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A discussion of nonnegative solutions of elliptic equations on symmetric domains*

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Abstract. In this note we summarize our recent results on nonnegative solutions of nonlinear elliptic equations on reflectionally symmetric domains. We discuss symmetry properties of such solutions, the structure of their nodal set, and the existence and multiplicity of solutions with a nontrivial nodal set.

Contents

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1 Introduction

Consider a nonlinear elliptic problem of the form

$$F(x,u,Du,D^2u) = 0, \quad x \in \Omega, \tag{1.1}$$
$$u = 0, \quad x \in \partial\Omega. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, which is reflectionally symmetric about the hyperplane

$$H_0 = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 = 0\}$$

and convex in the direction $e_1 := (1, 0, \ldots, 0)$. The nonlinearity F is assumed to be sufficiently regular, elliptic, and symmetric, so that in particular the equation is invariant under the reflection in H_0 (see Section 2 for the precise hypotheses). For example, the semilinear problem

$$\Delta u + f(x', u) = 0, \quad x = (x_1, x') \in \Omega, \tag{1.3}$$
$$u = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where $f : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ is continuous in all variables and Lipschitz in u, is admissible for our results without any additional assumption on f.

By a celebrated theorem of Gidas, Ni, and Nirenberg [13], and its generalization to nonsmooth domains given by Berestycki and Nirenberg [3] (see also Dancer's result in [8]), each positive (classical) solution u of (1.1), (1.2) is even in x_1:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega), \tag{1.5}$$

and decreasing with increasing $|x_1|$:

$$u_{x_1}(x_1, x') < 0 \quad ((x_1, x') \in \Omega, x_1 > 0). \tag{1.6}$$

This result was proved using the method of moving hyperplanes introduced by Alexandrov [1] and further developed and applied in a symmetry problem by Serrin [25]. We refer the reader to the surveys [2, 16, 17, 18], the monographs [9, 12, 24], or the more recent paper [6], for perspectives on this theorem, related results, and many other references.

The above symmetry and monotonicity theorem is not valid in general if the solution u is assumed to be nonnegative, rather than strictly positive: consider, for example, the function $u(x) = 1 + \cos x$ as a solution of $u'' + u - 1 = 0$ on $\Omega = (-3\pi, 3\pi)$. Note, however, that in this example u still has several symmetry properties: it is even in x and, moreover, it is symmetric about the center of the interval between any two successive zeros. It is not hard to prove that a similar symmetry result is valid for the nonnegative solutions of any problem (1.3), (1.4) in one space dimension (in the one-dimensional case, $\Omega = (-\ell, \ell)$ for some $\ell > 0$, and there is no variable x').

It is natural to ask whether in higher dimension, nonnegative solutions also have some symmetry properties. One would also like to know how the nodal set of such solutions can look like and whether it has some symmetry itself. We address these problems in Section 3. The theorem we give there states, roughly speaking, that each nonnegative solutions u of (1.1), (1.2) has a similar symmetry structure as solutions in one dimension: it is symmetric about H_0 and, if $u \not\equiv 0$ and u is not strictly positive in Ω, the nodal set of u divides Ω into a finite number of reflectionally symmetric subdomains (nodal domains) in which u has the usual Gidas-Ni-Nirenberg symmetry and monotonicity properties.

Discussing nonnegative solutions with a nontrivial nodal set, we have an obligation to address the problem of existence of such solutions. Using the one-dimensional example mentioned above, it is not difficult to find such solutions for some problems on a rectangle. However, it is not at all a trivial matter to determine whether such solutions can be found on other domains and whether they can be found for more specific problems, like the spatially homogeneous semilinear equations. These issues are discussed in Section 4, where we summarize known examples of solutions with a nontrivial nodal set and mention several results on the nonexistence of such solutions under various additional conditions on the nonlinearity and/or the domain.

