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Diffusion-induced bifurcations from the stationary solutions and infinity

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Since A. Turing discovered the mechanism of periodic pattern generation induced by diffusion in 1952, various patterns in nature have been explained by his idea, which is called the *diffusion-induced instability* or *Turing's instability* [8]. To explain this in detail, let us consider a system of the ordinary differential equations

$$\begin{cases} U_t = f_1(U,V), \\ V_t = f_2(U,V) \end{cases} \tag{1}$$

and the corresponding reaction-diffusion system

$$\begin{cases} u_t = d_1\Delta v + f_1(u,v), \\ v_t = d_2\Delta v + f_2(u,v) \end{cases} \tag{2}$$

in a bounded domain Ω and $t>0$ with homogeneous Neumann boundary condition. We note that any homogeneous solutions of (2) can be regarded as those of (1). Suppose that $(0,0)$ is a stable equilibrium of (1). Then $(0,0)$ is also a stationary solution of (2). However, $(0,0)$ may not be stable. If the nonlinear terms f_1, f_2 and the diffusion coefficients d_1, d_2 are chosen appropriately, then it becomes unstable. The stability is transferred to a spatially inhomogeneous steady state which bifurcates from the homogeneous one.

However, the diffusion-induced instability is caused only by a linear effect from the mathematical viewpoint. We would like to extend this result to the changes of global dynamics. It is also known that diffusion can be influenced on the global existence of solutions. More precisely, we are concerned with finite-time blow-up of solutions caused by diffusion. We say that a solution of (2) blows up in finite time if there is $T\in(0,\infty)$ such that

$$\limsup_{t\nearrow T}(\|u_1(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)})=\infty.$$

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Blow-up of solutions has been studied extensively by many authors. Most of them study that a solution of (1) blow up in a finite time. Therefore the following question naturally arises: can blow-up occur in (2), while (1) possesses a compact global attractor ? Mizoguchi, Ninomiya and Yanagida [4] presented the affirmative answer. They constructed the following system:

$$\begin{cases} u_t = d_1\Delta u + |u-v|^{p-1}(u-v) - u, \\ v_t = d_2\Delta v + |u-v|^{p-1}(u-v) - v \end{cases} \tag{3}$$

with $p > 1$. and have shown that some solutions of (3) blow up in finite time if $0 < d_1 < d_2$, while all solutions of the corresponding ordinary differential equations (1) converge to $(0,0)$ as t tends to infinity. This phenomenon is called *diffusion-induced blowup.* We note that the diffusion-inhibited blowup is also studied in [3, 2].

Let us consider the change of the global dynamics more precisely. This result holds even for large diffusivity and says that a compact attractor for (1) imbedded in (3) is only local for (3). We notice that all solutions of (3) with $d_1 = d_2 > 0$ are bounded for all $t > 0$. We may expect that solutions bifurcate from the infinity when d_1 varies from d_2.

Along this context we refer to the bifurcation from infinity by Stuart [7] and Rabinowitz [6]. To explain their idea, let us consider the following nonlinear problem:

$$u_t = \Delta u + au + f(u) \tag{4}$$

with Dirichlet homogeneous boundary condition. Suppose that $\lambda_{1,D}$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition and

$$f(u) = o(|u|) \tag{5}$$

as $|u|$ tends to ∞. The typical example of nonlinear term f is $u/(1+u^2)$. Then the system possesses an unbounded bifurcation branch which includes $(\lambda_{1,D}, \infty)$.

We cannot apply this result to the system (3) because the condition (5) is not satisfied for (3). We need to construct the theory for the bifurcation from infinity for the system. The partial result will be presented in this short note. See [1] for the details.

To begin with, we restrict the nonlinear terms $\tilde{f}_1$ and $\tilde{f}_2$ to p-homogeneous polynomials where p is a positive integer. Namely,

$$\tilde{f}_1(su, sv) = s^p \tilde{f}_1(u,v), \quad \tilde{f}_2(su, sv) = s^p \tilde{f}_2(u,v)$$

for $s > 0$.

Consider

$$\begin{cases} u_t = d_1 u_{xx} + f_1(u,v), & x \in (0,1), t > 0, \\ v_t = d_2 v_{xx} + f_2(u,v), & x \in (0,1), t > 0, \\ u_x(0,t) = v_x(0,t) = u_x(1,t) = v_x(1,t) = 0, & t > 0, \end{cases} \tag{6}$$

where

$$f_1(u,v) = \tilde{f}_1(u,v) - \varepsilon u, \quad f_2(u,v) = \tilde{f}_2(u,v) - \varepsilon v.$$

Assume that all solutions of the ordinary differential equations

$$\begin{cases} U_t &= f_1(U,V), \\ V_t &= f_2(U,V) \end{cases} \tag{7}$$

exists globally in time and that (7) with $\varepsilon = 0$ possesses an stationary solution (u_*, v_*) where $u_* v_* \neq 0$. The homogeniety of nonlinearities implies

$$\{s(1,k_*) \mid s \in \mathbb{R}, f(1,k_*) = g(1,k_*) = 0\}$$

is a set of stationary solutions of (7) with $\varepsilon = 0$. Under these assumptions we have the following theorem.

