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Convergence results in order-preserving dynamical systems and applications to a molecular motor system

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1 Introduction

In this note I will investigate reaction-diffusion equations that satisfy the comparison principle and possess a mass conservation property.

Motivated from mathematical analysis of transport models by molecular motors and chemical reversible reaction models, recently we have obtained some fundamental results on the structure of stationary and time periodic solutions in a rather general framework of order-preserving dynamical systems ([12]). More precisely, our general results state that:

(1) if there exists at least one fixed point (which corresponds to a stationary or a time periodic solution of the model equation), then there exist infinitely many of them, and the set of all the fixed points is totally ordered, connected and unbounded;

(2) any bounded orbit converges to some element of this continua of fixed points as time tends to infinity.

In particular, our general results imply that if the model equation possesses a trivial stationary or time periodic solution (such as zero), then there are automatically infinitely many nontrivial stationary or time periodic solutions.

Results on the existence of stationary (or time periodic) solutions and the convergence to stationary (or time periodic) solutions for the above mentioned molecular motor models and chemical reversible reaction models have been already somewhat known, though our theorems give an exceedingly simple proof. Furthermore, we do not need specific assumptions (such as the existence of a Lyapunov function or analyticity, and so on), which makes our theorems applicable to a wide range of problems.

This is joint work with Hiroshi Matano (University of Tokyo) and Danielle Hilhorst (CNRS and Université de Paris Sud).

2 Basic concepts and results

Let $(X, d, \leq)$ be an ordered metric space, that is, a complete metric space with partial order relation $\leq$ which is closed under the limiting procedure:

$$u_n \leq v_n \ (n = 1, 2, \ldots), \quad u_n \to u_\infty, \quad v_n \to v_\infty \quad \Longrightarrow \quad u_\infty \leq v_\infty.$$

For u, $v \in X$, we write

$$u < v \qquad \text{if} \quad u \leq v \ \text{ and } \ u \neq v$$

and let $[u, v]$ denote the order interval $\{w \in X \mid u \leq w \leq v\}$.

We assume that, for any $u \in X$ and any $\delta > 0$, there exists some v satisfying $u < v$ and $d(u, v) < \delta$.

We also assume that any pair of points u, $v \in X$ has the least upper bound $u \vee v$, namely the minimal element of the set $\{w \in X \mid u \leq w, v \leq w\}$. We further assume that the map $(u, v) \mapsto u \vee v \colon X \times X \to X$ is continuous.

Let F be a compact map from X to X, that is, F is a continuous map that maps any bounded set into a relatively compact set. We assume that F is order-compact, namely for any ordered pair $u < v \in X$ the image of the order interval $[u, v]$ by F is relatively compact. We also assume that

(F1) F is order-preserving, namely, $u \leq v$ implies $F(u) \leq F(v)$;

(F2) $F(u \vee v) > F(u) \vee F(v)$ if $u \not\leq v$ and $u \not\geq v$.

Let $M \colon X \to \mathbb{R}$ be a continuous map satisfying

(M1) $u < v$ implies $M(u) < M(v)$;

(M2) $M(F(u)) = M(u)$ for $u \in X$.

As we will describe in Section 4, in the application to reaction-diffusion equations, condition (F1) corresponds to the comparison principle, and combination of (F1) with (F2) are slightly stronger version of the comparison principle which is weaker than the strong comparison principle (the strong maximum principle). We also note that condition (M2) is fulfilled if the equation under consideration has a mass conservation property and (M1) is the assumption that the conserved quantity $M(u)$ is monotone in u.

We obtain the following theorems:

Theorem 1. Let E denote the set of all the fixed points of F. If $E \neq \emptyset$, then E is a totally ordered and connected set. Furthermore, E is unbounded from above, that is, E has no upper bound.

Theorem 2. Any bounded orbit $F^n(u)$ converges to some fixed point of F as $n \to \infty$.

The following is an immediate consequence of Theorem 2.

Corollary 3. For any integer $m \geq 2$, let E and E_m denote the set of all the fixed points of F and F^m, respectively. Then, $E = E_m$.

The above corollary states that F possesses no periodic points other than fixed points. Such a statement is not necessarily true for general order-preserving maps. It is a remarkable feature of an order-preserving map satisfying the conservation law.

3 Proof of the theorems

In this section, we prove the theorems. Since the space is limited, we only present an outline of the proofs. We refer to the forthcoming paper [12] for more details.

