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Diffusive relaxation limit in Besov spaces for damped compressible Euler equations

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1 Introduction

This article is a survey of the paper [20]. We consider the following damped compressible Euler equations

$$\begin{cases} \partial_t\rho+\nabla\cdot(\rho\mathbf{v})=0, \\ \partial_t(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})+\nabla p(\rho)=-\rho\mathbf{v}/\tau \end{cases} \tag{1.1}$$

for $(t,x)\in[0,+\infty)\times\mathbb{R}^d$ with $d\geq 1$. Here $\rho=\rho(t,x)$ is the fluid density function; $\mathbf{v}=(v^1,v^2,\cdots,v^d)^\top$($\top$ represents the transpose) denotes the fluid velocity. The pressure $p(\rho)$ satisfies the classical assumption

$$p'(\rho)>0, \quad \text{for any} \quad \rho>0.$$

A usual simplicity $p(\rho):=\rho^\gamma(\gamma\geq 1)$, where the adiabatic exponent $\gamma>1$ corresponds to the isentropic flow and $\gamma=1$ corresponds to the isothermal flow. The nondimensional number $0<\tau\leq 1$ is a (small) relaxation time. The notation $\nabla,\otimes$ are the gradient operator (in x) and the symbol for the tensor products of two vectors, respectively.

System (1.1) is complemented by the initial conditions

$$(\rho,\mathbf{v})(0,x)=(\rho_0,\mathbf{v}_0). \tag{1.2}$$

The main objective in this paper is to justify the singular limit as $\tau\to 0$ in (1.1) rigorously. To do this, we define the new variables by considering an "$\mathcal{O}(1/\tau)$" time scale as in [15]:

$$(\rho^\tau,\mathbf{v}^\tau)(s,x)=\Big(\rho,\mathbf{v}\Big)\Big(\frac{s}{\tau},x\Big). \tag{1.3}$$

Then

$$\begin{cases} \partial_s \rho^\tau + \nabla \cdot (\frac{\rho^\tau \mathbf{v}^\tau}{\tau}) = 0, \\ \tau^2 \partial_s (\frac{\rho^\tau \mathbf{v}^\tau}{\tau}) + \tau^2 \nabla \cdot (\frac{\rho^\tau \mathbf{v}^\tau \otimes \mathbf{v}^\tau}{\tau^2}) + \frac{\rho^\tau \mathbf{v}^\tau}{\tau} = -\nabla p(\rho^\tau) \end{cases} \tag{1.4}$$

with the initial data

$$(\rho^\tau, \mathbf{v}^\tau)(x, 0) = (\rho_0, \mathbf{v}_0). \tag{1.5}$$

At the formal level, at least, if we assume that $\frac{\rho^\tau \mathbf{v}^\tau}{\tau}$ is uniformly bounded, it will be shown that the limit $\mathcal{N}$ of ρ^τ as $\tau \to 0$ satisfies the classical porous medium equation

$$\begin{cases} \partial_s \mathcal{N} - \Delta p(\mathcal{N}) = 0, \\ \mathcal{N}(x, 0) = \rho_0, \end{cases} \tag{1.6}$$

which is a parabolic equation since $p(\mathcal{N})$ is strictly increasing.

This singular limit problem has served as a paradigm for the theory of diffusive relaxation [13]. For entropy weak solutions, the paper of Marcati and Milani [12] concerned the pourous media flow as the limit of the Euler equation in 1-D, later generalized by Marcati and Rubino [15] to the multi-D case. Their main analysis tools are the techniques of compensated compactness. Very recently, Lattanzio and Tzavaras [9] gave the convergence to the porous media equation away from vacuum, which is based on a Lyapunov type of functional provided by a calculation of the relative entropy. Junca and Rascle [7] proved the convergence for arbitrarily large $BV(\mathbb{R})$ solution away from vacuum.