Our next concern is the multiplicity of nonnegative solutions with a nontrivial nodal set, in case such solutions do exist. For one-dimensional problems (1.3), (1.4), a phase-plane analysis reveals that if a solution has interior zeros, then it's derivative has to vanish at the boundary points, that is, such a solution satisfies simultaneously the Dirichlet and Neumann boundary conditions. The uniqueness for the Cauchy problem for the second order ODE therefore implies that the solution is uniquely determined. Surprisingly perhaps, a similar uniqueness result holds for a large class of domains, not necessarily smooth, in any dimension. For general domains, the number of solutions with interior zeros is finite. See Section 5, for a discussion of these issues.

2 Notation and hypotheses

In this section we state the hypotheses used throughout the paper. First recall that the *standing hypothesis* on $\Omega \subset \mathbb{R}^N$ is that it is a bounded domain, which is x_1-convex (or convex in the direction $e_1 = (1, 0, \ldots, 0)$) and symmetric about the hyperplane $H_0 = \{(x_1, x') \in \mathbb{R}^N : x_1 = 0\}$.

To formulate our hypotheses on the nonlinearity F, let $\mathcal{S}$ denote the space of $N \times N$ symmetric (real) matrices and $\mathcal{B} := \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}$. Let Q be the transformation on $\mathcal{B}$ defined by

$$Q(u, p, q) = (u, -p_1, p_2, \ldots, p_N, \bar{q}), \tag{2.1}$$
$$\bar{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i,\ j \text{ equals } 1, \\ q_{ij} & \text{otherwise.} \end{cases}$$

We assume that $F : (x, u, p, q) \mapsto F(x, u, p, q) : \bar{\Omega} \times \mathcal{B} \to \mathbb{R}$, satisfies the following conditions.

(F1) (Regularity) F is continuous on $\bar{\Omega} \times \mathcal{B}$ and Lipschitz in (u, p, q): there is $\beta_0 > 0$ such that

$$|F(x, u, p, q) - F(x, \tilde{u}, \tilde{p}, \tilde{q})| \leq \beta_0 |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})|$$
$$((x, u, p, q), (x, \tilde{u}, \tilde{p}, \tilde{q}) \in \bar{\Omega} \times \mathcal{B}). \tag{2.2}$$

Moreover, F is differentiable with respect to q on $\Omega \times \mathcal{B}$ and its derivatives $F_{q_{ij}}$, $i, j = 1, \ldots, N$, are Lipschitz (in all variables) on $\Omega \times \mathcal{B}$.

(F2) (Ellipticity) There is a constant $\alpha_0 > 0$ such that

$$F_{q_{ij}}(x, u, p, q)\xi_i \xi_j \geq \alpha_0 |\xi|^2 \quad ((x, u, p, q) \in \Omega \times \mathcal{B},\ \xi \in \mathbb{R}^N). \tag{2.3}$$

Here and below we use the summation convention (summation over repeated indices). For example, in the above formula the left hand side represents the sum over $i, j = 1, \ldots, N$.

(F3) (Symmetry) F is independent of x_1 and for any $(x, u, p, q) \in \Omega \times \mathcal{B}$ one has

$$F(x, Q(u, p, q)) = F(x, u, p, q) \quad (= F((0, x'), u, p, q)).$$

We consider classical solutions u of (1.1), (1.2). By this we mean functions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that (1.1), (1.2) are satisfied everywhere.

When considering fully nonlinear equations, we shall require the following stronger regularity of the solutions:

(U) For $i,j = 1,\ldots,N$, the derivatives $u_{x_ix_j}$ are locally Lipschitz continuous on Ω.

We remark that one can often establish the validity of (U) for each classical solution if additional assumptions are made on F. A sufficient condition is that F is differentiable (in all variables) on $\Omega \times \mathcal{B}$ and all its first order derivatives are locally Hölder continuous (see [14, Lemma 17.16]).

The main reason for the condition (U) is that some theorems stated below depend on the unique continuation and related results for linear equations related to (1.1), such as the linear equation for the difference of two solutions of (1.1). For the unique continuation to apply, the leading coefficients in the linear equation must be (locally) Lipschitz continuous. This is guaranteed by the Lipschitz continuity of the derivatives $F_{q_{ij}}$, as assumed in (F1), and condition (U). For more specific equations, condition (F1) alone is sufficient. This is the case, for example, if (1.1) is quasilinear, that is,

$$F(x,u,p,q) = A_{ij}(x,u,p)q_{ij} + f(x,u,p) \quad ((x,u,p) \in (\bar{\Omega} \times \mathcal{B})) \tag{2.4}$$

for some functions A_{ij} and f. Note that in this case, the last requirement in (F1) translates to the Lipschitz continuity of the functions A_{ij}, $i,j = 1,\ldots,N$, in $(x,u,p) \in \Omega \times \mathbb{R}^{N+1}$.