First we prove Corollary 3 as a consequence of Theorem 2.

Proof of Corollary 3. We only prove that $E_m \subset E$ since the opposite inclusion is obvious. Let $\overline{u} \in E_m$. Then we have $F^m(\overline{u}) = \overline{u}$. This shows that the orbit

$$\{F^n(\overline{u}) \mid n \in \mathbb{N}\} = \{\overline{u},\, F(\overline{u}),\, \ldots,\, F^{m-1}(\overline{u})\}$$

is bounded. Therefore, applying Theorem 2, we see that $F^n(\overline{u})$ converges to some fixed point $\overline{v}$ of F as $n \to \infty$. Thus we have

$$F^n(\overline{u}) = \overline{v}, \qquad n = 0, 1, \ldots, m-1$$

and hence $\overline{u} = \overline{v} \in E$. The proof is completed. □

Next we prove Theorem 2.

Proof of Theorem 2. Let u be an element of X such that $\{F^n(u)\}_{n=1,2,\ldots}$ is bounded. Then, as is well-known, since F is a compact map, the omega-limit set $\omega(u)$ of u defined by

$$\omega(u) = \bigcap_{n=1}^{\infty} \{F^k(u) \mid k \geq n\}$$

is a nonempty compact set. Therefore, by Lemma 1 below, there exists some $\overline{u} \in X$ such that $\omega(u) = \{\overline{u}\}$, which shows that $F^n(u)$ converges to a fixed point $\overline{u}$ of F as $n \to \infty$. □

Lemma 1. If an omega limit set $\omega(u)$ is not empty, then $\omega(u) = \{\overline{u}\}$ for some $\overline{u} \in X$.

Proof. Suppose that $\omega(u)$ is not a singleton. Define a continuous map $G\colon X \times X \to \mathbb{R}$ by

$$G(v, w) = M(v \vee w).$$

Since $\omega(u)$ is a compact set, the restriction of G on the set $\omega(u) \times \omega(u)$ attains its maximum value at some point, which we denote by (v_1, w_1). By $F(\omega(u)) = \omega(u)$, there exists $v_0, w_0 \in \omega(u)$ satisfying $F(v_0) = v_1$, $F(w_0) = w_1$, respectively.

Note that (M2) implies

$$M(F^n(u)) = M(u) \qquad \text{for all } n \in \mathbb{N}$$

and therefore

$$M(w) = M(u) \qquad \text{for all } w \in \omega(u).$$

From this and (M1) we see that any two points $w, w' \in \omega(u)$ are non-ordered, namely, $w \not> w'$ and $w \not< w'$. Hence $v_0 \not< w_0$ and $v_0 \not> w_0$.

If $v_0 = w_0$, then $v_1 = w_1$ and we have

$$v_1 \vee w_1 = v_1 \vee v_1 = v_1 < v_1 \vee u_1$$

for $u_1 \in \omega(u)$ satisfying $u_1 \neq v_1$. However, by (M1), $v_1 \vee w_1 < v_1 \vee u_1$ implies

$$G(v_1, u_1) > G(v_1, w_1) = \max\{G(v, w) \mid v, w \in \omega(u)\}$$

and we are lead to a contradiction. Thus $v_0 \not\leq w_0$ and $v_0 \not\geq w_0$ hold.

Therefore by (F2) we have

$$F(v_0 \vee w_0) \; > \; F(v_0) \vee F(w_0) = v_1 \vee w_1$$

and hence

$$G(v_0, w_0) = M(v_0 \vee w_0) = M(F(v_0 \vee w_0)) \; > \; M(v_1 \vee w_1) = G(v_1, w_1),$$

which again contradicts the definition of v_1, w_1. The proof is completed. □

Before proving Theorem 1, we define the notion of stability from above of fixed points of F.

A fixed point $\overline{u}$ of F is called *stable from above* if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$d(u, \overline{u}) < \delta, \;\; u > \overline{u} \quad \Longrightarrow \quad d(F^n(u), \overline{u}) < \varepsilon \;\; (n = 1, 2, \ldots).$$

It is called *asymptotically stable from above* if it is stable from above and if there exists some $\delta > 0$ such that

$$d(u, \overline{u}) < \delta, \ \ u > \overline{u} \quad \Longrightarrow \quad \lim_{n \to \infty} F^n(u) = \overline{u}.$$

Proof of Theorem 1. By Lemma 2 below E is totally ordered. Furthermore Lemma 3 shows that any fixed point $\overline{u}$ of F is stable from above and is not asymptotically stable from above.

