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STURM GLOBAL ATTRACTORS OF HAMILTONIAN TYPE FOR SEMILINEAR PARABOLIC EQUATIONS

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Dedicated to Hiroshi Matano on the occasion of his 60th birthday

ABSTRACT. In this note we review the characterization of global attractors for dissipative semiflows generated by scalar semilinear parabolic equations of the form $u_t = u_{xx} + f(x, u, u_x)$ defined on the interval $0 \leq x \leq \pi$ with Neumann boundary conditions. We outline the characterization results for these global attractors – the Sturm attractors – obtained by a permutation of the equilibrium solutions – the Sturm permutation – associated to the second order ODE satisfied by the stationary solutions of the parabolic equation. In particular we consider the characterization results for the class of nonlinearities of the form $f = f(u)$ – the Hamiltonian class. Using this characterization we then outline some results on the geometry of Sturm attractors for the case of periodic boundary conditions.

1. STURM ATTRACTORS

In the following notes we survey recent results on the characterization of global attractors for dissipative semiflows generated by scalar semilinear parabolic equations. In the first three sections we review the notions of Sturm attractor and Sturm permutation for the case of Neumann boundary conditions. In these sections we illustrate the characterization of Sturm attractors by means of Sturm permutations, that is, in purely combinatorial terms. In the fourth section we consider recent results on the characterization of Sturm attractors for a restricted class that we call of Hamiltonian type. Then in the fifth section we present some consequences of this characterization for the case of periodic boundary conditions. In particular we obtain a characterization of Sturm attractors for a class of spatially S^1-equivariant problems.

We then start with the characterization of global attractors for dissipative semiflows generated by scalar semilinear parabolic equations of the form

$$u_t = u_{xx} + f(x, u, u_x) \ , \ x \in I \ , \tag{1}$$

defined on the interval $I = [0, \pi]$, with Neumann boundary conditions.

If the nonlinearity $f = f(x, u, u_x)$ satisfies adequate smooth and dissipative conditions then (1) generates a dynamical system in an appropriate phase space X. Here we assume

$$f \in C^2 \text{ is dissipative .} \tag{2}$$

Sufficient, but not necessary, conditions for dissipativeness take the following form of sign and growth conditions

$$vf(x, v, 0) < 0 \text{ for all large } |v| \ , \tag{3}$$

$$|f(x,v,p)| < C(|v|)(1+|p|^\gamma) , \tag{4}$$

for all (x, v, p), some constant $0 < \gamma < 2$, and some continuous function C. Under these conditions (1) generates a semiflow $S_f(t) : X \to X$

$$S_f(t) : u_0 \mapsto S_f(t)u_0 = u(t, \cdot) , \; t \geq 0 , \tag{5}$$

where $u(0, \cdot) = u_0$ and X is the Sobolev space, (see, e.g., [2, 19, 28]),

$$X = H^2(I) \cap \{u_x = 0 ; \, x \in \partial I\} \hookrightarrow C^1(I) . \tag{6}$$

By dissipativeness, the semiflow $S_f(t)$ has a global attractor

$$\mathcal{A} = \mathcal{A}_f \subset X , \tag{7}$$

which is a nonempty compact invariant set attracting every bounded subset of X. For references see, e.g., [16, 7, 17]. The global attractor $\mathcal{A}$ contains all the information on the asymptotic behavior of the semiflow $S_f(t)$. In fact, any solution $u(t) = u(t, \cdot)$ of (1) has a nonempty ω-limit in $\mathcal{A}$. Therefore the prime objective in the study of the semiflow generated by (1) is the geometric characterization of $\mathcal{A}$.

In the context of the parabolic partial differential equation (1) the set $\mathcal{A}$ is called the *Sturm attractor*. Undoubtedly the most useful tool in this framework is the Sturm *zero number* decay property; see [34, 24, 4]. In essence it states that the number of zeros of the difference $u_1 - u_2$ between any two different solutions of (1),

$$t \mapsto z(u_1(t, \cdot) - u_2(t, \cdot)) , \tag{8}$$

is a monotone nonincreasing function of t. This property is essential not only for establishing the asymptotic behavior of the solutions of (1) but also to determine the geometric characterization of the Sturm attractor. Moreover, under the appropriate adaptations the same result holds for other types of boundary conditions that include periodic boundary conditions.

In the case of Neumann boundary conditions the system $S_f(t)$ has a gradient-like behavior due to the existence of a Lyapunov function; see [37, 25]. Therefore, as $t \to +\infty$ any solution of (1) approaches the set $\mathcal{E} = \mathcal{E}_f$ of equilibrium solutions, which in this case is the set of solutions of the ODE Neumann boundary value problem

$$\begin{gathered} v'' + f(x, v, v') = 0 , \; x \in I = [0, \pi] , \\ v'(0) = v'(\pi) = 0 . \end{gathered} \tag{9}$$

In particular, the Sturm attractor is composed of the equilibria and the set $\mathcal{H}$ of heteroclinic orbits, i.e. the set of orbits with ω and α-limit in $\mathcal{E}$ (homoclinicity being excluded by the gradient-like behavior)

$$\mathcal{A} = \mathcal{E} \cup \mathcal{H} . \tag{10}$$

To guarantee finiteness and nondegeneracy of the equilibria $v \in \mathcal{E}$ we assume hyperbolicity: an equilibrium $v \in \mathcal{E}$ is *hyperbolic* if $\lambda = 0$ is not an eigenvalue of the linearization at v

$$\lambda u = u_{xx} + \partial_p f(x, v, v_x) u_x + \partial_v f(x, v, v_x) u , \; x \in I , \tag{11}$$

with Neumann boundary conditions. Then, if all the equilibria are hyperbolic, $\mathcal{E} = \{v_j, 1 \leq j \leq n\}$ is a finite set and the Sturm attractor is the union of the corresponding unstable manifolds (see [19]),

$$\mathcal{A} = \bigcup_{v_j \in \mathcal{E}} W^u(v_j) \ . \tag{12}$$

The set of nonlinearities $f = f(x, u, u_x)$ satisfying (2) for which all equilibria $v \in \mathcal{E}_f$ are hyperbolic is called the *Sturm class*

$$\text{Sturm}(x, u, u_x) \ . \tag{13}$$

In the obvious way the classes Sturm(u), Sturm(x, u), Sturm(u, u_x), etc, of nonlinearities f are also defined.