Condition (F1) implies that F differentiable with respect to u, p, q almost everywhere. In Section 5, we shall need the stronger differentiability property:

(F1a) F is everywhere differentiable with respect to u, p, q.

The reason for this condition is an application of the chain rule, which does not always hold for Lipschitz functions. However, for semilinear equations (1.3), condition (F1a) is not needed (see the remark at the end of Section 5).

The following notation is used throughout the paper (here $\lambda \in \mathbb{R}$ and $U \subset \Omega$):

$$\begin{aligned}
H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\},\\
\Gamma_\lambda &:= H_\lambda \cap \Omega,\\
\ell &:= \sup\{x_1 \in \mathbb{R} : (x_1, x') \in \Omega \text{ for some } x' \in \mathbb{R}^{N-1}\}.\\
\Sigma_\lambda^U &:= \{x \in U : x_1 > \lambda\}.
\end{aligned}$$

When $U = \Omega$, we omit the superscript $U = \Omega$, thus

$$\Sigma_\lambda := \Sigma_\lambda^\Omega.$$

Let P_λ stand for the reflection in the hyperplane H_λ. Note that since Ω is convex in x_1 and symmetric in the hyperplane H_0, $P_\lambda(\Sigma_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$ and Σ_0 is connected (for $\lambda > 0$, Σ_λ may not be connected).

For any function z on $\bar{\Omega}$, we define $V_\lambda z$ by

$$V_\lambda z(x) = z(P_\lambda x) - z(x) \quad (x \in \bar{\Sigma}_\lambda). \tag{2.5}$$

3 Symmetry properties of nonnegative solutions

The following theorem describes the symmetry structure of nonnegative solutions of (1.1), (1.2).

Theorem 3.1 ([20]). *Assume that* (F1)–(F3) *hold and let* u *be a nonnegative solution of* (1.1), (1.2). *Further assume that* (U) *holds or* F *is of the form* (2.4). *Then either* $u \equiv 0$ *(hence, necessarily,* $F(\cdot, 0, 0, 0) \equiv 0$*) or else there exist* $m \in \mathbb{N}$ *and constants* $\lambda_1, \dots, \lambda_m$ *with the following properties:*

(i) $0 = \lambda_m < \lambda_{m-1} < \cdots < \lambda_1 < \ell$.

(ii) For $i = 1, \dots, m$, $V_{\lambda_i} u \equiv 0$ *on a connected component of* Σ_{λ_i}. *In particular, as* Σ_0 *is connected,* $V_0 u \equiv 0$ *in* Σ_0*, that is,* u *is even in* x_1.

(iii) There are mutually disjoint open sets $G_i \subset \Omega$, $i = 1, \dots, m$, *with* G_m *possibly empty, such that the following statements are true:*

(a) $\emptyset \neq G_i \subset \Sigma_0 \quad (i = 1, \dots, m-1)$.

(b) $\bar{\Omega} = \bar{G}_m \cup \bigcup_{i=1}^{m-1} (\bar{G}_i \cup P_0(\bar{G}_i))$.

(c) For $i = 1, \dots, m$, *the set* G_i *is* x_1*-convex and* $P_{\lambda_i}(G_i) = G_i$.

(d) For $i = 1, \dots, m$, *one has* $u > 0$ *in* G_i, $u = 0$ *on* ∂G_i, $V_{\lambda_i} u \equiv 0$ *in* G_i^{λ}, *and* $u_{x_1} < 0$ *in* $\Sigma_{\lambda_i}^{G_i}$.

