xi \in \partial\Omega, |t| < \delta_0. \tag{3.2}$$

This formula is also written as

$$g(\xi + t\nu(\xi)) = \rho(\xi)\nu(\xi) \quad (\xi \in \partial\Omega, |t| < \delta_0).$$

Take the derivative of the both side of this expression with respect to t and put $t = 0$, we have the following property of g.

Lemma 3.2. $(\partial g/\partial y)\nu = 0$ on $\partial\Omega$.

We start the calculation of the variational equation. We denote the unknown variable by $\mathbf{\Phi}$ and the transformed unknown variable by $\widetilde{\mathbf{\Phi}}$. Their relation is

$$\widetilde{\mathbf{\Phi}}(y) = (\mathbf{\Phi} \circ \gamma_\epsilon)(y) \quad (y \in \Omega).$$

We express the unknown variable $\mathbf{\Phi}$ by its components as follows.

$$\mathbf{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))^t, \quad \widetilde{\mathbf{\Phi}}(y) = (\widetilde{\Phi}_1(y), \widetilde{\Phi}_2(y), \widetilde{\Phi}_3(y))^t$$

Accordingly we have

$$\widetilde{\Phi}_i(y) = (\Phi_i \circ \gamma_\epsilon)(y) \quad (y \in \Omega, \quad i = 1, 2, 3).$$

We calculate the system of equations for $\widetilde{\mathbf{\Phi}}$ with the boundary condition. $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal vector on $\partial\Omega$. We extend this field ν up to some neighborhood of $\partial\Omega$ for later convenience such that $\nu(x) = \nu(\xi)$ for $x = \xi + t\nu(\xi)$ with $\xi \in \partial\Omega$, $|t| < \delta_0$. A direct calculation gives

$$\nabla_y\widetilde{\Phi}_i(y) = \nabla_x\Phi_i(x)\ (\frac{\partial\gamma_\epsilon}{\partial y}(y)), \quad \nabla_x\Phi_i = (\frac{\partial\Phi_i}{\partial x_1}, \frac{\partial\Phi_i}{\partial x_2}, \frac{\partial\Phi_i}{\partial x_3}), \quad \nabla_y\widetilde{\Phi}_i = (\frac{\partial\widetilde{\Phi}_i}{\partial y_1}, \frac{\partial\widetilde{\Phi}_i}{\partial y_2}, \frac{\partial\widetilde{\Phi}_i}{\partial y_3}).$$

$$\gamma_\epsilon(y) = (\gamma_{1,\epsilon}(y), \gamma_{2,\epsilon}(y), \gamma_{3,\epsilon}(y))^t, \quad \frac{\partial\gamma_\epsilon}{\partial y}(y) = \begin{pmatrix} \partial\gamma_{1,\epsilon}/\partial y_1 & \partial\gamma_{1,\epsilon}/\partial y_2 & \partial\gamma_{1,\epsilon}/\partial y_3 \\ \partial\gamma_{2,\epsilon}/\partial y_1 & \partial\gamma_{2,\epsilon}/\partial y_2 & \partial\gamma_{2,\epsilon}/\partial y_3 \\ \partial\gamma_{3,\epsilon}/\partial y_1 & \partial\gamma_{3,\epsilon}/\partial y_2 & \partial\gamma_{3,\epsilon}/\partial y_3 \end{pmatrix}.$$

We get the transformed equation.

$$\mathrm{div}_y\left(\det(\frac{\partial\gamma_\epsilon}{\partial y})\nabla_y\widetilde{\Phi}_i[\frac{\partial\gamma_\epsilon}{\partial y}]^{-1}([\frac{\partial\gamma_\epsilon}{\partial y}]^{-1})^t\right) + \lambda\det(\frac{\partial\gamma_\epsilon}{\partial y})\widetilde{\Phi}_i = 0 \quad \text{in} \quad \Omega \quad (i = 1, 2, 3). \tag{3.3}$$

The "div-free" condition is written as

$$\sum_{k=1}^{3}\sum_{\ell=1}^{3} \frac{\partial\widetilde{\Phi}_k}{\partial y_\ell}([\frac{\partial\gamma_\epsilon}{\partial y}(y)]^{-1})_{\ell k} = 0 \quad \text{in} \quad \Omega. \tag{3.4}$$