Now we show that E is unbounded from above. Assume the contrary. Then E has an upper bound, which we denote by u^+. Fix $u_0 \in E$ arbitrarily and set

$$A = [u_0, u^+] \cap E.$$

Since $F(A) = A$ and since F is order-compact, A is a compact subset of E. Therefore, by Lemma 4, A possesses the maximal element, which is also the maximal element of E. This contradicts Lemma 5. Thus E is unbounded from above.

Finally we show that E is connected. Suppose that E is not connected. Then there exist open subsets O_1, O_2 in the relative topology of E such that

$$O_1, O_2 \neq \emptyset, \qquad O_1 \cap O_2 = \emptyset, \qquad O_1 \cup O_2 = E.$$

Take $u_1 \in O_1$ and $u_2 \in O_2$. Since E is totally ordered, without loss of generality we may assume that $u_1 < u_2$. Put

$$B = [u_1, u_2] \cap O_1.$$

Clearly $u_1 \in B$, $u_2 \notin B$ and B is a totally ordered set in X. Furthermore, since $B = ([u_1, u_2] \cap E) \setminus O_2$ and since $F([u_1, u_2] \cap E) = [u_1, u_2] \cap E$, B is compact. Hence, by Lemma 4, the maximal element of B, denoted by $\max B$, exists. Clearly $u_1 \leq \max B < u_2$ since $u_1 \in B$ and $u_2 \notin B$. By Lemma 5, there exists some convergent sequence $\overline{v}_k \in O_1 \to \max B$ satisfying $\max B < \overline{v}_k$. The inequality $\max B < \overline{v}_k$ implies $\overline{v}_k \notin B$. Therefore, $\overline{v}_k$ cannot belong to $[u_1, u_2]$. Since E is totally ordered, this implies $u_2 < \overline{v}_k$. Letting $k \to \infty$ yields $u_2 \leq \max B$. This contradicts the fact that $\max B < u_2$. Thus E is connected. $\square$

Lemma 2. Let $\overline{u}_1$, $\overline{u}_2$ be fixed points of F satisfying $\overline{u}_1 \neq \overline{u}_2$. Then either $\overline{u}_1 < \overline{u}_2$ or $\overline{u}_1 > \overline{u}_2$.

Lemma 3. Any fixed point $\overline{u}$ of F is stable from above and not asymptotically stable from above.

Lemma 4. Let A be a totally ordered and compact subset of X. Then A has the maximal element.

Lemma 5. For any $\overline{u} \in E$ and any $\delta > 0$ there exists some $\overline{v} \in E$ satisfying $\overline{u} < \overline{v}$ and $d(\overline{u}, \overline{v}) < \delta$.

Since the space is limited, we omit the proof of Lemmas 2-5. See the forthcoming paper [12] for details.

4 Applications

In this section we apply our general theory to reaction-diffusion equations and study the existence of stationary (or time periodic) solutions and the asymptotic behavior of solutions.

4.1 General strategy

In this subsection, we consider partial differential equations in a rather general setting to explain how our theory is applied. Let X be an ordered metric space satisfying the conditions in Section 2. First we consider an initial value problem for an abstract evolution equation on X of the form:

$$\begin{cases} \dfrac{du}{dt} = A(u), & t > 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where A is a map from some subset of X to X.

We assume that (1) is well-posed on X and defines a compact and order compact semiflow $\Phi = \{\Phi_t\}_{t \geq 0}$ on X, namely Φ is defined by

$$\Phi_t(u_0) = u(t; u_0) \qquad \text{for } u_0 \in X,\ t \geq 0,$$

where $u(t; u_0)$ denotes the solution of (1) with initial data $u(0) = u_0$, and the map $\Phi_t \colon X \to X$ is compact and order compact if $t > 0$. We also assume that there exists a continuous map $M \colon X \to \mathbb{R}$ satisfying condition (M1) in Section 2. We further assume that, for each $t > 0$, the maps $F = \Phi_t$ and M satisfy conditions (F1), (F2) and (M2).