For smooth solutions, Coulombel et al. [3, 10] fell back on the energy approach and constructed the (uniform) small smooth solutions to (1.1) pertaining to data in the usual Sobolev spaces $H^s(\mathbb{R}^d)(s > 1+d/2)$, furthermore, the diffusive limit was justified by the standard weak convergence method together with the Aubin-Lions compactness lemma in [16]. Inspired by the Maxwell iteration, the first author [18] obtained the definite convergence order for the diffusive relaxation limit. Lattanzio and Yong [8] studied the diffusion phenomenon when initial layers appear and gave a rigorous characterization, by using the matched expansion approach. These works for smooth solutions fell in the framework of the classical existence theory of Kato and Majda [5, 11]. Recently, the first author and Wang [21] investigated the limit case of regularity index ($s = 1+d/2$) where the basic theory fails and it was justified that the (scaled) density converges to the strong solution of the porous medium equation in the critical Besov space $B^{1+d/2}_{2,1}(\mathbb{R}^d)$.

Note that the works of Coulombel et al., it is a meaningful problem to seek for more general functional spaces such that the Cauchy problem (1.1)-(1.2) is well posed near constant equilibrium and to build a bridge for those relaxation results in both Sobolev spaces with higher regularity and the critical Besov space. Therefore, the Besov space $B^{\alpha}_{2,r}(\mathbb{R}^d)$ seems to be an *optimal* candidate for the motivation, whose indices satisfy the following condition:

$$\alpha > 1 + d/2,\ 1 \le r \le 2 \ \text{ or } \ \alpha = 1 + d/2,\ r = 1. \tag{1.7}$$

Let us sketch the technical obstructions briefly and the strategy to overcome them. Due to the partial damping of (1.1)-(1.2), the dissipation rate for the density produced by the "Shizuta-Kawashima" algebraic condition (see [17]) is absent, which leads to the difficulty in closing the uniform a priori estimates with respect to τ. To overcome this, a recent elementary fact (see Lemma 2.2) will be used, which indicates the connection between homogeneous and inhomogeneous Chemin-Lerner spaces.

On the other hand, to take care of the regularity for the extension, we develop a more general version of commutator estimates (see Proposition 2.2) in comparison with those in [4], which relaxes the restriction on the couple (s, r) and enables us to construct the desired a priori estimate in general Besov spaces.

Denote the functional spaces

$$\widetilde{\mathcal{C}}_T(B^s_{p,r}) := \widetilde{L}^\infty_T(B^s_{p,r}) \cap \mathcal{C}([0,T], B^s_{p,r})$$

and

$$\widetilde{\mathcal{C}}^1_T(B^s_{p,r}) := \{f \in \mathcal{C}^1([0,T], B^s_{p,r}) | \partial_t f \in \widetilde{L}^\infty_T(B^s_{p,r})\},$$

where the index T will be omitted when $T = +\infty$.

Now, we state main results as follows.

Theorem 1.1. *Let the couple (α, r) satisfy the condition (1.7) and let $\bar{\rho} > 0$ be a constant density. There exists a positive constant δ_0 independent of τ such that if*

$$\|(\rho_0 - \bar{\rho}, \mathbf{m}_0)\|_{B^{\alpha}_{2,r}(\mathbb{R}^d)} \le \delta_0$$

with $\mathbf{m}_0 := \rho_0 \mathbf{v}_0$, then the Cauchy problem (1.1)-(1.2) has a unique global classical solution $(\rho, \mathbf{m}) \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^d)$ satisfying $(\rho - \bar{\rho}, \mathbf{m}) \in \widetilde{\mathcal{C}}(B^{\alpha}_{2,r}(\mathbb{R}^d)) \cap \widetilde{\mathcal{C}}^1(B^{\alpha-1}_{2,r}(\mathbb{R}^d))$. Furthermore, the global solutions satisfy the following energy *inequality*