For a matrix M, $M_{\ell k}$ denotes the (ℓ, k) component of M and M^t is the transpose of M. We calculate the boundary condition for $\widetilde{\mathbf{\Phi}}$ on $\partial\Omega$. The unit outward normal vector ν_ϵ at $x = \gamma_\epsilon(y)$ on $\partial\Omega(\epsilon)$ is given by

$$\nu_\epsilon(\gamma_\epsilon(y)) = [(\partial\gamma_\epsilon/\partial y)^t(y)]^{-1}\nu(y)/|[(\partial\gamma_\epsilon/\partial y)^t(y)]^{-1}\nu(y)| \quad \text{for} \quad y \in \partial\Omega.$$

Since $\Phi(\gamma_\epsilon(y)) = \widetilde{\Phi}(y)$ at $y \in \partial\Omega$ is parallel to $\nu_\epsilon(\gamma_\epsilon)$, we have that $[(\partial\gamma_\epsilon/\partial y)^t(y)]\widetilde{\Phi}(y)$ is parallel to $\nu(y)$ due to the above expression of $\nu_\epsilon(\gamma_\epsilon(y))$. So we get the following boundary condition for $\widetilde{\Phi}$.

$$[(\partial\gamma_\epsilon/\partial y)^t(y)]\widetilde{\mathbf{\Phi}}(y) \times \nu(y) = \mathbf{0} \quad \text{on} \quad \partial\Omega$$

We write each component as follows.

$$\begin{aligned}(3.5)\qquad &\widetilde{\Phi}_1(-\nu_2(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_2}) + \widetilde{\Phi}_2(-\nu_2(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_2})\\ &+\widetilde{\Phi}_3(-\nu_2(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_1} + \nu_1(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_2}) = 0 \quad \text{on} \quad \partial\Omega,\end{aligned}$$

$$\begin{aligned}(3.6)\qquad &\widetilde{\Phi}_1(\nu_3(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_3}) + \widetilde{\Phi}_2(\nu_3(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_3})\\ &+\widetilde{\Phi}_3(\nu_3(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_1} - \nu_1(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_3}) = 0 \quad \text{on} \quad \partial\Omega,\end{aligned}$$

$$\begin{aligned}(3.7)\qquad &\widetilde{\Phi}_1(-\nu_3(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_2} + \nu_2(y)\frac{\partial\gamma_{1,\epsilon}}{\partial y_3}) + \widetilde{\Phi}_2(-\nu_3(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_2} + \nu_2(y)\frac{\partial\gamma_{2,\epsilon}}{\partial y_3})\\ &+\widetilde{\Phi}_3(-\nu_3(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_2} + \nu_2(y)\frac{\partial\gamma_{3,\epsilon}}{\partial y_3}) = 0 \quad \text{on} \quad \partial\Omega.\end{aligned}$$

We have obtained the reduced system (3.3)-(3.7) in Ω.

§4. Pertubation Analysis

The positive eigenvalues $\{\lambda_k(\epsilon)\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\{\mathbf{\Phi}_\epsilon^{(k)}\}_{k=1}^{\infty}$ have the have the following properties

$$(4.1)\qquad (\mathbf{\Phi}_\epsilon^{(p)}, \mathbf{\Phi}_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathbb{R}^3)} = \delta(p,q), \quad (\mathrm{rot}\,\mathbf{\Phi}_\epsilon^{(p)}, \mathrm{rot}\,\mathbf{\Phi}_\epsilon^{(q)})_{L^2(\Omega(\epsilon);\mathbb{R}^3)} = \delta(p,q)\lambda_p(\epsilon).$$

By the aid of the max-min principle (Proposition 2.4) for $\lambda_k(\epsilon)$ in (2.6) with the (almost) test functions $\mathbf{\Phi}_0^{(k)}(\gamma_\epsilon^{-1}(x))$ $(k \geqq 1)$, we can derive an upper estimate

$$(4.2)\qquad \lambda_k(\epsilon) \leqq \Lambda_k + O(\epsilon).$$

To obtain the lower estimate, we first note that there exists a constant $\delta_k > 0$ and $c_k > 0$ (from (4.1),(4.2)) such that

$$(4.3)\qquad ||\mathbf{\Phi}_\epsilon^{(k)}||_{H^1(\Omega(\epsilon);\mathbb{R}^3)} \leqq c_k \quad \text{for} \quad |\epsilon| \leqq \delta_k.$$

Recall $\widetilde{\mathbf{\Phi}}_\epsilon^{(k)}(y) = (\mathbf{\Phi}_\epsilon^{(k)} \circ \gamma_\epsilon)(y)$. As the tranformation $x = \gamma_\epsilon(y)$ smoothly approach the identity map, we have the following estimates (with the aid of Lemma 2.2 and Lemma 2.3).