Now, for an arbitrarily fixed $\tau > 0$, let $\overline{u} \in X$ be a fixed point of Φ_τ. We put $F = \Phi_{\tau/m}$, where m is an arbitrary positive integer. Then, $\overline{u}$ is a fixed point of $\Phi_\tau (= F^m)$ and therefore, by virtue of Corollary 3, it is also a fixed point of $\Phi_{\tau/m} (= F)$. Hence it is a fixed point of $\Phi_{l\tau/m}$ for all $l, m \in \mathbb{N}$. In other words, $\overline{u}$ is a fixed point of $\Phi_{q\tau}$ for any rational number $q > 0$. By continuity, this implies $\Phi_t(\overline{u}) = \overline{u}$ for all $t \geq 0$. Therefore $\overline{u}$ is a stationary solution of (1). Thus, Theorems 1 and 2 imply the following:

Theorem 4. (autonomous case)

(i) If there exists at least one stationary solution for (1), then (1) possesses infinitely many stationary solutions and the set of stationary solutions is a totally ordered, unbounded and connected subset of X.

(ii) Any bounded solution of (1) converges to some stationary solution of (1) as $t \to \infty$.

By statement (i) of Theorem 4, if (1) possesses some trivial stationary solution, such as 0, then there exist nontrivial stationary solutions for (1). Particularly, in the case where X is a linear space and $A(u)$ is linear, since 0 is a trivial stationary solution, we obtain the existence of nontrivial stationary solutions for (1).

From statement (ii) of Theorem 4, we see that (1) does not possess a time periodic solution that is not stationary.

Next we apply our results to the time periodic problem. Let us consider the problem of the form:

$$\begin{cases} \dfrac{du}{dt} = A(u,t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{2}$$

where A is T-periodic in t for some $T > 0$.

We assume that (2) is well-posed on X and let F be the time T-map associated with (2), namely,

$$F(u_0) = u(T; u_0) \qquad \text{for } u_0 \in X,$$

where $u(t; u_0)$ denotes the solution of (2) with initial data $u_0 \in X$. We assume that all the assumptions in Section 2 is fulfilled. Then, Theorems 1 and 2 imply the following:

Theorem 5. (time periodic case)

(i) If there exists at least one T-periodic solution for (2), then (2) possesses infinitely many T-periodic solutions and the set of T-periodic solutions is a totally ordered, unbounded and connected subset of X.

(ii) Any bounded solution of (2) converges to some T-periodic solution of (2) as $t \to \infty$.

By statement (i) of Theorem 5, the existence of at least one trivial T-periodic solution implies the existence of nontrivial T-periodic solutions. Especially, in the case where X is a linear space and $A(u,t)$ is linear in u, since 0 is a trivial T-periodic solution for (2), we obtain the existence of nontrivial T-periodic solutions for (2).

From statement (ii) of Theorem 5, we see that (2) possesses no subharmonic solution; in other words, there exists no periodic solution whose minimal period is mT with $m \in \mathbb{N}$, $m \geq 2$.

4.2 Molecular motor model

First let us consider the following cooperative system, which comes from a model for intracellular transportation by molecular motors:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial}{\partial x}\left(\sigma_1 \dfrac{\partial u_1}{\partial x} + \psi_1'(x)u_1\right) - a_1(x)u_1 + a_2(x)u_2, & x \in (0,1),\ t > 0, \\ \dfrac{\partial u_2}{\partial t} = \dfrac{\partial}{\partial x}\left(\sigma_2 \dfrac{\partial u_2}{\partial x} + \psi_2'(x)u_2\right) + a_1(x)u_1 - a_2(x)u_2, & x \in (0,1),\ t > 0, \\ \sigma_i \dfrac{\partial u_i}{\partial x} + \psi_i'(x)u_i = 0, \qquad x = 0, 1,\ t > 0,\ i = 1, 2, & \end{cases} \tag{3}$$

where $\sigma_i > 0$ is a constant and $a_i(x) \geq 0, \not\equiv 0$, $\psi_i(x)$ are smooth functions. It is assumed that the molecular motor is two-headed and its state switches between state 1 and state 2. For each $t \geq 0$, $u_1(x,t)$ and $u_2(x,t)$ denote the probability density at position x. Thus one has $u_1(x,t), u_2(x,t) \geq 0$ and

$$\int_0^1 \big(u_1(x,t) + u_2(x,t)\big)dx = 1, \qquad t \geq 0. \tag{4}$$

Derivation of system (3) from a mass transport viewpoint is given in the paper [6] by Chipot, Kinderlehrer and Kowalczyk. For a mathematical analysis and for further references we refer to [4], [9], [10], [14] and [15]. In what follows, for convenience, we consider all nonnegative solutions of (3) without setting (4).

