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Extinction of solutions of the fast diffusion equation

Marek Fila
Comenius University

1 Introduction

In this survey we consider the Cauchy problem for the fast diffusion equation:

$$\begin{cases} u_\tau = \nabla \cdot (u^{m-1} \nabla u), & y \in \mathbb{R}^n,\ \tau \in (0,T), \\ u(y,0) = u_0(y) \geq 0, & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m < 1$ and $T > 0$. It is known that for m below the critical exponent $m_c := (n-2)/n$ all solutions with initial data in some suitable space, like $L^p(\mathbb{R}^n)$ with $p := n(1-m)/2$, vanish in finite time. We discuss results on the asymptotic behaviour of solutions near extinction in the range

$$m \leq m_* := \frac{n-4}{n-2}, \qquad n > 2.$$

The exponent m_* plays an important role in [1, 2, 3, 4, 6, 7, 9].

The book [11] contains a general description of the phenomenon of extinction. It is explained there that the size of the initial data at infinity (the tail of u_0) is very important in determining both the extinction time and the extinction rates.

For $m < m_c$ we have explicit self-similar solutions $U_{D,T}$ called *generalized Barenblatt solutions*, given by the formula

$$U_{D,T}(y,\tau) := \frac{1}{R(\tau)^n} \left(D + \frac{\beta(1-m)}{2} \left| \frac{y}{R(\tau)} \right|^2 \right)^{-\frac{1}{1-m}}, \tag{1.2}$$

where

$$R(\tau) := (T-\tau)^{-\beta}, \qquad \beta := \frac{1}{n(1-m)-2} = \frac{1}{n\,(m_c - m)} = \frac{\mu}{2(n-\mu)}.$$

Here $T \geq 0$ (extinction time) and $D > 0$ are free parameters. These solutions have a decay rate near extinction of the form $\|u(\cdot,\tau)\|_\infty = O((T-\tau)^{n\beta})$.

A very interesting limit case occurs if we take $D = 0$ in formula (1.2), and we find the singular solution

$$U_{0,T}(y,\tau) := k_*\,(T-\tau)^{\mu/2} |y|^{-\mu}, \qquad k_* := (2(n-\mu))^{\mu/2}.$$

whose attracting properties were studied in [6] where we obtained a continuum of extinction rates for suitable bounded data u_0.

To study the behaviour of solutions near extinction one can rewrite (1.1) by introducing the change of variables

$$t := \frac{1-m}{2}\log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{\beta(1-m)}{2}}\,\frac{y}{R(\tau)},$$

with R as above, and the rescaled function

$$v(x,t) := R(\tau)^n\, u(y,\tau).$$

If u is a solution of (1.1) then v solves the equation

$$v_t = \nabla\cdot(v^{m-1}\nabla v) + \mu\nabla\cdot(x\,v), \quad t>0, \quad x\in\mathbb{R}^n, \tag{1.3}$$

which is a nonlinear Fokker-Planck equation. The generalized Barenblatt solutions $U_{D,T}$ are transformed into *generalized Barenblatt profiles* V_D which are stationary solutions of (1.3):

$$V_D(x) := (D+|x|^2)^{\frac{1}{m-1}}, \quad x\in\mathbb{R}^n.$$

The singular Barenblatt solution becomes

$$V_0(x) = |x|^{-\mu}, \quad x\in\mathbb{R}^n\setminus\{0\}.$$

The critical exponent m_* has the property that the difference of two generalized Barenblatt profiles is integrable for $m\in(m_*,m_c)$, while it is not integrable for $m\le m_*$.

We discuss convergence to V_0 for $m<m_*$ in Section 2, convergence to V_D when $D>0$, $m<m_*$ in Section 3, and convergence to V_D when $D>0$, $m=m_*$ in Section 4.

2 Convergence to the singular Barenblatt profile

The following was shown in [6].