2. Sturm permutations

For $f \in \text{Sturm}(x, u, u_x)$ the graphs $\{(x, v_j(x), v_j'(x)) : x \in I\}$ of the n equilibria $v_j \in \mathcal{E}_f$ define a braid of n strands in the space $I \times \mathbb{R}^2$. For an illustration see Figure 1. Associated to this braid of equilibria we have a

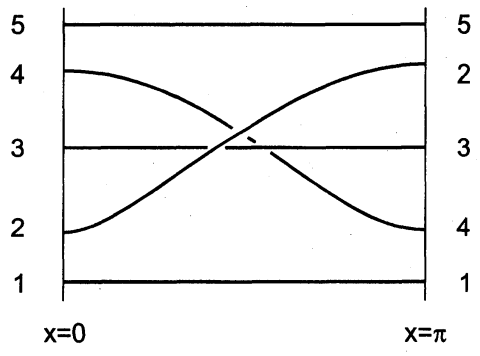

FIGURE 1. The $(x, v(x))$-projection of a braid of 5 equilibria.

permutation $\sigma = \sigma_f \in \mathcal{S}(n)$. This permutation is called the *Sturm permutation* and is defined by the ordering of the (hyperbolic) solutions $v_j(x)$ at $x = 0$ and $x = \pi$. If the equilibria are labeled such that the ordering at $x = 0$ is

$$v_1(0) < v_2(0) < \cdots < v_n(0) \ , \tag{14}$$

then $\sigma \in \mathcal{S}(n)$ is defined by the ordering at $x = \pi$

$$v_{\sigma(1)}(\pi) < v_{\sigma(2)}(\pi) < \cdots < v_{\sigma(n)}(\pi) \ . \tag{15}$$

The Sturm permutation corresponding to the illustration in Figure 1 (as a permutation of $\{1, 2, 3, 4, 5\}$ and also in cycle notation) is given by

$$\sigma = \{1, 4, 3, 2, 5\} = (2\ 4) \ . \tag{16}$$

The Sturm permutations are central objects of consideration in the characterization of Sturm attractors. In fact, many of the main geometric features of a Sturm attractor $\mathcal{A}_f$ are explicitly determined by the Sturm permutation σ_f. For references see [15, 11].

To obtain the Sturm permutation σ_f we solve the boundary value problem (9) by the shooting method. We consider the ODE initial value problem for $w = w(x,a)$ with initial Neumann condition at $x = 0$,

$$w'' + f(x,w,w') = 0 \ , \quad w(0,a) = a \ , \ w'(0,a) = 0 \ , \tag{17}$$

where $' = \partial/\partial x$, and compute the curve $\Gamma = \Gamma_f$ corresponding to the solutions at $x = \pi$ (i.e. the shooting curve)

$$\Gamma = \{\gamma(a), a \in \mathbb{R}\} \subset \mathbb{R}^2 \ , \quad \gamma : a \mapsto (w(\pi,a), w'(\pi,a)) \ . \tag{18}$$

This is a plane Jordan curve which intersects, with strict crossings, the straight line of Neumann conditions at a sequence of n points. Such a curve is appropriately called a *meander* [6] and the permutation that arises from the ordering of the intersection points, first along the straight line and then along the meander, is a *meander permutation.* Then the intersection points correspond to the solutions of the Neumann boundary value problem (9), i.e. the equilibria $v_j \in \mathcal{E}$, and the Sturm permutation σ_f corresponds to the meander permutation of Γ. See the illustration in Figure 2.

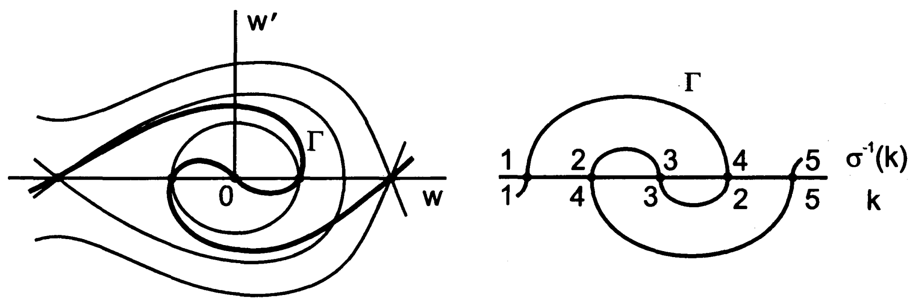

FIGURE 2. A shooting curve Γ (left) and the corresponding meander permutation (right) illustrating the Sturm permutation $\sigma = \{1,4,3,2,5\} = (2\ 4)$.

The Morse index of an equilibrium is one of the many geometric features of $\mathcal{A}_f$ which is entirely determined in terms of the Sturm permutation σ_f. Recall that the Morse index $i(v_j)$ of an equilibrium $v_j \in \mathcal{E}$ is the number of positive eigenvalues of the linearization (11) at v_j. Hence the Morse index of v_j is the dimension of the corresponding unstable manifold,

$$i(v_j) = \dim W^u(v_j) \ . \tag{19}$$

We next illustrate its computation. Given a permutation $\sigma \in \mathcal{S}(n)$ we define the *Morse numbers*

$$i_j(\sigma) := \sum_{k=1}^{j-1} (-1)^{k+1} \operatorname{sign}(\sigma^{-1}(k+1) - \sigma^{-1}(k)) \ , \ 1 \le j \le n \ , \tag{20}$$

where empty sums denote zero. Then, in terms of the Sturm permutation σ_f the Morse indices are explicitly given by $i(v_j) = i_j(\sigma_f)$, see again [15, 11].

A meander permutation $\sigma \in \mathcal{S}(n)$ is a Sturm permutation if there exists a nonlinearity $f \in \text{Sturm}(x,u,u_x)$ such that $\sigma = \sigma_f$. Obviously this places restrictions on the set of Sturm permutations for which we give a purely combinatorial characterization. For this purpose we introduce the following definitions: A meander permutation $\sigma \in \mathcal{S}(n)$ is called *dissipative* if n is odd and σ satisfies:

$$\sigma(1) = 1\ ,\ \sigma(n) = n\ ; \tag{21}$$

Furthermore, σ is called *Morse* if all its Morse numbers (20) are non-negative,

$$i_j(\sigma) \geq 0\ ,\ 1 \leq j \leq n\ . \tag{22}$$

Then we have the following combinatorial characterization of Sturm permutations:

Theorem 1 ([12], Theorem 1.2). *A permutation $\sigma \in \mathcal{S}(n)$ is a Sturm permutation $\sigma = \sigma_f$ in the Sturm class $f \in \mathrm{Sturm}(x, u, u_x)$ if, and only if, σ is a dissipative Morse meander permutation.*

Most remarkably the heteroclinic orbits in a Sturm attractor $\mathcal{A}_f$ are also obtained from the Sturm permutation σ_f. In fact it is possible to determine solely in terms of σ_f if two equilibria in $\mathcal{E}_f$ are connected by a heteroclinic orbit or not. This result is one of the major successes of the theory and closed a long-standing open problem that involved the research of many authors. For details and references see [11, 36].

3. A model problem

The special case of the Chafee-Infante problem [8], which considers equation (1) for the family of cubic nonlinearities

$$f_\lambda(u) = \lambda u - u^3\ ,\ \lambda \in \mathbb{R}\ , \tag{23}$$

was the first example where the Sturm attractor $\mathcal{A} = \mathcal{A}_\lambda$ was characterized (see [20]). This model problem has always been a source of motivation for the development of the theory and still provides a good show case where many details can be explicitly computed. For example, the shooting curve $S = \Gamma(\lambda)$ for this family of nonlinearities can be expressed in terms of elliptic integrals (see e.g. [18]). The case corresponding to a parameter value $1 < \lambda < 4$ is illustrated in Figure 3.