Lemma 4.1. For each $k \in \mathbb{N}$, there exists a constant $c(k) > 0$ such that

$$||\widetilde{\mathbf{\Phi}}_\epsilon^{(k)}||_{H^1(\Omega;\mathbb{R}^3)} \leqq c(k) \quad \text{for small} \quad \epsilon > 0.$$

As $\Omega(\epsilon)$ depends smoothly on ϵ, we can apply the regularity argument for $\Phi_\epsilon^{(k)}$ in the boundary value problem (2.6) which is developed in the famous Morrey's book [14]. We can have the following regularity.

Lemma 4.2. For each $k \in \mathbb{N}$, there exists a constant $c'(k) > 0$ such that

$$\|\widetilde{\Phi}_\epsilon^{(k)}\|_{C^2(\overline{\Omega};\mathbb{R}^3)} \leqq c'(k) \quad \text{for small} \quad \epsilon > 0.$$

Take an arbitrary sequence $\{\epsilon(p)\}_{p\geqq 1}$ such that $\lim_{p\to\infty}\epsilon(p) = 0$. Then, there exists a subsequence $\{\epsilon(p(m))\}_{m=1}^\infty$ and an orthonormal system $\{\Theta^{(k)}\}_{k=1}^\infty$ in $L^2(\Omega;\mathbb{R}^3)$ such that

$$\widetilde{\Phi}_{\epsilon(p(m))}^{(k)} \longrightarrow \Theta^{(k)} \quad (m \to \infty) \tag{4.4}$$

strongly in $L^2(\Omega;\mathbb{R}^3)$ and weakly in $H^1(\Omega;\mathbb{R}^3)$ and $\operatorname{div}\Theta^{(k)} = 0$ in Ω, $\Theta^{(k)} \times \nu = \mathbf{0}$ on $\partial\Omega$. From (4.2), (4.4), we have

$$\begin{aligned}\Lambda_k &\geqq \liminf_{m\to\infty} \lambda_k(\epsilon(p(m))) = \liminf_{m\to\infty} \int_{\Omega(\epsilon(p(m)))} |\operatorname{rot}\Phi_{\epsilon(p(m))}^{(k)}(x)|^2 dx \\ &= \liminf_{m\to\infty} \int_\Omega |\operatorname{rot}\widetilde{\Phi}_{\epsilon(p(m))}^{(k)}(y)|^2 dy \geqq \int_\Omega |\operatorname{rot}\Theta^{(k)}(y)|^2 dy.\end{aligned} \tag{4.5}$$

From the orghogonality of $\{\Theta^{(k)}\}_{k=1}^\infty$ in $L^2(\Omega;\mathbb{R}^3) \cap X_0^\perp$ with (4.5), we have $\int_\Omega |\operatorname{rot}\Theta^{(k)}|^2 dy = \Lambda_k$ for $k \geqq 1$. This implies $\Theta^{(k)}$ is necessarily a $k-$th eigenfunction. Eventually we get the convergence $\lim_{m\to\infty}\lambda_k(\epsilon(p(m))) = \Lambda_k$. Since $\{\epsilon(p)\}$ was arbitrary, we have the following result.

Proposition 4.3. $\lim_{\epsilon\to 0}\lambda_k(\epsilon) = \Lambda_k$ $(k \geqq 1)$.

To study the detailed asymptotics of $\lambda_k(\epsilon)$ for $\epsilon \to 0$, we need to find a candidate of $(d\lambda_k(\epsilon)/d\epsilon)_{|\epsilon=0}$.

To calculate the derivative of the equation of (3.2)-(3.3) and the boundary condition (3.4),(3.3),(3.4), we prepare some formulas.