Set $X = (C([0,1])_+)^2$, where $C([0,1])_+$ denotes the set of nonnegative continuous functions on $[0,1]$. Then X is an ordered metric space endowed with metric induced by the uniform convergence topology and order relation defined by

$$u \leq v \qquad \text{if} \quad u_i(x) \leq v_i(x), \ \ x \in [0,1],\ i = 1, 2 \tag{5}$$

for $u = (u_1, u_2)$, $v = (v_1, v_2) \in X$. Note that the symbol $u < v$ then means that $u \leq v$ and that either $u_1(x_0) < v_1(x_0)$ or $u_2(x_0) < v_2(x_0)$ holds at some point $x_0 \in [0,1]$. The least upper bound $u \vee v$ of u, v is defined by

$$u \vee v\,(x) = \big(\max\{u_1(x), v_1(x)\},\ \max\{u_2(x), v_2(x)\}\big), \quad x \in [0,1].$$

By the standard a priori estimate it is known that (3) defines a compact semiflow on X, which we denote by $\{\Phi_t\}_{t\geq 0}$. Furthermore, it follows from the comparison principle and the strong maximum principle that, if

$$u_i(x,0) \leq v_i(x,0), \ \ x \in [0,1],\ i = 1, 2, \qquad u(x,0) \not\equiv v(x,0)$$

hold for solutions $u(x,t) = (u_1(x,t), u_2(x,t))$, $v(x,t) = (v_1(x,t), v_2(x,t))$ of (3), then

$$u_i(x,t) < v_i(x,t), \quad x \in [0,1],\ t > 0,\ i = 1, 2.$$

We define a continuous map $M\colon X \to \mathbb{R}$ by

$$M(u) = \int_0^1 (u_1(x) + u_2(x))dx \qquad \text{for } u = (u_1, u_2) \in X. \tag{6}$$

Since (3) is a linear problem, $0 = (0,0)$ is a stationary solution of (3) and therefore, by statement (i) of Theorem 4, there exists some non-zero stationary solution $\overline{u} = (\overline{u}_1(x), \overline{u}_2(x)) > 0$ of (3). Clearly $\lambda\overline{u}$ is also a stationary solution of (3) for any $\lambda > 0$ and from the strong maximum principle it follows that

$$\overline{u}_i(x) > 0, \qquad x \in [0,1],\ i = 1,2.$$

Let $u(x,t) = (u_1(x,t), u_2(x,t))$ be an arbitrary solution of (3) and put

$$\mu = \max\{u_i(x,0)/\overline{u}_i(x) \mid x \in [0,1],\ i = 1,2\}.$$

Then

$$u(x,0) \le \mu\overline{u}(x), \quad x \in [0,\infty)$$

and hence, by virtue of the comparison principle,

$$u(x,t) \le \mu\overline{u}(x), \quad x \in [0,\infty),\ t > 0,$$

which shows that $u(\cdot,t)$ is bounded from above by $\mu\overline{u}$ for all $t \ge 0$.
Thus Theorem 4 implies the following:

Proposition 1. (autonomous model)
(i) (3) possesses a unique (up to multiplication by positive constant) positive stationary solution $\overline{u}(x) = (\overline{u}_1(x), \overline{u}_2(x))$.
(ii) Any solution $u(x,t)$ of (3) converges to a stationary solution $\lambda\overline{u}(x)$ in $(C([0,1])_+)^2$ as $t \to \infty$, where a constant λ is determined by the initial data $u(\cdot,0)$ as $\lambda = M(u(\cdot,0))/M(\overline{u})$.

Next we apply our result to a time periodic model, that is, a flashing ratchet model, proposed by [11], [7], which is represented as Fokker-Plank equation with a time periodic potential $\psi(x,t)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\sigma\dfrac{\partial u}{\partial x} + \psi_x(x,t)\,u\right), & x \in (0,1),\ t > 0, \\ \sigma\dfrac{\partial u}{\partial x} + \psi_x(x,t)\,u = 0, & x = 0,1,\ t > 0, \end{cases} \tag{7}$$

where $\sigma > 0$ is a constant and $\psi(x,t)$ is a smooth function which is T-periodic in t for some $T > 0$. Here the molecular motors are represented by the probability density $u(x,t)$ and thus $u(x,t) \ge 0$ and

$$\int_0^1 u(x,t)\,dx = 1, \qquad t \ge 0.$$

In what follows, as for (3), we consider all nonnegative solutions of (7).