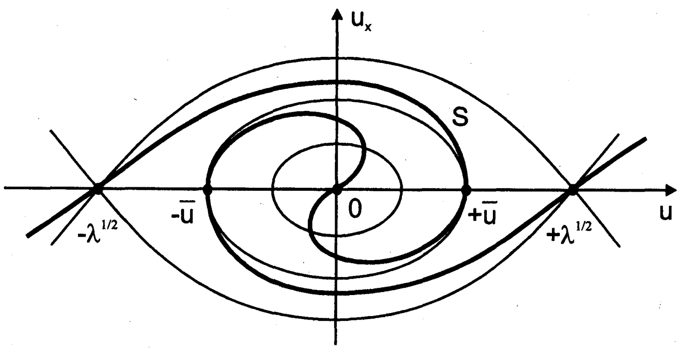

FIGURE 3. Shooting curve $S = \Gamma(\lambda)$ for the Chafee-Infante problem (24) with $1 < \lambda < 4$.

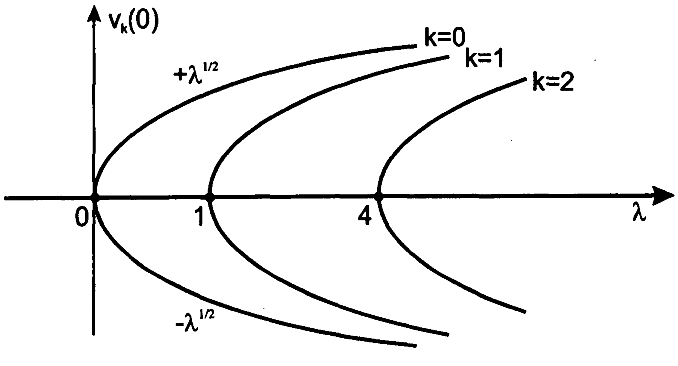

FIGURE 4. Bifurcation diagram for the Chafee-Infante equilibria.

¿From the analysis of the curve $S = \Gamma(\lambda)$ one immediately obtains relevant information on the set of equilibria $\mathcal{E}_\lambda \subset \mathcal{A}_\lambda$ for our model problem

$$\begin{aligned} u_t &= u_{xx} + \lambda u - u^3 \ , \ 0 < x < \pi \ , \\ u_x(t,0) &= u_x(t,\pi) = 0 \ . \end{aligned} \tag{24}$$

The diagram representing the set of equilibria $\mathcal{E}_\lambda$ as a function of $\lambda \in \mathbb{R}$ is known as the bifurcation diagram of the equilibria and is illustrated in Figure 4. For $\lambda = k^2$, with $k = 0, 1, \ldots$, the equilibrium $u \equiv 0$ corresponding to the origin in X is not hyperbolic. Moreover, as λ increases across a squared integer the origin undergoes a supercritical pitchfork bifurcation and two hyperbolic nonhomogeneous solutions of (24) arise from the trivial solution $u \equiv 0$. Then, for $\lambda \neq k^2, k = 0, 1, \ldots$, all the equilibria in $\mathcal{E}_\lambda$ are hyperbolic and the nonlinearities (23) belong to the special Sturm class $f_\lambda \in \text{Sturm}(u)$.

The Sturm permutations σ_λ for the Chafee-Infante problem (24) are easily obtained from the meanders $\Gamma(\lambda)$. In fact, $\mathcal{E}_\lambda$ has exactly $n = 2k + 1$ equilibria for $k^2 < \lambda < (k+1)^2$ and we have

$$\sigma_\lambda = \{1, 2k, 3, 2k-2, \ldots, 2, 2k+1\} = (2\ 2k)(4\ 2k-2)\ldots(\underline{k}\ \overline{k}) \ , \tag{25}$$

where $\overline{k} = 2k + 2 - \underline{k}$ and $\underline{k}$ is the largest even integer not exceeding k.

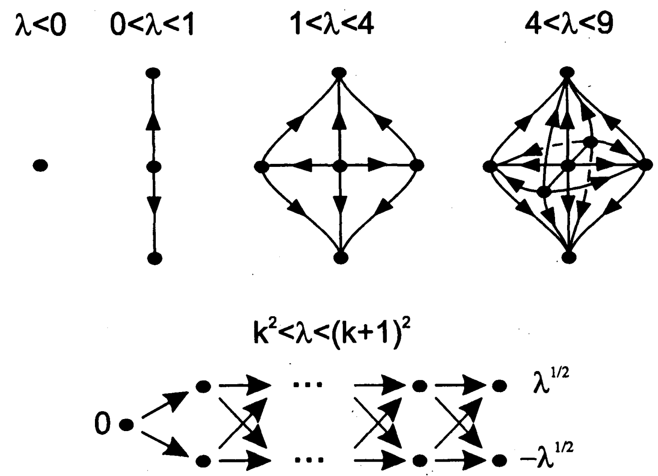

FIGURE 5. Chafee-Infante Sturm attractors $\mathcal{A}_\lambda$.

The heteroclinic orbit connections of the Chafee-Infante Sturm attractors $\mathcal{A}_\lambda$ are then obtained from the Sturm permutations $\sigma_\lambda \in \mathcal{S}(n)$. Figure 5 illustrates some graphic representations of these Sturm attractors $\mathcal{A}_\lambda$.

$$
\begin{aligned}
n &= 1 : \\
\sigma_{1,1} &= \{1\} = \mathrm{id} \; ; \\[1em]
n &= 3 : \\
\sigma_{3,1} &= \{1,2,3\} = \mathrm{id} \; ; \\[1em]
n &= 5 : \\
\sigma_{5,1} &= \{1,2,3,4,5\} = \mathrm{id} \; ; \\
\sigma_{5,2} &= \{1,4,3,2,5\} = (2\ 4) \; ; \\[1em]
n &= 7 : \\
\sigma_{7,1} &= \{1,2,3,4,5,6,7\} = \mathrm{id} \; ; \\
\sigma_{7,2} &= \{1,2,3,6,5,4,7\} = (4\ 6) \; ; \\
\sigma_{7,3} &= \{1,4,5,6,3,2,7\} = (2\ 4\ 6)(3\ 5) \; ; \\
\sigma_{7,4} &= \{1,6,3,4,5,2,7\} = (2\ 6) \; ; \\
\sigma_{7,5} &= \{1,6,5,4,3,2,7\} = (2\ 6)(3\ 5) \; ;
\end{aligned}
$$

TABLE 1. List of all Sturm permutations $\sigma_{n,k} \in \mathcal{S}(n)$ with $n \leq 7$.