Lemma 4.4. Let $A(\epsilon)$ be an invertible square matrix which is differentiable in ϵ. Then we have

$$\frac{d}{d\epsilon}A(\epsilon)^{-1} = -A(\epsilon)^{-1}\frac{d}{d\epsilon}A(\epsilon)A(\epsilon)^{-1}. \tag{4.6}$$

Moreover, if $A(0) = I$ (Identity matrix), then

$$\frac{d}{d\epsilon}\det A(\epsilon)_{|\epsilon=0} = \operatorname{Tr}\left(\frac{dA(\epsilon)}{d\epsilon}_{|\epsilon=0}\right). \tag{4.7}$$

(Proof) This is proved by a direct calculation.

Since $\gamma_0(y) = y$ (Identity map), it follows $(\partial\gamma_0/\partial y) = I$. Hence we can apply the above formulas (4.6), (4.7) to the Jacobian matrix $\partial\gamma_\epsilon/\partial y$, we have

$$\frac{d}{d\epsilon}\left(\frac{\partial\gamma_\epsilon}{\partial y}\right)^{-1}_{|\epsilon=0} = -\frac{\partial g(y)}{\partial y}. \tag{4.8}$$

$$\frac{d}{d\epsilon}\det\left(\frac{\partial\gamma_\epsilon}{\partial y}\right)_{|\epsilon=0} = \operatorname{div}_y g(y) = \sum_{j=1}^3 \frac{\partial g_j(y)}{\partial y_j}. \tag{4.9}$$

[Variational equation]

Fix a natural number k hereafter. Drop the index k and denote $\boldsymbol{\Phi}_\epsilon = \boldsymbol{\Phi}_\epsilon^{(k)}$, $\widetilde{\boldsymbol{\Phi}}_\epsilon = \widetilde{\boldsymbol{\Phi}}_\epsilon^{(k)}$, $\lambda(\epsilon) = \lambda_k(\epsilon)$. Note that $\widetilde{\boldsymbol{\Phi}}_0 = \boldsymbol{\Phi}_0$ because γ_0 is the identity map. Assume that $\widetilde{\boldsymbol{\Phi}}_\epsilon$, $\lambda(\epsilon)$ is differentiable in ϵ at 0 and put

$$\text{(4.10)} \qquad \boldsymbol{\Psi}(y) = (\Psi_1(y), \Psi_2(y), \Psi_3(y))^t = (\partial\widetilde{\boldsymbol{\Phi}}_\epsilon^{(k)}/\partial\epsilon)_{\epsilon=0}, \qquad \boldsymbol{\kappa} = (d\lambda_k(\epsilon)/d\epsilon)(0).$$

We seek for the relation which $\boldsymbol{\Psi}$ and $\boldsymbol{\kappa}$ should satisfy if they exist. Take the derivative of (3.3),(3.4),(3.5),(3.6), (3.7) and put $\epsilon = 0$ and calculate by the formula (4.8) and (4.9) and substitute $\epsilon = 0$, we get

$$\text{(4.11)} \qquad \operatorname{div}(\nabla\Psi_i) + \operatorname{div}_y((\operatorname{div} g)\nabla_y\Phi_{0i}) - \operatorname{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t))$$
$$+\boldsymbol{\kappa}\Phi_{0i} + \lambda(0)(\operatorname{div} g)\Phi_{0i} + \lambda(0)\Psi_i = 0 \quad (y \in \Omega, i = 1,2,3),$$

$$\text{(4.12)} \qquad \operatorname{div}\boldsymbol{\Psi} = \sum_{i=1}^{3}\sum_{\ell=1}^{3}\frac{\partial\Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i} \quad \text{in} \quad \Omega.$$

From (3.5), (3.6), (3.7), we have the boundary condition for $\boldsymbol{\Psi}$ which gives the values of $\nu \times \boldsymbol{\Psi}$ on $\partial\Omega$,

$$\text{(4.13)} \quad \Psi_2\nu_1 - \Psi_1\nu_2 = \Phi_{01}(\nu_2\frac{\partial g_1}{\partial y_1} - \nu_1\frac{\partial g_1}{\partial y_2}) + \Phi_{02}(\nu_2\frac{\partial g_2}{\partial y_1} - \nu_1\frac{\partial g_2}{\partial y_2}) + \Phi_{03}(\nu_2\frac{\partial g_3}{\partial y_1} - \nu_1\frac{\partial g_3}{\partial y_2}),$$