Theorem 2.1 *Assume that*

$$n\ge 5 \quad \textit{and} \quad 0<m<m_*=\frac{n-4}{n-2}, \tag{2.1}$$

and let the initial function u_0 be continuous, bounded, and satisfy the conditions:

$$0\le u_0(y)\le A\,|y|^{-\mu} \quad \textit{for all } y\ne 0$$

and

$$A\,|y|^{-\mu}-c_1|y|^{-l}\le u_0(y)\le A\,|y|^{-\mu}-c_2|y|^{-l} \qquad \textit{for } \ |y|\ge 1$$

for some $A, c_1, c_2>0$, and

$$\mu+2<l\le L:=\mu+\sqrt{2(n-\mu)}. \tag{2.2}$$

Then the solution u of problem (1.1) has complete extinction precisely at the time $T:=(A/k_*)^{1-m}>0$, and there are positive constants K_1, K_2 such that for $0<\tau<T$ we have

$$K_1(T-\tau)^{\theta_l}\le\|u(\cdot,\tau)\|_\infty\le K_2(T-\tau)^{\theta_l},$$

where

$$\theta_l := \frac{n\mu - \gamma_l}{2(n-\mu)} > 0, \qquad \gamma_l := \frac{\mu(l-\mu-2)(n-l)}{l-\mu}. \tag{2.3}$$

One of the main aims of [9] is to show that Theorem 2.1 does not hold for $l > L$.

The main result from [6] can be formulated as follows.

Theorem 2.2 *Let* (2.1) *hold. Assume that* $v_0 \geq 0$ *is continuous, bounded and such that*

$$|x|^{-\mu} - c_1|x|^{-l} \leq v_0(x) \leq |x|^{-\mu} - c_2|x|^{-l} \qquad \text{for } |x| \geq 1,$$

where l *is as in* (2.2) *and* $c_1, c_2 > 0$. *Assume also that* $v_0(x) \leq |x|^{-\mu}$ *for all* $x \neq 0$. *Let* v *denote the solution of* (1.3) *with initial condition*

$$v(x,0) = v_0(x), \qquad x \in \mathbb{R}^n. \tag{2.4}$$

Then:

(i) *There exist* $K_1, K_2 > 0$ *such that for* $t \geq 1$ *we have*

$$K_1 e^{\gamma_l t} \leq \|v(\cdot,t)\|_\infty \leq K_2 e^{\gamma_l t}, \tag{2.5}$$

here γ_l *is as in* (2.3).

(ii) *For each* $r_0 > 0$ *one can find* $C_1, C_2 > 0$ *such that for* $t \geq 1$ *and* $|x| \geq r_0$ *the following holds*

$$C_1 e^{-\alpha_l t} \leq |x|^{-\mu} - v(x,t) \leq C_2 e^{-\alpha_l t}, \qquad \alpha_l := (l-\mu-2)(n-l). \tag{2.6}$$

The reason why we assume that $l > \mu + 2$ is that the difference $|x|^{-\mu} - V_D(x)$ behaves like $|x|^{-(\mu+2)}$ as $|x| \to \infty$. It was shown in [9] that the condition $\mu + 2 < l \leq L$ is optimal for Theorem 2.2 (i) but not for Theorem 2.2 (ii) which holds for a larger range

$$l \in (\mu+2, l_\star), \qquad l_\star := \frac{1}{2}(n+\mu+2). \tag{2.7}$$

More precisely, the following results were established in [9]:

Theorem 2.3 *Assume that* $m < m_*$, $n > 2$, *and* $v_0 \geq 0$ *is continuous.*

(i) *If*

$$v_0(x) < |x|^{-\mu}, \qquad x \neq 0, \tag{2.8}$$

and

$$v_0(x) \leq |x|^{-\mu} - c|x|^{-l}, \qquad |x| > 1,$$

with some l *as in* (2.7) *and* $c > 0$ *then for any* $r_0 > 0$ *there exists* $C(r_0) > 0$ *such that the solution of* (1.3), (2.4) *satisfies*

$$v(x,t) \leq |x|^{-\mu} - C(r_0)e^{-\alpha_l t}|x|^{-l}, \qquad |x| \geq r_0, \quad t \geq 0.$$

(ii) *Assume that*

$$v_0(x) \geq |x|^{-\mu} - c|x|^{-l}, \qquad |x| > 1,$$

with some l *as in* (2.7) *and* $c > 0$. *Then one can find* $C > 0$ *such that the solution of* (1.3), (2.4) *satisfies*

$$v(x,t) \geq |x|^{-\mu} - Ce^{-\alpha_l t}|x|^{-l}, \qquad x \neq 0, \quad t > 0.$$