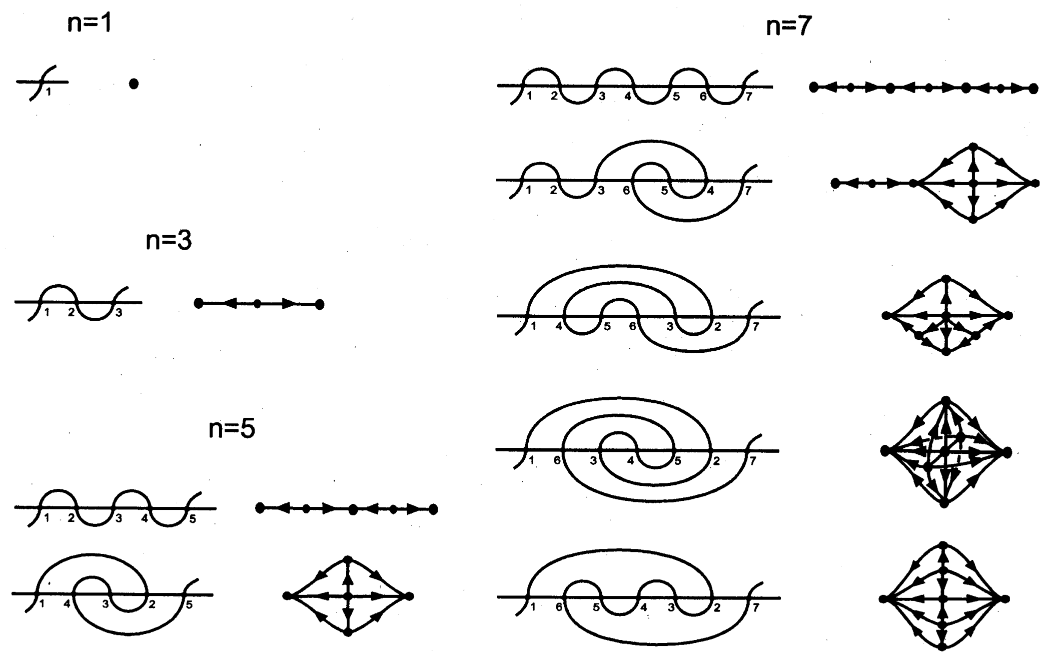

FIGURE 6. List of stylized meanders corresponding to the Sturm permutations $\sigma_{n,k} \in \mathcal{S}(n)$ (left) and the Sturm attractors $\mathcal{A}_{n,k}$ (right) with $n \leq 7$ equilibria.

¿From the characterization of Sturm permutations provided by Theorem 1 we also obtain the complete list of Sturm permutations $\sigma_f \in \mathcal{S}(n)$. In Table

1 we list all the Sturm permutations with $n \leq 7$ up to trivial equivalence. Moreover, for completeness we also show in Figure 6 stylized illustrations of the corresponding meanders Γ_f and graphic representations of the Sturm attractors $\mathcal{A}_f$.

4. Hamiltonian type

In the restricted class $f \in \text{Sturm}(u)$ already considered in the previous section, viz. the Chafee-Infante problem, the stationary problem (9) for the equilibria of (1) has the form

$$v'' + f(v) = 0 . \tag{26}$$

This nonlinear pendulum equation corresponds to the Hamiltonian planar system

$$v' = p \ , \ p' = -f(v) \ , \tag{27}$$

with Hamiltonian function $H(v,p) = \frac{1}{2}p^2 + F(v)$, where F is a potential satisfying $dF(u)/du = f(u)$. For this reason, when $f \in \text{Sturm}(u)$ we say that the Sturm attractor $\mathcal{A}_f$ is of *Hamiltonian type.*

In this section we present the characterization of Sturm attractors $\mathcal{A}_f$ of Hamiltonian type. This characterization has obvious important implications for the strict modeling question concerning spatial inhomogeneity or existence of drifting terms. Moreover, it is also useful for the characterization of Sturm attractors in the case of periodic boundary conditions. We postpone to the last section a demonstration of the implications for this case.

Let then $f \in \text{Sturm}(u)$. Due to the reversibility of (26) with respect to the reflection $x \mapsto -x$, any equilibrium solution $v_j \in \mathcal{E}_f$ can be extended to $-\infty < x < \infty$ by reflection through the boundaries. With this extension we obtain from each $v_j(x)$ a periodic solution of (26) with possibly non-minimal period 2π. Conversely, from any 2π-periodic solution of (26) after an appropriate x-shift we obtain an equilibrium $v_j(x)$ of (1). This correspondence between equilibria $v_j \in \mathcal{E}_f$ of (1) and 2π-periodic solutions of (26) is the key to the characterization of the Sturm attractors $\mathcal{A}_f$ of Hamiltonian type. For details see [14].

In fact, the Sturm permutation σ_f necessarily reflects the nesting structure of the 2π-periodic orbits of (1) in the phase plane (v, v'). Figure 7 illustrates a phase portrait obtained from a potential F with five stationary points. Each stationary point of F is a zero of $f = F'$, hence the Sturm attractor $\mathcal{A}_f$ possesses exactly $m = 5$ spatially homogeneous equilibria. The phase portrait also illustrates the bounded regions of periodic orbits. In particular it shows two punctured disk regions, $\mathcal{C}_2, \mathcal{C}_3$, and one annular region $\mathcal{C}_1$. These are the regions where the 2π-periodic orbits of (27) are to be found. Moreover, recalling the Neumann boundary conditions, for each 2π-periodic orbit in the phase portrait of (27) we obtain two nonhomogeneous equilibrium solutions of (1). Therefore, if the phase portrait obtained from $f \in \text{Sturm}(u)$ has m fixed points and q 2π-periodic orbits, the corresponding Sturm attractor $\mathcal{A}_f$ has exactly $n = m + 2q$ equilibria.

The standard way of studying the 2π-periodic orbits of (26) uses the *period map* $T_f = T_f(a)$, which is defined as the minimal period of the solution

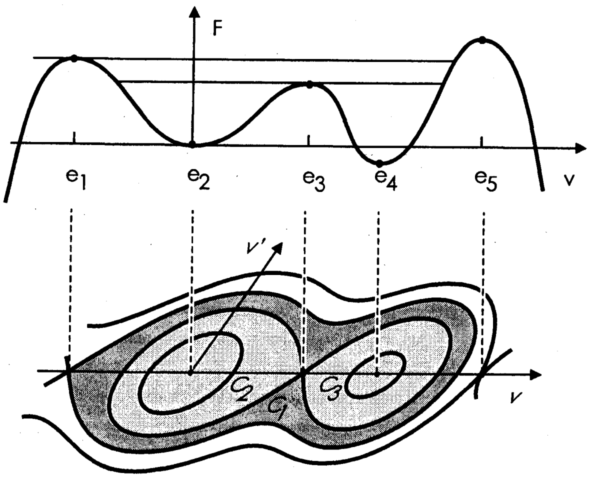

FIGURE 7. Phase portrait corresponding to a potential F with five stationary points, e_1–e_5. The regions $\mathcal{C}_1$–$\mathcal{C}_3$ of periodic orbits are shaded.

$v = v(x,a)$ of (26) with $v(T_f(a),a) = v(0,a) = a$. In its domain $D \subset \mathbb{R}$ of definition $T_f : D \to \mathbb{R}$ is given by

$$T_f(a) = \sqrt{2}\int_a^{b(a)} [F(a) - F(s)]^{-1/2}\, ds\ . \tag{28}$$

This period map has been widely used to characterize the solutions of boundary value problems of the form (26), see e.g. [35, 33, 32] and their references. It was also used in [30] to characterize the set of 2π-periodic orbits of (26).