$$\text{(4.14)} \quad \Psi_1\nu_3 - \Psi_3\nu_1 = \Phi_{01}(\nu_1\frac{\partial g_1}{\partial y_3} - \nu_3\frac{\partial g_1}{\partial y_1}) + \Phi_{02}(\nu_1\frac{\partial g_2}{\partial y_3} - \nu_3\frac{\partial g_2}{\partial y_1}) + \Phi_{03}(\nu_1\frac{\partial g_3}{\partial y_3} - \nu_3\frac{\partial g_3}{\partial y_1}),$$

$$\text{(4.15)} \quad \Psi_3\nu_2 - \Psi_2\nu_3 = \Phi_{01}(\nu_3\frac{\partial g_1}{\partial y_2} - \nu_2\frac{\partial g_1}{\partial y_3}) + \Phi_{02}(\nu_3\frac{\partial g_2}{\partial y_2} - \nu_2\frac{\partial g_2}{\partial y_3}) + \Phi_{03}(\nu_3\frac{\partial g_3}{\partial y_2} - \nu_2\frac{\partial g_3}{\partial y_3}).$$

For the domain derivative of solution of poisson equations, we can learn a lot of things in Murat-Simon [15,16]. For later convenience we define the vector field $\boldsymbol{\psi}_0$ by

$$\boldsymbol{\psi}_0 = -(\frac{\partial g}{\partial y})^t\boldsymbol{\Phi}_0 \quad \text{in} \quad \Omega.$$

Using $\boldsymbol{\psi}_0$, the boundary condition for $\boldsymbol{\Psi}$ (i.e. (4.13),(4.14),(4.15)) is equivalently written by

$$\text{(4.16)} \qquad \boldsymbol{\Psi} \times \nu = \boldsymbol{\psi}_0 \times \nu \quad \text{on} \quad \partial\Omega.$$

We multiply both sides of the equation (4.11) by Φ_{0i} and sum for $i = 1,2,3$.

$$\sum_{i=1}^{3}\int_\Omega\left\{\Phi_{0i}\Delta\Psi_i + \Phi_{0i}\operatorname{div}((\operatorname{div} g)\nabla\Phi_{0i}) - \Phi_{0i}\operatorname{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t))\right\}dy$$
$$+\sum_{i=1}^{3}\int_\Omega(\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\operatorname{div} g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i})dy = 0.$$

Denote the left hand side by J. Substitute $\Delta\boldsymbol{\Psi} = -\mathrm{rot}\,\mathrm{rot}\boldsymbol{\Psi} + \nabla\,\mathrm{div}\,\boldsymbol{\Psi}$ into J with (4.12) and integrate by parts, we get

$$
\begin{aligned}
J = &\int_\Omega \langle \boldsymbol{\Phi}_0, \nabla(\sum_{i,\ell=1}^3 \frac{\partial \Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i})\rangle dy + \int_{\partial\Omega} \langle \boldsymbol{\Phi}_0, (-\nu)\times \mathrm{rot}\,\boldsymbol{\Psi}\rangle dS - \int_\Omega \langle \mathrm{rot}\,\boldsymbol{\Phi}_0, \mathrm{rot}\,\boldsymbol{\Psi}\rangle dy \\
&+ \sum_{i=1}^3 \int_\Omega \{\Phi_{0i}\mathrm{div}((\mathrm{div} g)\nabla\Phi_{0i}) - \Phi_{0i}\mathrm{div}(\nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t))\} dy \\
&+ \sum_{i=1}^3 \int_\Omega (\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div} g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i}) dy \\
= &\int_{\partial\Omega} \langle \boldsymbol{\Phi}_0, \nu\rangle (\sum_{i,\ell=1}^3 \frac{\partial \Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i}) dS - \int_\Omega (\mathrm{div}\,\boldsymbol{\Phi}_0)(\sum_{i,\ell=1}^3 \frac{\partial \Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i}) dy \\
&- \int_{\partial\Omega} \langle \mathrm{rot}\,\boldsymbol{\Psi}, \boldsymbol{\Phi}_0\times\nu\rangle dS - \int_{\partial\Omega} \langle \mathrm{rot}\,\boldsymbol{\Phi}_0, \nu\times\boldsymbol{\Psi}\rangle dS - \int_\Omega \langle \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\Phi}_0, \boldsymbol{\Psi}\rangle dy \\
&+ \sum_{i=1}^3 \int_{\partial\Omega} \Phi_{0i}(\mathrm{div} g)\langle \nu, \nabla\Phi_{0i}\rangle dS - \sum_{i=1}^3 \int_\Omega (\mathrm{div} g)|\nabla\Phi_{0i}|^2 dy \\
&- \sum_{i=1}^3 \int_{\partial\Omega} \Phi_{0i}\langle \nu, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dS + \sum_{i=1}^3 \int_\Omega \langle \nabla\Phi_{0i}, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dy \\
&+ \sum_{i=1}^3 \int_\Omega (\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div} g)\Phi_{0i}^2 + \lambda(0)\Psi_i\Phi_{0i}) dy
\end{aligned}
$$