(iii) *Set*

$$\alpha_\star := \alpha_{l_\star} = \frac{(n-\mu-2)^2}{4}. \tag{2.9}$$

If (2.8) *holds then for any* $\alpha > \alpha_\star$ *and each* $r_0 > 0$ *there exists* $C(\alpha, r_0) > 0$ *such that the solution of* (1.3), (2.4) *satisfies*

$$\sup_{|x|\geq r_0} \left(|x|^{-\mu} - v(x,t)\right) \geq Ce^{-\alpha t}, \qquad t > 0.$$

Theorem 2.4 *Let* $m < m_*$, $n > 2$. *Assume* (2.8) *and* $v_0 \geq 0$ *is continuous. Then for any*

$$\gamma > \gamma_L := \mu\left(n + 2 - \mu - 2\sqrt{2(n-\mu)}\right)$$

there exists $C(\gamma) > 0$ *such that the solution of* (1.3), (2.4) *satisfies*

$$v(x,t) \leq C(\gamma)e^{\gamma t}, \qquad x \in \mathbb{R}^n, \quad t > 0.$$

The fact that the optimal condition on l is different for (2.5) and (2.6) is in contrast with corresponding results for the equation $u_t = \Delta u + u^p$, see [5, 8, 10].

3 Convergence to regular Barenblatt profiles

The basin of attraction of V_D, $D > 0$ and the rates of convergence to V_D, $D > 0$ was studied in [1, 2] using certain functional inequalities of Hardy-Poincaré type. It was established there that the basin of attraction of V_D in the range $m < m_*$ contains functions v_0 such that

$$V_{D_0} \leq v_0 \leq V_{D_1}, \quad 0 < D_1 < D < D_0, \qquad |v_0 - V_D| \in L^1(R^n).$$

We call this set the variational basin, and for this the entropy method from [1, 2] gives precise decay rates (the variational rates).

The main result in [7] is the following:

Theorem 3.1 *Let* $m < m_*$, $n > 2$. *Assume that* $c, D > 0$ *and* $\mu + 2 < l < l_\star$, *here* $l_\star$ *is as in* (2.7).

(i) *If*

$$|v_0(x) - V_D(x)| \leq c|x|^{-l}, \qquad |x| \geq 1,$$

and

$$0 < v_0(x) \leq V_\delta(x), \qquad x \in \mathbb{R}^n$$

for some $\delta < D$*, then there exists* $C_1 > 0$ *such that the solution* v *of* (1.3) *with the initial condition* (2.4) *satisfies*

$$\sup_{x\in\mathbb{R}^n} |v(x,t) - V_D(x)| \le C_1 e^{-\alpha_l t}, \qquad t \ge 0,$$

where α_l *is as in* (2.6).

(ii) *If*

$$v_0(x) \le V_D(x) - c|x|^{-l}, \qquad |x| \ge 1,$$

and

$$0 < v_0(x) \le V_D(x), \qquad x \in \mathbb{R}^n,$$

then there exists $C_2 > 0$ *such that the solution* v *of* (1.3), (2.4) *satisfies*

$$\sup_{x\in\mathbb{R}^n} \Big(V_D(x) - v(x,t)\Big) \ge C_2 e^{-\alpha_l t}, \qquad t \ge 0.$$

(iii) *If*

$$v_0(x) \ge V_D(x) + c|x|^{-l}, \qquad |x| \ge 1,$$

and

$$v_0(x) \ge V_D(x), \qquad x \in \mathbb{R}^n,$$

then there exists $C_3 > 0$ *such that the solution* v *of* (1.3), (2.4) *satisfies*

$$\sup_{x\in\mathbb{R}^n} \Big(v(x,t) - V_D(x)\Big) \ge C_3 e^{-\alpha_l t}, \qquad t \ge 0.$$

This result gives a sharp description of the basin of attraction of generalized Barenblatt profiles for $m < m_*$. It shows that non-integrable perturbations of V_D may still yield convergence to V_D. The condition $l > \mu+2$ is optimal since the difference of two Barenblatt profiles is of the order $|x|^{-(\mu+2)}$.