Now we set $X = C([0,1])_+$ endowed with metric induced by the uniform convergence topology and order relation defined by

$$u \le v \qquad \text{if} \qquad u(x) \le v(x),\; x \in [0,1]$$

for $u, v \in X$ and put

$$M(u) = \int_0^1 u(x)\,dx \qquad \text{for } u \in X.$$

Then, applying Theorem 5 we obtain the following:

Proposition 2. (time periodic model)

(i) (7) possesses a unique (up to multiplication by positive constant) positive time T-periodic solution $\overline{u}(x,t)$.

(ii) Any solution $u(x,t)$ of (7) converges to a time T-periodic solution $\lambda\overline{u}(x,t)$ in $C([0,1])_+$ as $t \to \infty$, where a constant λ is determined by the initial data as $\lambda = M(u(\cdot,0))/M(\overline{u}(\cdot,0))$.

The above proposition implies, in particular, that (7) possesses no subharmonic solution, that is, no periodic solution whose minimal period is mT with $m \in \mathbb{N}$, $m \ge 2$.

We remark that, we can relax the smoothness assumption on the coefficients of equations (3) and (7), by setting, for example, $X = L^2([0,1])_+$ or $X = (L^2([0,1])_+)^2$ instead of $C([0,1])_+$ or $(C([0,1])_+)^2$, where $L^2([0,1])_+$ denotes the set of square-integrable nonnegative functions on $[0,1]$.

We also remark that there are earlier related results concerning (3) and (7). The paper [4] deals with (3) and proves results that are basically the same as our Proposition 1 above. Their proof relies on the spectrum theory of compact linear operator. The paper [7], [16] deal with (7) and proves results that are basically the same as our Proposition 2 above. The proof in [7] relies on the entropy analysis. Furthermore, though the paper [16] proves its homogenization, it mentions briefly the existence and stability of time periodic solutions by Floquet theory. On the other hand, Propositions 1 and 2 follow immediately from a much more general result without relying on further information such as spectrum, entoropy and Floquet exponents. Therefore it is easy to extend the results in Propositions 1 and 2 to more general equations including nonlinear equations.

4.3 Reversible chemical reaction model

In this subsection we consider the following reaction-diffusion system which models a reversible chemical reaction between two mobile reactants A and B. See

[2] and [8] and references therein for details.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 - \alpha \left(r_A(u_1) - r_B(u_2) \right), & x \in \Omega,\ t > 0 \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + \beta \left(r_A(u_1) - r_B(u_2) \right), & x \in \Omega,\ t > 0 \\ \dfrac{\partial u_i}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\ i = 1, 2, \end{cases} \tag{8}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν is the outward normal at each point of $\partial\Omega$, $d_1, d_2, \alpha, \beta > 0$ are constants and $r_A(u)$, $r_B(u)$ are strictly increasing functions satisfying $r_A(0), r_B(0) = 0$. Here, for each $t \geq 0$, $u_1(x,t), u_2(x,t) \geq 0$ represents concentration of A, B at $x \in \overline{\Omega}$, respectively.

Now we set $X = (C(\overline{\Omega})_+)^2$ and define the metric induced by the uniform convergence topology and order relation by (5) replaced $[0,1]$ by $\overline{\Omega}$. We further put

$$M(u) = \int_\Omega \big(u_1(x)/\alpha + u_2(x)/\beta\big)dx \qquad \text{for } u = (u_1, u_2) \in X.$$

Take $v_0 = (a_0, b_0) \in [0,\infty)^2$ arbitrarily and let $v(t) = (v_1(t), v_2(t))$ denote a solution of (8) satisfying $v(0) = (a_0, b_0)$. Then, since condition (M1) in Section 2 holds, we have

$$v(t) \in \{(a,b) \in [0,\infty)^2 \mid a/\alpha + b/\beta = a_0/\alpha + b_0/\beta\}, \quad t > 0,$$

which shows that a solution whose initial value is a constant function is bounded. Furthermore, for any $u_0 \in X$, if we we choose $v_0 \in [0,\infty)^2$ satisfying

$$u_0 \leq v_0,$$

then the comparison principle implies

$$u(\cdot, t) \leq v(t) \qquad t > 0,$$

where $u(x,t)$, $v(t)$ is a solution of (8) with initial data $u(\cdot, 0) = u_0$, $v(0) = v_0$, respectively. This shows that any solution of (8) is bounded.