Using $\boldsymbol{\Phi}_0 \times \nu = \mathbf{0}$ on $\partial\Omega$ and $\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\Phi}_0 - \lambda(0)\,\boldsymbol{\Phi}_0 = \mathbf{0}$ and $\mathrm{div}\,\boldsymbol{\Phi}_0 = 0$ in Ω, we can simplify this expression and get

$$
\begin{aligned}
J = &\int_{\partial\Omega} \langle \boldsymbol{\Phi}_0, \nu\rangle (\sum_{i,\ell=1}^3 \frac{\partial \Phi_{0i}}{\partial y_\ell}\frac{\partial g_\ell}{\partial y_i}) dS - \int_{\partial\Omega} \langle \mathrm{rot}\,\boldsymbol{\Phi}_0, \nu\times\boldsymbol{\Psi}\rangle dS \\
&+ \sum_{i=1}^3 \int_{\partial\Omega} \Phi_{0i}(\mathrm{div} g)\langle \nu, \nabla\Phi_{0i}\rangle dS - \sum_{i=1}^3 \int_\Omega (\mathrm{div} g)|\nabla\Phi_{0i}|^2 dy \\
&- \sum_{i=1}^3 \int_{\partial\Omega} \Phi_{0i}\langle \nu, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dS + \sum_{i=1}^3 \int_\Omega \langle \nabla\Phi_{0i}, \nabla\Phi_{0i}(\frac{\partial g}{\partial y} + (\frac{\partial g}{\partial y})^t)\rangle dy \\
&+ \sum_{i=1}^3 \int_\Omega (\boldsymbol{\kappa}\Phi_{0i}^2 + \lambda(0)(\mathrm{div} g)\Phi_{0i}^2) dy
\end{aligned}
$$

$$
\begin{aligned}
J = &- \int_{\partial\Omega} A\, dS + \int_{\partial\Omega} B\, dS - \sum_{i=1}^3 \int_{\partial\Omega} \langle g, \nu\rangle |\nabla\Phi_{0i}|^2 dS + 2\sum_{i=1}^3 \int_{\partial\Omega} \langle g, \nabla\Phi_{0i}\rangle\langle \nu, \nabla\Phi_{0i}\rangle dS \\
&+ \lambda(0)\sum_{i=1}^3 \int_{\partial\Omega} \langle g, \nu\rangle \Phi_{0i}^2 dS + \boldsymbol{\kappa}\sum_{i=1}^3 \int_\Omega \Phi_{0i}^2\, dy
\end{aligned}
$$

Here A, B are given as follows. Note that the expression of $\nu \times \Psi$ is substituted.