Using the relation between the Neumann boundary value problem (9) with $f = f(u)$ and the phase portrait of (27) we obtain a characterization of the Sturm permutations σ_f for $f \in \text{Sturm}(u)$. The first immediate conclusion is that σ_f in this class must be an involution. In fact, consider the braid of equilibria associated to the Sturm permutation $\sigma = \sigma_f \in \mathcal{S}(n)$. If we perform the change of variables $x \to \pi - x$ we obtain a new problem for which the corresponding braid of equilibria runs from $x = \pi$ to $x = 0$. Hence the corresponding Sturm permutation is $\sigma^{-1} \in \mathcal{S}(n)$. On the other hand, due to the reversibility of (26) our problem is invariant under the given change of variables. Therefore we conclude that σ is an involution:

$$\sigma = \sigma^{-1} \qquad \text{or} \qquad \sigma^2 := \sigma \circ \sigma = \text{id}\ . \tag{29}$$

As a result of this involutive property the Sturm permutations $\sigma = \sigma_f$ in the restricted Hamiltonian class $f \in \text{Sturm}(u)$ have unique combinatorial representations as products of 2-cycles:

$$\sigma = (\underline{c}_1\ \overline{c}_1)\dots(\underline{c}_r\ \overline{c}_r)\ . \tag{30}$$

Recalling our list of Sturm permutations $\sigma_{n,k} \in \mathcal{S}(n)$ with $n \leq 7$ in Table 1, we observe that $\sigma_{7,3}$ is not an involution and hence cannot be realized in the Hamiltonian class $f \in \text{Sturm}(u)$. All the other permutations in this list are involutions and, in fact, can be realized in the Hamiltonian class.

However, due to the topological restrictions that arise from the nesting structure of 2π-periodic orbits in the phase plane, not all Sturm involutions can be realized in the class of $f \in \text{Sturm}(u)$. In fact, σ_f must also respect this nesting structure. The topological phase plane restrictions are essentially the following (for more specific details see [14]):

(a) different 2π-periodic orbits cannot intersect;
(b) on each connected region of periodic orbits $\mathcal{C}_j$, the 2π-periodic orbits are nested with a total ordering;
(c) the boundaries of the connected regions $\mathcal{C}_j$ contain saddle points of the ODE planar system (27).

For the characterization of the Hamiltonian class $f \in \mathrm{Sturm}(u)$ in purely combinatorial terms we introduce some definitions. Recall the Morse numbers (20) for a given permutation $\sigma \in \mathcal{S}(n)$. Then the point k is called *σ-stable* if

$$i_k(\sigma) = 0 \ . \tag{31}$$

Moreover, if $\sigma \in \mathcal{S}(n)$ is an involution, having in mind the cycle representation (30) we define the *σ-stable core* C_α of the 2-cycle $(\underline{c}_\alpha \ \overline{c}_\alpha)$ as the set

$$C_\alpha = \{k : i_k(\sigma) = 0, \ \underline{c}_\alpha < k < \overline{c}_\alpha\} \ . \tag{32}$$

We remark that, for $f \in \mathrm{Sturm}(u)$, the ODE saddle points of (27) are in one-to-one correspondence with the stable (and hence homogeneous) equilibria $v_j \in \mathcal{E}_f$ of the PDE problem (1). For details we refer again to [14].

In view of the topological restrictions (a–c) the integer boundary labels of the 2-cycles composing the involution $\sigma = \sigma_f$ have to satisfy some conditions which we condense in the following additional definitions. For $\alpha \neq \beta$ we say that the 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are *intersecting* if the corresponding open intervals in $\mathbb{R}$ have a nonempty intersection

$$(\underline{c}_\alpha \, , \overline{c}_\alpha) \cap (\underline{c}_\beta \, , \overline{c}_\beta) \neq \emptyset \ ; \tag{33}$$

Intersecting 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are called *nested* if one of these intervals contains the other, i.e. if

$$(\underline{c}_\beta - \underline{c}_\alpha)(\overline{c}_\alpha - \overline{c}_\beta) > 0 \ ; \tag{34}$$

Nested 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are called *centered* if the mid-points of these intervals coincide, i.e. if

$$\underline{c}_\beta - \underline{c}_\alpha = \overline{c}_\alpha - \overline{c}_\beta \ . \tag{35}$$

We also say that two nested 2-cycles $(\underline{c}_\alpha \ \overline{c}_\alpha)$ and $(\underline{c}_\beta \ \overline{c}_\beta)$ are *core-equivalent* if they share the same σ-stable core, $C_\alpha = C_\beta$.

Finally, we reach the central definition for the permutation characterization of Sturm attractors $\mathcal{A}_f$ of Hamiltonian type. An involution permutation $\sigma \in \mathcal{S}(n)$ is called *integrable* if:

(I.1) intersecting 2-cycles are nested;
(I.2) core-equivalent 2-cycles are centered;
(I.3) non-nested 2-cycles are separated by σ-stable points.

Then, we have the following characterization of Sturm permutations in the Hamiltonian class of $f \in \mathrm{Sturm}(u)$.

Theorem 2 ([14], Theorem 1). *A Sturm permutation $\sigma \in \mathcal{S}(n)$ is in the Hamiltonian class of $f \in$ Sturm(u) if and only if σ is an integrable involution.*

The necessity of the involutive condition is asserted by (29). Then the topological restrictions imposed by the phase plane nesting of the 2π-periodic orbits show the necessity of the integrability condition. On the other hand, the sufficiency of both conditions for a realization of σ in the Hamiltonian class is much more difficult to establish. This essentially follows from the characterization presented in [30] of the period maps for planar Hamiltonian systems of the form (26).

Theorem 2 precisely identifies the Sturm permutations which correspond to Sturm attractors of Hamiltonian type. Using this theorem we can, in fact, ensure that all the Sturm involutions listed in Table 1 can be realized in the Hamiltonian class of $f \in \text{Sturm}(u)$. This excluded only the non-involutive Sturm permutation $\sigma_{7,3}$.

In the case of the Sturm permutations $\sigma_{n,k} \in \mathcal{S}(n)$ with $n = 9$, all listed in Table 2 up to trivial equivalences, the integrable involutions are also easily identified. From the 18 Sturm permutations with $n = 9$ equilibria we exclude

$$\sigma_{9,4} \ , \ \sigma_{9,8} \ , \ \sigma_{9,9} \ , \ \sigma_{9,10} \ , \ \sigma_{9,12} \ , \ \sigma_{9,16} \tag{36}$$

which clearly are not involutions. Moreover, we also exclude

$$\sigma_{9,11} \ , \ \sigma_{9,14} \tag{37}$$

which are not integrable. In fact, the 2-cycles of $\sigma_{9,11}$ are intersecting but not nested, and the 2-cycles of $\sigma_{9,14}$ are core-equivalent but not centered. The remaining 10 Sturm permutations then correspond to Sturm attractors of Hamiltonian type.