Theorem 3.1 yields a continuum of convergence rates which depend explicitly on the tail of initial data. The rate $\alpha_l = (l - \mu - 2)(n - l)$ converges to zero as $l \to \mu + 2$ and to the maximum value $\alpha_\star$ (see (2.9)) as $l \to l_\star$. Here $\alpha_\star$ is the rate found in [1, 2] for solutions emanating from integrable perturbations of V_D. This fastest rate is the best constant in a Hardy-Poincaré inequality (see [2]). This best constant is also the bottom of the continuous spectrum of the linearization on a suitable weighted space (see [1, 2]).

In Theorem 3.1, the assertion (i) is no longer true if $l > l_\star$. In fact, the following result about the optimality of the range of l was obtained in [7].

Theorem 3.2 *Let* $m < m_*$, $n > 2$. *Assume that* $D > 0$ *and*

$$0 < v_0(x) < V_D(x), \qquad x \in \mathbb{R}^n$$

or

$$v_0(x) > V_D(x), \qquad x \in \mathbb{R}^n.$$

Then for any $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that the solution* v *of* (1.3), (2.4) *satisfies*

$$\sup_{x\in\mathbb{R}^n} \Big|V_D(x) - v(x,t)\Big| \ge C_\varepsilon e^{-(\alpha_\star+\varepsilon)t}, \qquad t \ge 0. \tag{3.1}$$

It follows from (3.1) that Theorem 2 (i) in [1] is optimal if $m < m_*$, $n > 2$. The sharpness of the rate given by $\alpha_\star$ was discussed in [2] in terms of relative entropy which can be written as

$$\mathcal{F}[w] := \frac{1}{1-m}\int_{\mathbb{R}^n}\left[w - 1 - \frac{1}{m}(w^m - 1)\right] V_D^m dx, \qquad w := \frac{v}{V_D}.$$

The statement on the sharp rate in [2] says that $\alpha = \alpha_\star$ is the best possible rate for which

$$\mathcal{F}[w(\cdot,t)] \le \mathcal{F}[w(\cdot,0)]e^{-\alpha t}$$

holds for all $t \ge 0$ if $V_{D_0} \le v_0 \le V_{D_1}$ for some $D_0 > D > D_1 > 0$ and $v_0 - V_D$ is integrable. Theorem 3.2 implies that solutions starting from positive or negative perturbations of V_D cannot converge to V_D (in L^∞) at exponential rates faster than $e^{-\alpha_\star t}$.

4 Critical case

The case $m = m_*$ was treated in [3] by functional analytic methods. A suitable linearization of the non–linear Fokker–Planck equation (1.3) was viewed as the plain heat flow on a suitable Riemannian manifold and then non–linear stability was studied by entropy methods. One of the main results of [3] says that if $0 < D_1 < D_0$, $D \in [D_1, D_0]$ and

$$\begin{aligned} &V_{D_0}(x) \le v_0(x) \le V_{D_1}(x), \qquad x \in \mathbb{R}^n, \\ &|v_0(x) - V_D(x)| \le f(|x|), \qquad x \in \mathbb{R}^n, \qquad f(|\cdot|) \in L^1(\mathbb{R}^n), \end{aligned} \tag{4.1}$$

then for the solution v of (1.3) with the initial condition $v(x,0) = v_0(x)$ it holds that

$$\|v(\cdot,t) - V_D\|_{L^\infty(\mathbb{R}^n)} \le K(t+1)^{-\frac{1}{4}}, \qquad t \ge 0, \tag{4.2}$$

for some $K > 0$.

No lower bound for the rate was given in [3] and the question of whether the rate from (4.2) is optimal for a class of data was posed there as an open problem together with the question of whether one can prove convergence, maybe with worse rates or without rates, for more general initial data. The aim in [4] is to provide some answers to these questions by establishing optimal results on rates of convergence for a class of initial data which do not satisfy (4.1).