$$A = \langle \mathrm{rot}\, \mathbf{\Phi}_0, \nu \times \mathbf{\Psi} \rangle$$

$$= (\frac{\partial \Phi_{03}}{\partial y_2} - \frac{\partial \Phi_{02}}{\partial y_3})[\Phi_{01}(\nu_3 \frac{\partial g_1}{\partial y_2} - \nu_2 \frac{\partial g_1}{\partial y_3}) + \Phi_{02}(\nu_3 \frac{\partial g_2}{\partial y_2} - \nu_2 \frac{\partial g_2}{\partial y_3}) + \Phi_{03}(\nu_3 \frac{\partial g_3}{\partial y_2} - \nu_2 \frac{\partial g_3}{\partial y_3})]$$
$$+(\frac{\partial \Phi_{01}}{\partial y_3} - \frac{\partial \Phi_3}{\partial y_1})[\Phi_{01}(\nu_1 \frac{\partial g_1}{\partial y_3} - \nu_3 \frac{\partial g_1}{\partial y_1}) + \Phi_{02}(\nu_1 \frac{\partial g_2}{\partial y_3} - \nu_3 \frac{\partial g_2}{\partial y_1}) + \Phi_{03}(\nu_1 \frac{\partial g_3}{\partial y_3} - \nu_3 \frac{\partial g_3}{\partial y_1})]$$
$$+(\frac{\partial \Phi_{02}}{\partial y_1} - \frac{\partial \Phi_1}{\partial y_2})[\Phi_{01}(\nu_2 \frac{\partial g_1}{\partial y_1} - \nu_1 \frac{\partial g_1}{\partial y_2}) + \Phi_{02}(\nu_2 \frac{\partial g_2}{\partial y_1} - \nu_1 \frac{\partial g_2}{\partial y_2}) + \Phi_{03}(\nu_2 \frac{\partial g_3}{\partial y_1} - \nu_1 \frac{\partial g_3}{\partial y_2})]$$

$$B = \langle \mathbf{\Phi}_0, \nu \rangle \sum_{i,\ell=1}^{3} \frac{\partial \Phi_{0i}}{\partial y_\ell} \frac{\partial g_\ell}{\partial y_i} + \sum_{i=1}^{3} (\mathrm{div} g) \Phi_{0i} \frac{\partial \Phi_{0i}}{\partial \nu} - \sum_{i,j,\ell=1}^{3} \nu_\ell (\frac{\partial g_\ell}{\partial y_j} + \frac{\partial g_j}{\partial y_\ell}) \Phi_{0i} \frac{\partial \Phi_{0i}}{\partial y_j}$$

We mention some useful property for the boundary condition of $\mathrm{rot}\, \mathbf{\Phi}_0$.

Lemma 4.5. We have $\langle \mathrm{rot}\, \mathbf{\Phi}_0, \nu \rangle = 0$ on $\partial\Omega$.

(Proof) From the direct calculation near $\partial\Omega$, the boundary condition $\mathbf{\Phi}_0 \times \nu = \mathbf{0}$ on $\partial\Omega$ gives this property of $\mathrm{rot}\, \mathbf{\Phi}_0$. $\square$

[Evaluation of A, B]

We see the values A and B in terms of Ω, $\mathbf{\Phi}_0$, ρ. For that purpose, we take an arbitray point of $\partial\Omega$ and a special coordinate around the point to calculate A and B. Take any point $O \in \partial\Omega$ and take the orthogonal coordinate $y = (y_1, y_2, y_3)$ centered at O such that $\nu(O) = (1, 0, 0)$. We express $\partial\Omega$ by a graph $y_1 = h(y_2, y_3)$ near O. There exists a $\delta > 0$ and C^2 function such that

$$\Omega \cap U(O, \delta) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid |y| < \delta,\ y_1 < h(y_2, y_3)\}.$$

It holds that $(\partial h/\partial y_2)(0,0) = 0$, $(\partial h/\partial y_2)(0,0) = 0$. We can assume that two vectors $(0,1,0)$ and $(0,0,1)$ are principal directions in the tangent space of $\partial\Omega$ at O. In this case

$$\frac{\partial \nu}{\partial y_2}(O) = \alpha\,(0,1,0), \quad \frac{\partial \nu}{\partial y_3}(O) = \beta\,(0,0,1),$$

where α and β are the principal curvatures of $\partial\Omega$ at O. Put $\phi(y) = \langle \mathbf{\Phi}_0(y), \nu(y) \rangle$ for $y \in \partial\Omega$ for simplicity.