$$\begin{array}{lll}
\sigma_{9,1} = \{1,2,3,4,5,6,7,8,9\} = \text{id} \ ; & \\
\sigma_{9,2} = \{1,2,3,4,5,8,7,6,9\} = (6\ 8) \ ; & \\
\sigma_{9,3} = \{1,2,3,6,5,4,7,8,9\} = (4\ 6) \ ; & \\
\sigma_{9,4} = \{1,2,3,6,7,8,5,4,9\} = (4\ 6\ 8)(5\ 7) \ ; & * \\
\sigma_{9,5} = \{1,2,3,8,5,6,7,4,9\} = (4\ 8) \ ; & \\
\sigma_{9,6} = \{1,2,3,8,7,6,5,4,9\} = (4\ 8)(5\ 7) \ ; & \\
\sigma_{9,7} = \{1,4,3,2,5,8,7,6,9\} = (2\ 4)(6\ 8) \ ; & \\
\sigma_{9,8} = \{1,4,5,6,7,8,3,2,9\} = (2\ 4\ 6\ 8)(3\ 5\ 7) \ ; & * \\
\sigma_{9,9} = \{1,4,5,8,7,6,3,2,9\} = (2\ 4\ 8)(3\ 5\ 7) \ ; & * \\
\sigma_{9,10} = \{1,6,7,8,3,4,5,2,9\} = (2\ 6\ 4\ 8)(3\ 7\ 5) \ ; & * \\
\sigma_{9,11} = \{1,6,7,8,5,2,3,4,9\} = (2\ 6)(3\ 7)(4\ 8) \ ; & * \\
\sigma_{9,12} = \{1,6,7,8,5,4,3,2,9\} = (2\ 6\ 4\ 8)(3\ 7) \ ; & * \\
\sigma_{9,13} = \{1,8,3,4,5,6,7,2,9\} = (2\ 8) \ ; & \\
\sigma_{9,14} = \{1,8,3,4,7,6,5,2,9\} = (2\ 8)(5\ 7) \ ; & * \\
\sigma_{9,15} = \{1,8,3,6,5,4,7,2,9\} = (2\ 8)(4\ 6) \ ; & \\
\sigma_{9,16} = \{1,8,5,6,7,4,3,2,9\} = (2\ 8)(3\ 5\ 7)(4\ 6) \ ; & * \\
\sigma_{9,17} = \{1,8,7,4,5,6,3,2,9\} = (2\ 8)(3\ 7) \ ; & \\
\sigma_{9,18} = \{1,8,7,6,5,4,3,2,9\} = (2\ 8)(3\ 7)(4\ 6) \ ; &
\end{array}$$

TABLE 2. List of all Sturm permutations $\sigma_{n,k} \in \mathcal{S}(n)$ with $n = 9$. Permutations marked with $*$ are not integrable involutions.

It is shown in [14] that the characterization of Sturm permutations in the Hamiltonian class of $f \in \text{Sturm}(u)$ contained in Theorem 2 also holds in the larger Sturm class of reversible $f \in \text{Sturm}(u, u_x)$. This is the class of nonlinearities $f \in \text{Sturm}(u, u_x)$ which satisfy

$$f(v, -p) = f(v, p) \ . \tag{38}$$

For f in this class – which contains Sturm(u) – the ODE corresponding to the stationary problem (9) has the form

$$v'' + f(v, v') = 0 \ . \tag{39}$$

By (38), this equation is reversible with respect to the change of variables $x \to -x$, like in the case of (26). Moreover, (39) is also integrable and there exists a period map $T = T_f : D \to \mathbb{R}$, which extends to (39) the period map introduced for (26). For more information see [13, 29].

Then, using this period map and arguing as in the case of $f \in \text{Sturm}(u)$, we obtain the following extension of Theorem 2; see [14] for details.

Theorem 2′. *A Sturm permutation $\sigma \in \mathcal{S}(n)$ is in the class of reversible $f \in \text{Sturm}(u, u_x)$ if and only if σ is an integrable involution.*

5. Periodic boundary conditions

We finally address the case of periodic boundary conditions. Consider then solutions $u(t, x)$ of equation (1) defined on the interval $x \in I = [0, 2\pi]$ and satisfying

$$u(t, 0) = u(t, 2\pi) \ , \ u_x(t, 0) = u_x(t, 2\pi) \ , \tag{40}$$

or, equivalently, defined on the circle $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In this case, under the dissipative conditions (2), equation (1) generates a dissipative semiflow $S_f(t) : X \to X, t \geq 0$, in the phase space $X = H^2(S^1)$. However, in general, $S_f(t)$ is no longer gradient-like. In fact, in this case the global attractor $\mathcal{A}_f^p$ may contain periodic orbits, or homoclinic orbits between equilibria. For examples of such behavior see [5, 10] and [31].

On the other hand the Poincaré-Bendixson property holds in this case: the ω-limit set of any solution $u(t) = u(t, \cdot)$ of (1), (40), either contains an equilibrium point, or is a periodic orbit; see [10] for details.

Let $p(t) = p(t, \cdot)$ denote a periodic orbit of $S_f(t)$ with period $\tau > 0$ and initial value $p(0) = p_0 \in X$. The *characteristic multipliers* of $p(t)$ are the eigenvalues of the evolution operator $P_\tau : X \to X$ defined by the linearization of (1), (40), around the periodic orbit $p(t)$, at time $t = \tau$;

$$P_\tau = DS_f(\tau)p_0 \ . \tag{41}$$

To require nondegeneracy of a periodic orbit we assume hyperbolicity as in the case of equilibria. In this case the periodic orbit $p(t)$ is *hyperbolic* if $\mu = 1$ is a simple eigenvalue of P_τ and is the unique characteristic multiplier of $p(t)$ on the complex unit circle. For more details see, e.g., [19, 9].

To distinguish from the previous case we introduce the Sturm class

$$\text{Sturm}^p(x, u, u_x) \tag{42}$$

to denote the set of nonlinearities $f = f(x, u, u_x)$ satisfying (2), for which all equilibria and periodic orbits of (1), (40), are hyperbolic.

The characterization of Sturm attractors $\mathcal{A}_f^p$ for $f \in \text{Sturm}^p(x,u,u_x)$ is essentially an open problem. A most remarkable general result available in this case is the following transversality property of the stable and unstable manifolds of hyperbolic periodic orbits:

Theorem 3 ([9], Theorem 8.2). *The stable and unstable manifolds of two hyperbolic periodic orbits $p^\pm$ of (1), (40), always intersect transversely,*

$$W^u(p^-) \pitchfork W^s(p^+) . \tag{43}$$

For details and further results on the geometric properties of $\mathcal{A}_f^p$ in the Sturm class of $f \in \text{Sturm}^p(x,u,u_x)$ see [9, 21, 22] and their references.

In contrast with the general case, the geometry of Sturm attractors $\mathcal{A}_f^p$ in the restricted Sturm class of

$$f \in \text{Sturm}^p(u,u_x) \tag{44}$$

is much better understood. Therefore we turn to the problem

$$u_t = u_{xx} + f(u,u_x) \ , \ x \in S^1 \ , \tag{45}$$

for nonlinearities f satisfying (2). In this case, due to the S^1-equivariance of (45) with respect to x-shifts any periodic solution has the form $u(t,x) = v(x-ct)$, that is, a *rotating wave* which rotates around the circle $x \in S^1$ with constant speed $c \neq 0$. Moreover, as $t \to +\infty$ any solution of (45) either approaches a (hyperbolic) equilibrium solution or a (hyperbolic) rotating wave. These results again follow from the Sturm property (8), as shown in [5, 25, 26]. In addition, homoclinic behavior to equilibria or periodic orbits is excluded; see [26]. Therefore, denoting by $\mathcal{R} = \mathcal{R}_f$ the set of rotating wave solutions of (45), the Sturm attractor $\mathcal{A}^p = \mathcal{A}_f^p$ for $f \in \text{Sturm}^p(u,u_x)$ decomposes as

$$\mathcal{A}^p = \mathcal{E} \cup \mathcal{R} \cup \mathcal{H} \ . \tag{46}$$

Below we will see that hyperbolicity, in addition to implying finiteness of the set $\mathcal{E}_f$ of equilibria, also implies finiteness of the set $\mathcal{R}_f$ of rotating waves.

We sidetrack to remark that in the general case of $f \in \text{Sturm}^p(x,u,u_x)$ homoclinic behavior to periodic orbits is also excluded, as shown in [27, 9], in blatant contrast with homoclinic behavior to equilibria which may occur; see [31].