If $m = 1$ (and $\lambda_1 = 0$), statements (ii) and (iii) give the usual symmetry and monotonicity properties of a positive solution u. In the general case, (ii), (iii) show that the nodal set of u, $u^{-1}(0)$, divides Ω into a finite number of open reflectionally symmetric subsets G_m, G_i, $P_0(G_i)$, $i = 1, \dots, m-1$, in each of which u is positive, and has the usual Gidas-Ni-Nirenberg symmetry and monotonicity properties. In is also proved in [20] that each of the sets G_i has finitely many connected components. We remark that, although in [20] the formulation of condition (U) is stronger in that the Lipschitz continuity of the functions $u_{x_i x_j}$ on Ω is required, just the local Lipschitz continuity is needed in the proof.

A related symmetry result for nonnegative solutions of variational problems is proved in [4]. It says that for each subdomain U of Ω in which $u > 0$ and $u_{x_1} > 0$, the graph of u contains a part reflectionally symmetric to $\{(x, u(x)) : x \in U\}$. The basic method of [4] is the continuous Steiner symmetrization. In [20], a modification of the method of moving hyperplanes is used. The latter applies to more general equations, but requires stronger regularity assumptions.

4 Existence and nonexistence results

As we will see shortly, there are domains Ω and nonlinearities $f = f(x', u)$, such that the semilinear problem (1.3), (1.4) admits a solution with a nontrivial nodal set in Ω (here "nontrivial" means different from Ω and $\emptyset$). On the other hand, there are domains on which there are no such solutions, no matter how the nonlinearity is chosen. An example is any C^1 convex domain in $\mathbb{R}^2$ whose boundary contains a line segment parallel to the x_2 axis. This was shown in [20, Proposition 2.7] for semilinear problems (1.3), (1.4). By similar arguments, one can prove that on such a domain there can be no solutions with a nontrivial nodal set for any fully nonlinear problem (1.1), (1.2) (assuming that conditions (F1)-(F3), (F1a), and (U) are in effect). We refer the reader to [20] for some explanations as to why the existence of solutions with a nontrivial nodal set imposes restriction on the domain and how this is related to some results concerning overdetermined problems.

We do not have a good understanding of domains which support solutions with a nontrivial nodal set, let alone any general classification of such domains. A classification problem of this sort can be formulated in the context of general fully nonlinear problems (1.1), (1.2) or more specific problems, such as (1.3), (1.4). We cannot say much about either. However, we do have some general nonexistence results concerning the spatially homogeneous problem

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{4.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{4.2}$$

see Section 4.2 below. In Section 4.1, we summarize known examples of semilinear problems (1.3), (1.4) admitting solutions with a nontrivial nodal set. As of today, there seem to be no known examples of such solutions for the homogeneous multidimensional problem (4.1), (4.2). Results in Section 4.2 completely rule out such examples with smooth domains, or in the case of $\Omega \subset \mathbb{R}^2$, even with piecewise smooth domains.

4.1 Examples

In all examples given in this section, Ω is a planar domain, hence we use the simplified notation $(x, y) = (x_1, x')$. We consider problems of the form

$$\Delta u + \mu u + h(y) = 0, \quad (x, y) \in \Omega, \tag{4.3}$$
$$u = 0, \quad (x, y) \in \partial\Omega, \tag{4.4}$$

where $\Omega \subset \mathbb{R}^2$ satisfies the standing hypothesis, μ is a positive constant, and h a continuous function of y only. Thus this is a problem of the form (1.3), (1.4). For suitable Ω, μ, and h, as specified below, there is a nonnegative solution u with interior nodal curves. In Figures 1-4, the solid lines indicate the nodal curves of the solution u and the dashed lines indicate the symmetry hyperplanes (lines) for the nodal domains of u (cp. Theorem 3.1).

We start with two explicit examples.

Example 4.1. Let $\mu = 2$, $h(y) = -\sin y$, $u_1(x, y) := (1 + \cos x)\sin y$, and $u_2(x, y) := (1 - \cos x)\sin y$. Then, for any $k \in \mathbb{N}$, the functions u_1 and u_2 are nonnegative solutions of (4.3), (4.4) on $\Omega = (-(2k + 1)\pi, (2k + 1)\pi) \times (0, \pi)$ and $\Omega = (-2k\pi, 2k\pi) \times (0, \pi)$, respectively.