We note that

$$\nu_1(O) = 1,\ \nu_2(O) = 0,\ \nu_3(O) = 0,\ \Phi_{01}(O) = \langle \mathbf{\Phi}_0(O), \nu(O) \rangle,\ \Phi_{02}(O) = 0,\ \Phi_{03}(O) = 0,$$

$$\frac{\partial g_1}{\partial y_1}(O) = 0,\ \frac{\partial g_2}{\partial y_1}(O) = 0,\ \frac{\partial g_3}{\partial y_1}(O) = 0,\ \frac{\partial g_1}{\partial y_2}(O) = \frac{\partial \rho}{\partial y_2}(O),$$
$$\frac{\partial g_1}{\partial y_3}(O) = \frac{\partial \rho}{\partial y_3}(O),\ \frac{\partial g_2}{\partial y_2}(O) = \rho(O) \frac{\partial \nu_2}{\partial y_2}(O) = \rho(O)\alpha,$$
$$\frac{\partial g_2}{\partial y_3}(O) = \frac{\partial g_3}{\partial y_2}(O) = 0,\ \frac{\partial g_3}{\partial y_3}(O) = \rho(O) \frac{\partial \nu_3}{\partial y_3}(O) = \rho(O)\,\beta,$$

From the condition $\mathbf{\Phi}_0 \times \nu = 0$ on the boundary, we have

$$\Phi_{0i}(\xi)\nu_j(\xi) - \Phi_{0j}(\xi)\nu_i(\xi) = 0 \quad (\xi \in \partial\Omega, 1 \leqq i, j \leqq 3).$$

We can operate $\partial/\partial y_2$, $\partial/\partial y_3$ (tangential derivative) on the above equations at O and get the following properties,

$$\frac{\partial \Phi_{02}}{\partial y_2}(O) = \alpha\,\Phi_{01}(O), \quad \frac{\partial \Phi_{03}}{\partial y_3}(O) = \beta\,\Phi_{01}(O), \quad \frac{\partial \Phi_{01}}{\partial y_1}(O) = -(\alpha+\beta)\Phi_{01}(O),$$
$$\frac{\partial \Phi_{02}}{\partial y_3}(O) = \frac{\partial \Phi_{03}}{\partial y_2}(O) = 0.$$

Substituting these quantities into A and B, we have

$$A(O) = (\frac{\partial \Phi_{01}}{\partial y_3} - \frac{\partial \Phi_{03}}{\partial y_1})\phi(O)\frac{\partial \rho}{\partial y_3}(O) - (\frac{\partial \Phi_{02}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_2})\phi(O)\frac{\partial \rho}{\partial y_2}(O) = \langle \mathrm{rot}\,\mathbf{\Phi}_0 \times \nabla\rho, \nu\rangle\langle \mathbf{\Phi}_0, \nu\rangle$$

$$\begin{aligned} B(O) &= \frac{\partial \rho}{\partial y_2}(O)\phi(O)(\frac{\partial \Phi_{02}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_2}) + \frac{\partial \rho}{\partial y_3}(O)\phi(O)(\frac{\partial \Phi_{03}}{\partial y_1} - \frac{\partial \Phi_{01}}{\partial y_3}) \\ &\quad + \alpha^2\phi(O)^2\rho(O) + \beta^2\phi(O)^2\rho(O) - \rho(O)\phi(O)^2(\alpha+\beta)^2 \\ &= \phi(O)\langle \nabla\rho \times \mathrm{rot}\,\mathbf{\Phi}_0, \nu\rangle - 2K(O)\rho(O)\phi(O)^2 \end{aligned}$$

Note that $K(O) = \alpha\,\beta$ is the Gaussian curvature of $\partial\Omega$ at O.

Summing up these quantities $A(O), B(O)$ and put them into $J = 0$ (recall that $\mathbf{\Phi}_0(y) = \mathbf{\Phi}_0^{(k)}(y)$), we get

$$(4.17) \quad \boldsymbol{\kappa} \int_\Omega |\mathbf{\Phi}_0^{(k)}|^2\,dx = \int_{\partial\Omega} \left(|\nabla \mathbf{\Phi}_0^{(k)}|^2 - 2|\frac{\partial \mathbf{\Phi}_0^{(k)}}{\partial \nu}|^2 + (2K(x) - \lambda_k(0))|\mathbf{\Phi}_0^{(k)}(x)|^2 \right) \rho\,dS$$
$$+2\int_{\partial\Omega} \langle \mathbf{\Phi}_0^{(k)}, \nu\rangle\langle \mathrm{rot}\,\mathbf{\Phi}_0^{(k)} \times \nabla\rho, \nu\rangle\; dS$$

Thus we have obtained the candidate of $(d\lambda_k(\epsilon)/d\epsilon)(0)$ which is the value $\boldsymbol{\kappa}$.

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