Denote by E the set of stationary solutions of (8). Clearly

$$\{(a,b) \in [0,\infty)^2 \mid r_A(a) = r_B(b)\} \subset E.$$

Applying Theorem 4, we obtain the following:

Proposition 3. Let E denote the set of all the stationary solutions of (8). Then,

(i) $E = \{(a,b) \in [0,\infty)^2 \mid r_A(a) = r_B(b)\}$.

(ii) Any solution $u(x,t) = \big(u_1(x,t), u_2(x,t)\big)$ of (8) converges to some stationary solution $(\overline{a}, \overline{b}) \in E$ in $(C(\overline{\Omega})_+)^2$ as $t \to \infty$, which is determined by the initial data as $M((u_1(\cdot,0), u_2(\cdot,0)) = |\Omega|(\overline{a}/\alpha + \overline{b}/\beta)$.

In [2], Bothe and Hilhorst studied (8) and proved the convergence of solutions as reaction rates α, β tend to infinity. The limiting problem is given by a single diffusion equation with nonlinear diffusion. They also described the asymptotic behavior of solutions as $t \to \infty$ by using the existence of Lyapunov function (entoropy).

On the other hand, our method is applicable to more general problems. For example, we can consider the case where functions r_A, r_B depend on x, t and they are T-periodic in t. In this case, applying Theorem 5 we can prove the existence of time T-periodic solutions and convergence to time T-periodic solutions.

4.4 Cooperative reaction-diffusion system

The last example is a cooperative reaction-diffusion system, whose special cases include (3) and (8). Now we consider the cooperative system of the form

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \operatorname{div}(\sigma_i \nabla u_i + u_i \nabla \psi_i) + \alpha_i \displaystyle\sum_{j=1}^{N} \lambda_{ij}\, r_j(u_j, x), & x \in \Omega,\ t > 0,\ i = 1, \ldots, N, \\ \sigma_i \dfrac{\partial u_i}{\partial \nu} + u_i \dfrac{\partial \psi_i}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\ i = 1, \ldots, N, \end{cases} \tag{9}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν is the outward normal at each point of $\partial\Omega$, $\sigma_i > 0$ and $\alpha_i > 0$ are constants and λ_{ij} is a constant such that

$$\lambda_{ii} \leq 0, \qquad \lambda_{ij} \geq 0 \quad \text{if } i \neq j, \qquad \sum_{i=1}^{N} \lambda_{ij} = 0$$

and that a matrix (λ_{ij}) is irreducible. We assume that functions $\psi_i(x)$, $r_i(u,x)$ are smooth and $r_i(u,x)$ is nondecreasing in u and satisfies $r_i(0,x) = 0$. As is the case of previous examples, we consider nonnegative solutions for (9).

Now we set $X = (C(\overline{\Omega})_+)^N$ associated with order relation

$$u \leq v \qquad \text{if} \quad u_i(x) \leq v_i(x), \ \ x \in \overline{\Omega},\ i = 1, \ldots, N$$

for $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N) \in X$. Put

$$M(u) = \int_\Omega \sum_{i=1}^{N} u_i(x)/\alpha_i \, dx \qquad \text{for } u = (u_1, \ldots, u_N) \in X.$$

Note that $0 = (0, \dots, 0)$ is a stationary solution of (9). Therefore Theorem 4 implies the following:

Proposition 4.

(i) The set of stationary solutions of (9) is a nonempty, totally ordered, unbounded connected subset of $(C(\overline{\Omega})_+)^N$.

(ii) Any bounded solution of (9) converges to some stationary solution in $(C(\overline{\Omega})_+)^N$ as $t \to \infty$.

In [5], Chipot, Hilhorst, Kinderlehrer and Olech proved L^1-contraction property for solutions of (9). They then proved the existence of stationary solutions for the case where (9) is linear, especially the case where $r_i(u, x) \equiv u$. Our theorems are also applicable to nonlinear problems.

Finally let us mention that, similarly to (8), applying Theorem 5 we can consider the case where problem (9) is time periodic, more precisely, the case where constants σ_i, λ_{ij} and functions ψ_i, r_i depend on t and they are T-periodic in t. In this case, our Theorem 5 immediately yields the existence of time T-periodic solutions and convergence to time T-periodic solutions.

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