Theorem 4.1 *Assume that* $n > 2$, $m = m_* = \frac{n-4}{n-2}$ *and* $D > 0$. *Let* v *be the solution of* (1.3) *with the initial condition*

$$v(x,0) = v_0(x) := \left(|x|^2 + D + \psi_0(x)\right)^{-\frac{n-2}{2}}, \qquad x \in \mathbb{R}^n, \tag{4.3}$$

where ψ_0 *is continuous and nonnegative on* $\mathbb{R}^n$, $\psi_0 \not\equiv 0$.
(i) *If there are* $B > 0$ *and* $\gamma \in (0,1)$ *such that*

$$\psi_0(x) \le B \ln^{-\gamma}|x|, \qquad |x| > 2,$$

then there exists $C > 0$ *such that*

$$V_D(x)\left(1 - CV_D^{\frac{2}{n-2}}(x)(t+1)^{-\frac{\gamma}{2}}\right) \le v(x,t) \le V_D(x), \qquad x \in \mathbb{R}^n, \quad t \ge 0.$$

(ii) *If there are* $b > 0$ *and* $\gamma \in (0,1)$ *such that*

$$\psi_0(x) \geq b \ln^{-\gamma} |x|, \qquad |x| > 2,$$

then there exists $c > 0$ *such that*

$$v(0,t) \leq V_D(0) - c(t+1)^{-\frac{\gamma}{2}}, \qquad t > 0.$$

This theorem says that if $V_D(x) - v_0(x)$ behaves like $|x|^{-n} \ln^{-\gamma} |x|$ for $|x|$ large and some $\gamma \in (0,1)$ then $\|v(\cdot,t) - V_D\|_{L^\infty(\mathbb{R}^n)}$ behaves like $t^{-\gamma/2}$ for t large. Hence, we obtain a continuum of algebraic rates for initial data which do not satisfy (4.1). It is also shown in [4] that convergence to V_D from below cannot occur at any rate faster than $t^{-1/2}$, so Theorem 4.1 (i) does not hold for $\gamma > 1$.

Theorem 4.2 *Let* $n > 2, m = m_\star$ *and* $D > 0$, *and assume that* ψ_0 *is continuous and nonnegative on* $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. *Then there exists* $c > 0$ *such that the solution* v *of* (1.3), (4.3) *satisfies*

$$v(0,t) \leq V_D(0) - c(t+1)^{-\frac{1}{2}} \qquad \text{for all } t > 0.$$

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References

[1] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates*, Arch. Rat. Mech. Anal. **191** (2009), 347–385.

[2] M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proc. Nat. Acad. Sciences **107** (2010), 16459–16464.

[3] M. Bonforte, G. Grillo and J. L. Vázquez, *Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold*, Arch. Rat. Mech. Anal. **196** (2010), 631–680.

[4] M. Fila, J. King and M. Winkler, *Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent*, preprint.

[5] M. Fila, J. King, M. Winkler and E. Yanagida, *Optimal lower bound of the grow-up rate for a supercritical parabolic equation*, J. Diff. Equations **228** (2006), 339–356.

[6] M. Fila, J. L. Vázquez and M. Winkler, *A continuum of extinction rates for the fast diffusion equation*, Comm. Pure Appl. Anal. **10** (2011), 1129–1147.

[7] M. Fila, J. L. Vázquez, M. Winkler and E. Yanagida, *Rate of convergence to Barenblatt profiles for the fast diffusion equation*, Arch. Rat. Mech. Anal. **204** (2012), 599–625.

[8] M. Fila and M. Winkler, *Rate of convergence to a singular steady state of a supercritical parabolic equation*, J. Evol. Equations **8** (2008), 673–692.

[9] M. Fila and M. Winkler, *Optimal rates of convergence to the singular Barenblatt profile for the fast diffusion equation*, preprint.

[10] M. Fila, M. Winkler and E. Yanagida, *Grow-up rate of solutions for a supercritical semilinear diffusion equation*, J. Diff. Equations **205** (2004), 365–389.

[11] J. L. Vázquez, "Smoothing and Decay Estimates for Nonlinear Diffusion Equations", Oxford Lecture Notes in Maths. and its Applications, vol. 33, Oxford University Press, Oxford, 2006.

Department of Applied Mathematics and Statistics
Comenius University
84248 Bratislava
Slovakia
E-mail address: `fila@fmph.uniba.sk`