Another important observation regarding the geometric properties of $\mathcal{A}_f^p$ in the S^1-equivariant case of $f \in \text{Sturm}^p(u,u_x)$ is the following:

Theorem 4 ([13], Proposition 3.2). *The stable and unstable manifolds of two hyperbolic elements $v^\pm \in \mathcal{E} \cup \mathcal{R}$ of (45), always intersect transversely,*

$$W^u(v^-) \pitchfork W^s(v^+) . \tag{47}$$

This extends the transversality result of Theorem 3 to all the hyperbolic elements of $\mathcal{E} \cup \mathcal{R}$ and shows that the Sturm attractor $\mathcal{A}_f^p$ for $f \in \text{Sturm}^p(u,u_x)$ has the *Morse-Smale property*; see for example [17].

This property, after a slight extension to include normal hyperbolicity, is used in [13] to determine the heteroclinic orbit connections between any pair of hyperbolic equilibria or rotating waves. For a hyperbolicity preserving homotopy $f_s, s \in [0,1]$, we obtain a continuous family of global attractors $\mathcal{A}_s^p$ which, due to the Morse-Smale property, is s-invariant up to orbit equivalence by a homeomorphism. Therefore, the heteroclinic connections in the Sturm attractor $\mathcal{A}_0^p$ are determined using a homotopy f_s taking $\mathcal{A}_0^p$ to a simpler Sturm attractor $\mathcal{A}_1^p$ for which the heteroclinic orbits are known. For the details on the homotopy construction and the heteroclinic connections see [13].

We proceed with the characterization of Sturm attractors $\mathcal{A}_f^p$ for $f \in \mathrm{Sturm}^p(u, u_x)$. As in the Neumann case of Hamiltonian type, $f \in \mathrm{Sturm}(u)$, the characterization of $\mathcal{A}_f^p$ is pursued in terms of a permutation obtained from a period map. In fact, all the required information on the PDE dynamics is encoded in ODE boundary value problems associated to the solutions $v \in \mathcal{E}_f \cup \mathcal{R}_f$, that is, the stationary equilibria and the rotating wave solutions. It turns out that the period map approach is quite appropriate to solve these boundary value problems.

Taking a unified approach, a solution $v = v(t,x)$ of (45) is an equilibrium or rotating wave, $v \in \mathcal{E}_f \cup \mathcal{R}_f$, if and only if there exists $c \in \mathbb{R}$ such that v satisfies the periodic boundary value problem

$$v'' + cv' + f(v, v') = 0 \ , \ \ x \in S^1 \ . \tag{48}$$

Then rotating waves $v \in \mathcal{R}_f$ correspond to solutions $u(t,x) = v(x - ct)$ with a rotation speed $c \neq 0$, and equilibria $v \in \mathcal{E}_f$ correspond to solutions with $c = 0$.

The set of equilibria $v \in \mathcal{E}_f$ is, in general, composed of two types of solutions: the spatially homogeneous solutions, $v \equiv e$, corresponding to the zeros of $f(\cdot, 0)$,

$$f(e, 0) = 0 \ ; \tag{49}$$

and the spatially nonhomogeneous solutions, $v = v(x)$ with $v' \not\equiv 0$.

We remark that spatially nonhomogeneous equilibrium solutions are always nonhyperbolic. Indeed, in this case v_x is an eigenfunction for the trivial eigenvalue $\lambda = 0$ of the linearization (11) with periodic boundary conditions. Due to S^1-equivariance these solutions always occur in families of x-shifted copies around $x \in S^1$. Therefore, this type of solutions is absent when $f \in \mathrm{Sturm}^p(u, u_x)$. When such solutions are present they can be considered as *frozen* or *standing waves* "rotating" around the circle $x \in S^1$ with zero speed, $c = 0$. This case can also be treated by requiring these solutions to be *normally hyperbolic*, i.e. with a simple trivial eigenvalue; see [13] for details.

Then to define the appropriate period map we consider the planar system corresponding to the rotating wave equation (48)

$$v' = p \ , \ \ p' = -f(v, p) - cp \ . \tag{50}$$

For the set $\mathcal{C} \subset \mathbb{R}^2$ of initial conditions (v, p) for which there is $c \in \mathbb{R}$ such that (v, p) is a periodic point of (50), we define:

(i) the unique wave speed $c = c(v, p)$ such that (v, p) is a periodic point;
(ii) the minimal period $T_f = T_f(v, p)$ of the periodic orbit through (v, p).

Then $T_f = T_f(v,p)$ is our *period map* for the permutation characterization of the Sturm attractor $\mathcal{A}_f^p$ for $f \in \mathrm{Sturm}^p(u,u_x)$.

First, all spatially nonhomogeneous equilibria $v \in \mathcal{E}_f$ are directly obtained from (49). Then, all rotating waves $v \in \mathcal{R}_f$ are obtained from T_f. A periodic point (v,p) of (50) is the initial value of a 2π-periodic orbit if and only if

$$T_f(v,p) = 2\pi/k \ , \ k \in \mathbb{N} \ . \tag{51}$$

Subsequently $c = c(v,p)$ determines if the orbit solution $(v(x), v'(x))$ corresponds to a rotating or a frozen wave of (45). If $c \neq 0$ then $v = v(x-ct)$ is a rotating wave, $v \in \mathcal{R}_f$. In addition, hyperbolicity of $v \in \mathcal{R}_f$ is equivalent to noncriticality of the corresponding value $2\pi/k$ for T_f. This result is entirely analogous to the one established for the usual period map defined by (28); see again [13, 33]. Since minimal periods are uniformly bounded from below [1], we conclude that hyperbolicity of all rotating waves $v \in \mathcal{R}_f$ implies finiteness of the set $\mathcal{R}_f$.

By restricting the period map $T_f = T_f(v,p)$ to the set $D_{\mathcal{N}}$ of initial Neumann conditions,

$$D_{\mathcal{N}} := \mathcal{C} \cap \{(a,0) : a \in \mathbb{R}\} \ , \tag{52}$$

we obtain a map $T_f : D_{\mathcal{N}} \to \mathbb{R}$, like in the case of Neumann boundary conditions. ¿From this map we then determine a Sturm permutation σ_f which characterizes the Sturm attractor $\mathcal{A}_f^p$ for $f \in \mathrm{Sturm}^p(u,u_x)$.

To obtain this characterization of $\mathcal{A}_f^p$ we use two homotopies $f_s, s \in [0,1]$, constructed in [13], which leave the period map T_f invariant.

The first is a *freezing homotopy* which preserves all rotating waves of (45) reducing their wave speeds to $c \equiv 0$ while preserving the period map T_f. ¿From this homotopy we obtain a nonlinearity $h = h(u,u_x)$ for which all the rotating waves of (45) become spatially nonhomogeneous equilibria, i.e. frozen waves. Hence h is not in $\mathrm{Sturm}^p(u,u_x)$, but all the equilibria are normally hyperbolic and $\mathcal{R}_h$ is empty, $\mathcal{A}_h^p = \mathcal{E}_h \cup \mathcal{H}_h$.