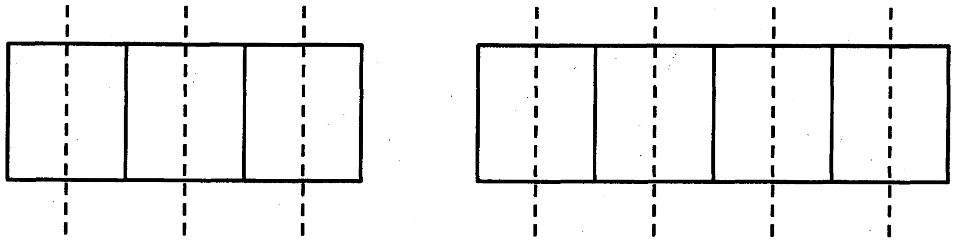

Figure 1: The nodal set (solid lines) and symmetry hyperplanes (dashed lines) for the solutions u_1, u_2 in Example 4.1.

Example 4.2. Let $\mu = 16/3$, $h(y) = -(32/3)\sin^2(2y)$,

$$u(x, y) := \left(\cos\frac{2x}{\sqrt{3}} - \cos 2y\right)^2.$$

The nodal lines of u are given by $y = \pm x/\sqrt{3} + k\pi$, $k \in \mathbb{Z}$, and the function u is a nonnegative solution of (4.3), (4.4) on any symmetric domain whose boundary consists of segments from these lines. Figure 2 shows two possibilities.

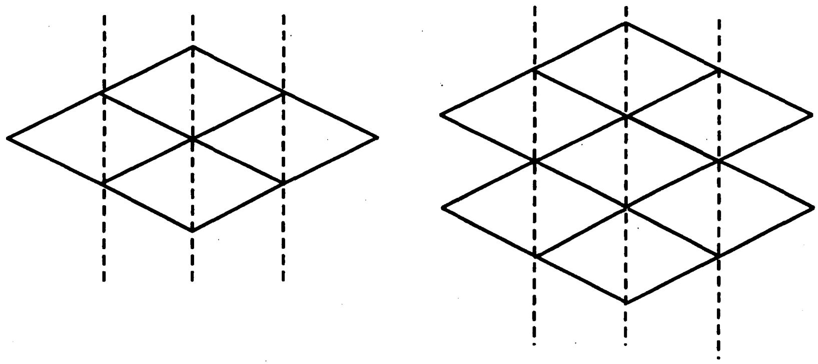

Figure 2: The nodal set and symmetry lines for solutions in Example 4.2.

In the previous two examples, the interior nodal set of u consists of line segments. This is different in the next example, where the nodal set consists of non-flat analytic curves.

Example 4.3. The domain Ω and the nodal curves of u are as in Figure 3. The definition of Ω, μ, and h is not so simple and explicit here; we refer the reader to [20, Section 5] for the detailed construction.

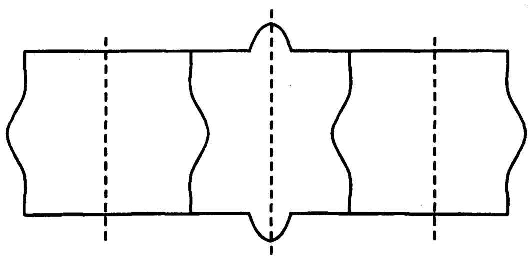

Figure 3: The domain and nonflat nodal lines of a solution.

The domains in the previous examples have corners. The next theorem shows that even on smooth domains one can find solutions with a nontrivial nodal set.

Theorem 4.4 ([23]). *There exist a constant $\mu > 0$, a continuous function $h : \mathbb{R} \to \mathbb{R}$, and a bounded analytic domain $\Omega \subset \mathbb{R}^2$ satisfying the standing hypothesis such that problem (4.3), (4.4) has a nonnegative solution u whose nodal set in Ω consists of two analytic curves (see Figure 4).*

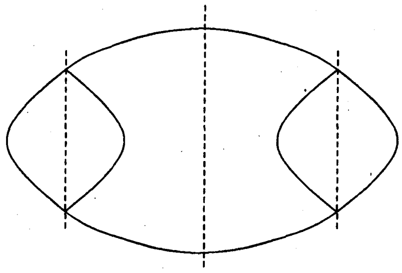

Figure 4: The domain Ω, and the nodal set and symmetry lines of the solution u from Theorem 4.4.