Theorem 1 *Consider the case* $\varepsilon = 0$ *in* (6). *Set* $d_2 = d_1\mu_* - \delta$. *Assume nonlinear terms* $f_1(u,v), f_2(u,v)$ *of* (6) *satisfy*

(i) $f_1(1,k_*) = f_2(1,k_*) = 0, (k_* \neq 0, \infty)$,

(ii) $\mu_* \neq \nu_*$,

(iii) p *is odd,*

(iv) $f_1(1, k_* + k_*\delta/d_1(\nu_* - \mu_*)) > 0$ *for any* δ *in some inetrval* Λ

where

$$\mu_* := \lim_{k \to k_*} \frac{f_2(1,k)}{k f_1(1,k)}, \quad \nu_* := \lim_{k \to k_*} \frac{d}{dk} \frac{f_2(1,k)}{f_1(1,k)}.$$

Then there exist an small interval $\Lambda^* \subset \Lambda$ *and a non-constant solution* $(u_*, v_*) = (\bar{u}, \bar{k}\bar{u})$ *of* (6) *for any* $\delta \in \Lambda^*$ *such that* $0 \in \bar{\Lambda}^*$ *and*

$$\begin{aligned} \bar{k} &= k_* + \frac{k_*}{d_1(\nu_* - \mu_*)}\delta + O(\delta^2), \\ \bar{u} &= \rho(\delta) U(x), \\ \rho(\delta) &= \left(\frac{d_1}{f_1(1,\bar{k})}\right)^{1/(p-1)} \end{aligned}$$

where $U(x)$ *is a solution of*

$$\begin{cases} U_{xx} + U^p = 0, & (0 \leq x \leq L), \\ U_x(0) = U_x(L) = 0. \end{cases}$$

Moreover,

$$\lim_{\delta \to 0, \delta \in \Lambda^*} \bar{u}(x) = \infty, \quad \lim_{\delta \to 0, \delta \in \Lambda^*} \bar{k} = k_*. \tag{8}$$

It follows from (8) that non-constant solutions bifurcates from the infinity.

Theorem 2 *Assume* (i)—(iv) *in Theorem* 1 *replaced with*

(ii)′ $\nu_* < \mu_* = 1$.

Take Λ^* *as in Theorem 1 and choose* d_1, d_2 *satisfying* $|(d_2 - d_1)/(d_1 d_2)| \leq \delta$. *Then there is a positive constant* ε^* *such that a non-constant stationary solution of* (6) *exists for any* $\varepsilon \in (0, \varepsilon^*)$.

Under the assumptions of Theorem 2, the non-constant solution in (6) bifurcates from infinity as δ and ε go to 0 due to Theorem 1.

We can confirm that

$$\mu_* = 1, \quad \nu_* = 0$$

for the case (3). Thus Theorems 1 and 2 imply the existence of solutions with large norm for (3). We can understand the diffusion-induced blowup in the following way. The global attractor when $d_1 = d_2$ is compact. However, when $d_1 \neq d_2$, these solutions come out from the infinity and these solutions and their unstable manifolds separate the basin of domain. Thus some solutions which can go to infinity apper.

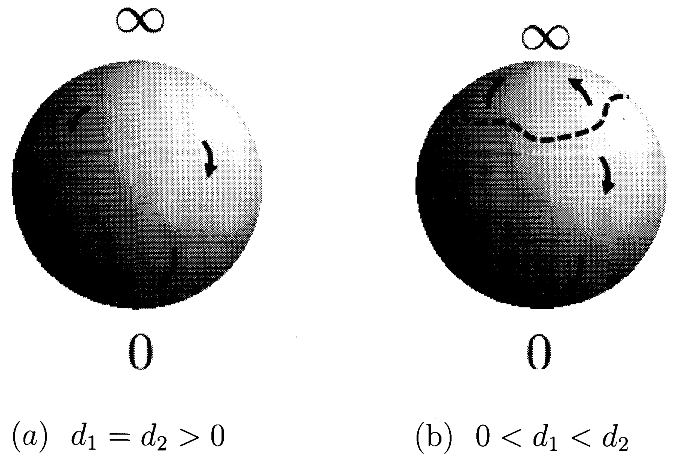

Figure 1: Heuristic understanding of the diffusion-induced blowup by the bifurcation from infinity. (a) all solutions converge to 0; (b) some large solution blows up in a finite time.

These theorems facilitate to construct the candidate systems for the diffusion-induced blowup. We only need to construct examples satisfying the assumptions of Theorem 2. For example, we have

$$\begin{cases} u_t = d_1 u_{xx} + (u^2 - v^2)v - \varepsilon u, & x \in (0,1), t > 0, \\ v_t = d_2 v_{xx} + (u^2 - v^2)u - \varepsilon v, & x \in (0,1), t > 0, \\ u_x(0,t) = v_x(0,t) = u_x(1,t) = v_x(1,t) = 0, & t > 0. \end{cases} \tag{9}$$

Since $\mu_* = 1,\ \nu_* = -1$, Theorem 2 immediately implies that the non-constant stationary solutions bifurcate from the infinity.

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