The second is a *symmetrizing homotopy* which, while preserving the period map T_f, symmetrizes the phase portrait of the planar system for the frozen $h = h(u,u_x)$. ¿From this homotopy we obtain a nonlinearity $g = g(u,u_x)$ which is even in u_x, that is

$$g(v,-p) = g(v,p) \ , \tag{53}$$

and since the period map is preserved we obtain

$$T_g(a,0) = T_f(a,0) \ . \tag{54}$$

Due to (53) the resulting problem

$$u_t = u_{xx} + g(u,u_x) \ , \ x \in S^1 \ , \tag{55}$$

is then equivariant also with respect to the reflection symmetry $x \mapsto -x$. Hence any equilibrium solution is reflection symmetric with respect to any of its local maxima or minima. In the case of frozen waves, choosing the appropriate representatives of each family of x-shifted copies we obtain two equilibrium solutions with a maximum and a minimum at $x = 0$, respectively. These solutions satisfy Neumann boundary conditions at $x = 0$ and $x = \pi$ due to reflection symmetry.

The flow $S_g(t)$ generated by (55) has an embedded flow which satisfies Neumann boundary conditions on the half-interval $I = [0, \pi]$. In fact, we have just identified the set of its spatially nonhomogeneous equilibria. These normally hyperbolic solutions, together with the homogeneous equilibria, are hyperbolic for the Neumann flow, and we conclude that $g \in \text{Sturm}(u, u_x)$ on $I = [0, \pi]$. The Sturm attractor for this Neumann flow is then our source of information on the Sturm attractor $\mathcal{A}_f^p$ for the initial problem (45) with $f \in \text{Sturm}^p(u, u_x)$.

We invoke the reversibility (53) to obtain the period map $T = T_g$ which is defined in the class of reversible $g \in \text{Sturm}(u, u_x)$. From this period map we obtain the Sturm permutation $\sigma = \sigma_g$ used in the characterization of the Sturm attractor for the Neumann problem. We recall that by Theorem 2′ the Sturm permutation σ_g is necessarily an integrable involution.

For $f \in \text{Sturm}^p(u, u_x)$ we define the *Sturm permutation* $\sigma = \sigma_f$ obtained from $T_f = T_f(a, 0)$ and notice that we have $\sigma_f = \sigma_g$ as shown by (54).

The Sturm attractor $\mathcal{A}_f^p$ for $f \in \text{Sturm}^p(u, u_x)$ is then characterized by the Sturm permutation $\sigma_f \in \mathcal{S}(n)$. Here again we have

$$n = m + 2q \ , \tag{56}$$

where now m denotes the number of (hyperbolic, spatially homogeneous) equilibria, and q denotes the number of (hyperbolic) rotating waves,

$$m = \#\mathcal{E}_f \ , \ q = \#\mathcal{R}_f \ . \tag{57}$$

To illustrate this characterization of $\mathcal{A}_f^p$ for $f \in \text{Sturm}^p(u, u_x)$ we mention that σ_f determines all the heteroclinic orbit connections in $\mathcal{A}_f^p$. As an example, consider heteroclinic orbits between rotating waves in $\mathcal{A}_f^p$. A pair of periodic orbits is connected by a heteroclinic orbit if, and only if, in the embedded Neumann flow the corresponding pair of frozen waves is connected by a heteroclinic orbit. Then we recall that all the heteroclinic connections in the Sturm attractor $\mathcal{A}_g$ for this Neumann flow are determined by the Sturm permutation $\sigma_g = \sigma_f$.

Since the period map $T_g = T_g(a, 0)$ introduced for (55) is then identical to the period map $T_g = T_g(a)$ for the Neumann problem, in view of (54) we obtain from Theorem 2′ the following characterization of the Sturm permutations for $f \in \text{Sturm}^p(u, u_x)$:

Theorem 5. *A Sturm permutation $\sigma = \sigma_f \in \mathcal{S}(n)$ is in the class of $f \in \text{Sturm}^p(u, u_x)$ if and only if σ is an integrable involution.*

An instructive example of $f \in \text{Sturm}^p(u, u_x)$ is again provided by the Chafee-Infante problem after a suitable adaptation. Consider the problem

$$u_t = u_{xx} + \lambda u - u^3 + c u_x \ , \ x \in S^1 \ . \tag{58}$$

In this case the frozen symmetrized nonlinearity g is obviously the cubic $\lambda u - u^3$, and the Neumann problem is given by (24). Hence the spatially nonhomogeneous Neumann solutions become rotating waves of (58) with rotation speed c. Then, for

$$f(u, u_x) = \lambda u - u^3 + c u_x \tag{59}$$

we obtain $f \in \mathrm{Sturm}^p(u, u_x)$ provided

$$c \neq 0 \quad \text{and } \lambda \neq k^2 \text{ for } k = 0, 1, \ldots \;. \tag{60}$$

When $\lambda \leq 0$ the Sturm attractor $\mathcal{A}_f^p = \mathcal{A}_\lambda^p$ is trivial, i.e. $\mathcal{A}_\lambda^p = \{0\}$, and when $\lambda > 0$ it has exactly three spatially homogeneous equilibrium solutions. Moreover, when $c \neq 0$, the flow $S_f(t)$ generated by (58) undergoes a sequence of supercritical Hopf bifurcations of the trivial solution $u \equiv 0$ at the parameter values $\lambda = k^2$, $k \in \mathbb{N}$. The corresponding Sturm attractors $\mathcal{A}_\lambda^p$ are then easily described.

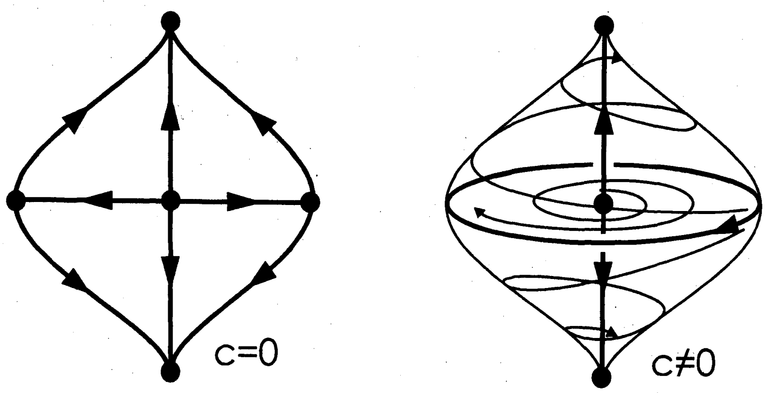

FIGURE 8. The Chafee-Infante spindle attractor for $1 < \lambda < 4$. Left: Chafee-Infante attractor for the Neumann flow section in the frozen case, $c = 0$; Right: Chafee-Infante spindle attractor for $c \neq 0$.

As an illustration we consider the Sturm attractor $\mathcal{A}_\lambda^p$ for $1 < \lambda < 4$. In this case we have $n = 5$, with $m = 3$ and $q = 1$. Moreover, by (25) the Sturm permutation $\sigma = \sigma_\lambda \in \mathcal{S}(5)$ is

$$\sigma_\lambda = \sigma_{5,2} = (2\ 4)\ , \tag{61}$$

The corresponding Sturm attractor $\mathcal{A}_\lambda^p$ is three dimensional and we show a graphical representation in Figure 8. In [23] an entirely similar global attractor occurring for a delay differential equation was appropriately called a *spindle attractor* since it visualizes as a smooth solid spindle.

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