A few words about how the above examples have been found. The constructions link the solutions of (4.3) to eigenfunction of the Laplacian. Specifically, if u is a solution of (4.3), then $v = u_x$ satisfies $\Delta v + \mu v = 0$ in Ω. Moreover, if $u \geq 0$ in Ω, then $v = 0$ on all nodal curves of u in Ω. Also, one has $v = 0$ on H_0 and all the other symmetry lines of u parallel to H_0. Thus a key prerequisite for our construction is an eigenvalue-eigenfunction pair (μ, v) of the Laplacian, such that v has a suitable nodal structure. The solution u of (4.3), (4.4), for some function h, is then found as an antiderivative of v with respect to x.

4.2 Nonexistence of nonnegative solutions with a nontrivial nodal set

Some results on the nonexistence of solutions with a nontrivial nodal set have been available for a long time, in particular for the homogeneous problem (4.1), (4.2). In [5], such a result is proved if Ω is a ball in $\mathbb{R}^N$ ($N \geq 2$) (see also the monographs [9, 12] for the proof and a discussion of this result; an extension to quasilinear radial problems can be found in [24]). More generally, nonexistence results for (4.1), (4.2) can be found in [15] or [7], where Ω is a C^2 domain satisfying, in addition to the standing hypothesis, a geometric condition: a sort of strict x_1-convexity in [15] and convexity in all directions in [7]. For nonsmooth domains, a sufficient condition for the strict positivity of nonnegative nonzero solutions was given in [10]. It requires, roughly speaking, that for any $\delta > 0$ there be a two-dimensional wedge W, such that if the tip of W is translated to any point of $\partial\Omega$ with $x_1 \geq \delta$, then W is contained in $\bar{\Omega}$. Note that a rectangle, or a rectangle with smoothed out corners, does not satisfy the geometric condition of [10]. The results of [10] apply to equations (1.3) (and to a class of of fully nonlinear equations), if they satisfy additional symmetry assumptions.

We now give two rather general nonexistence results for (4.1), (4.2). In the first one, we deal with general smooth domains in $\mathbb{R}^N$, $N \geq 2$.

Theorem 4.5 ([19]). *Let Ω be a C^2 bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the standing hypothesis. If $u \in C^2(\bar{\Omega})$ is a nonnegative solution of* (4.1), (4.2) *for some locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, then either $u \equiv 0$ or else $u > 0$, hence u has the symmetry and monotonicity properties* (1.5) *and* (1.6).

We remark that, by the Schauder theory, any classical solution of (1.1), (1.2) belongs to $C^2(\bar{\Omega})$ (even to $C^{2+\alpha}(\bar{\Omega})$) if Ω is a $C^{2+\alpha}$ domain for some $\alpha \in (0, 1)$.

The next theorem gives the nonexistence for a large class of planar domains.

Theorem 4.6 ([21]). *Assume that Ω is a bounded domain in $\mathbb{R}^2$ satisfying the standing hypothesis such that one of the following conditions is satisfied:*

- *(i) Ω is convex (not necessarily symmetric) in the direction $e_2 = (0, 1)$ (the direction of the x_2 axis),*
- *(ii) Ω is strictly convex in the direction e_1,*
- *(iii) Ω is piecewise $C^{1,1}$.*

Let $f : [0, \infty) \to \mathbb{R}$ be a locally Lipschitz function such that for some constants $\delta > 0$, $\alpha \in (0, 1]$ one has $f\big|_{[0,\delta)} \in C^{1,\alpha}[0, \delta)$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a nonnegative solution of (1.1), (1.2), *then either $u \equiv 0$ or else $u > 0$.*

Note that Ω is strictly convex in the direction e_1 if $\partial\Omega$ contains no horizontal line segments (that is, segments parallel e_1). Condition (iii) can be weakened; we only need the boundary of Ω to be piecewise $C^{1,1}$ near the end points of the horizontal line segments contained in $\partial\Omega$. If there are no horizontal line segments in $\partial\Omega$, then (ii) applies.

5 Uniqueness and multiplicity results

We mentioned in the introduction that in one-dimensional problems, the uniqueness of the Cauchy problem implies the uniqueness of solutions with a nontrivial nodal set. The same can be said of multidimensional problems under some geometric conditions on Ω, for example, if Ω is convex (in all directions). This may be surprising at the first glance, as we are making no smoothness assumptions on Ω. To explain, recall that the symmetries of u (see Theorem 3.1) imply that a portion of $\partial\Omega$ is the reflection of a nodal set of u. Now, the nodal set of u is at the same time the nodal set of u_{x_1} (as $u \geq 0$), and the latter has some partial regularity properties, thanks to well-known theorems for linear equations. One eventually shows that any two solutions with a nontrivial nodal set in Ω vanish on a smooth portion of $\partial\Omega$ together with their gradients. The uniqueness for the Cauchy problem for elliptic equations then implies that any two such solutions coincide on a nonempty open subset. Consequently, by unique continuation, they coincide everywhere in Ω, which gives the uniqueness.

The above arguments give the uniqueness if Ω is convex or if other geometric conditions are imposed. Without any additional conditions on Ω, we can prove that the number of solutions with a nontrivial nodal set is finite. To give a precise statement, let E_{nod} be the set of all nonnegative solutions u of (1.1), (1.2), which satisfy (U) and for which $u^{-1}(0) \cap \Omega \neq \emptyset$.

Theorem 5.1 ([22]). *Assume that* (F1)–(F3), (F1a) *hold. Then the set* E_{nod} *is finite. If the set* Σ^λ *is connected for each* $\lambda > 0$, *then* E_{nod} *has at most one element.*

Note that Σ^λ is connected for each $\lambda > 0$ if Ω is convex (in all directions) or, more generally, if it is convex in all directions perpendicular to e_1.

See [22] for the proof of this theorem and a more precise multiplicity result giving an estimate on $|E_{nod}|$ in terms of $N = \dim \Omega$, the constants β_0, α_0 from (F1), (F2), and some geometric characteristics of Ω.

The finite multiplicity result is of some importance in studies of the parabolic problem associated with (1.1), (1.2). The solutions of (1.1), (1.2) are equilibria for the parabolic problem, and the equilibria with a nontrivial nodal set play a distinguished role in the global dynamics (more details on this will appear in [11]).

The multiplicity result have also some symmetry consequences for the solutions of (1.1), (1.2) themselves. For example, if both Ω and F are invariant under a continuous group of rotations, then each solution with a nontrivial nodal set must be symmetric with respect to that group (otherwise its group orbit yields infinitely many such solutions).

We conclude with a remark concerning assumption (F1a). The arguments in [22] depend on the fact that the function u_{x_1}, which is of class $C^{1,1}$ by (U), satisfies almost everywhere a linear equation with bounded coefficients. To see this just differentiate (1.1) with respect to x_1 using the chain rule and the fact that F is independent of x_1 (see condition (F3)). The chain rule does not apply in general to Lipschitz functions, and this is the only reason why we need condition (F1a). However, one can use different arguments if the equation is semilinear, as in (1.3). Even without (F1a), one can show that $v = u_{x_1}$ is a solution of the equation

$$\Delta v + a(x)v = 0, \quad x \in \Omega, \tag{5.1}$$

where $a(x)$ is a bounded measurable function. More specifically, $a(x)$ is any bounded measurable which coincides with $f_u(x, u(x))$, except at the points x such that either $v(x) = u_{x_1}(x) = 0$ (in which case the value of $a(x)$ is irrelevant in (5.1)) or $u_{x_1} \neq 0$ and the derivative f_u does not exist at $(x', u(x))$. It is not difficult to show, using the Lipschitz continuity of f with respect to u, that the set of all points $x \in \Omega$ with the latter property has measure zero. One then proves that u_{x_1} satisfies (5.1) almost everywhere by considering the equation satisfied by $(u(x_1 + \epsilon, x') - u(x_1, x'))/\epsilon$ and taking the limit as $\epsilon \searrow 0$.

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