$$\begin{aligned}
&\|(\rho-\bar{\rho},\mathbf{m})\|_{\widetilde{L}^\infty(B^\alpha_{2,r}(\mathbb{R}^d))}\\
&+\mu_0\Big(\Big\|\frac{\mathbf{m}}{\sqrt{\tau}}\Big\|_{\widetilde{L}^2(B^\alpha_{2,r}(\mathbb{R}^d))}+\Big\|\sqrt{\tau}(\nabla\rho,\nabla\mathbf{m})\Big\|_{\widetilde{L}^2(B^{\alpha-1}_{2,r}(\mathbb{R}^d))}\Big)\\
\leq\ & C_0\|(\rho_0-\bar{\rho},\mathbf{m}_0)\|_{B^\alpha_{2,r}(\mathbb{R}^d)},
\end{aligned}\tag{1.8}$$

where $\mathbf{m}:=\rho\mathbf{v}$, μ_0 *and* C_0 *are some uniform positive constants independent of* $\tau(0<\tau\leq 1)$.

As a direct consequence, we obtain the diffusive relaxation limit towards to the porous media equation.

Theorem 1.2. *Let* $(\rho,\mathbf{m})$ *be the global solution of Theorem 1.1. Then it yields*

$$\begin{aligned}
&\rho^\tau-\bar{\rho}\quad \textit{is uniformly bounded in}\ \ \mathcal{C}(\mathbb{R}^+,B^\alpha_{2,r}(\mathbb{R}^d));\\
&\frac{\rho^\tau\mathbf{v}^\tau}{\tau}\quad \textit{is uniformly bounded in}\ \ L^2(\mathbb{R}^+,B^\alpha_{2,r}(\mathbb{R}^d)).
\end{aligned}$$

Furthermore, there exists some function $\mathcal{N}\in\mathcal{C}(\mathbb{R}^+,\bar{n}+B^\alpha_{2,r}(\mathbb{R}^d))$ *which is a unique solution of (1.6). For any* $0<T,R<\infty$, $\{\rho^\tau(s,x)\}$ *strongly converges to* $\mathcal{N}(s,x)$ *in* $\mathcal{C}([0,T],(B^{\alpha-\delta}_{2,r}(B_R))$ *as* $\tau\to 0$, *where* $\delta\in(0,1)$ *and* B_R *denotes the ball of radius* R *in* $\mathbb{R}^d$. *In addition, it holds that*

$$\|(\mathcal{N}(s,\cdot)-\bar{\rho}\|_{B^\alpha_{2,r}(\mathbb{R}^d)}\leq C_0\|(\rho_0-\bar{\rho},\mathbf{m}_0)\|_{B^\alpha_{2,r}(\mathbb{R}^d)},\ \ s\geq 0,\tag{1.9}$$

where $C_0>0$ *is a uniform constant independent of* τ.

2 Preliminary

Throughout the paper, $C>0$ is a generic constant. Denote by $\mathcal{C}([0,T],X)$ (resp., $\mathcal{C}^1([0,T],X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X. Also, $\|(f,g)\|_X$ means $\|f\|_X+\|g\|_X$, where $f,g\in X$. $\langle f,g\rangle$ denotes the inner product of two functions f,g in $L^2(\mathbb{R}^d)$.

The global a priori estimate requires a dyadic decomposition of Fourier variables, however, we omit the Littlewood-Paley decomposition and the theory of Besov spaces and Chemin-Lerner spaces. The interesting reader is referred to [1] or [19]. In what follows, we list partial facts used only.

Lemma 2.1. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then*

(1) If $s > 0$, then $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$;

(2) If $\tilde{s} \leq s$, then $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,\tilde{r}}$;

(3) $\dot{B}^{N/p}_{p,1} \hookrightarrow \mathcal{C}_0$, $\quad B^{N/p}_{p,1} \hookrightarrow \mathcal{C}_0 (1 \leq p < \infty)$;

where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

Proposition 2.1. *Let $1 \leqslant p, r \leqslant \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\phi \in C^\infty_c$, the map $f \mapsto \phi f$ is compact from $B^{s+\varepsilon}_{p,r}$ to $B^s_{p,r}$.*

The Chemin-Lerner spaces $\widetilde{L}^\theta_T(B^s_{p,r})$ may be linked with the usual spaces $L^\theta_T(B^s_{p,r})$ via the Minkowski's inequality.

Remark 2.1. *It holds that*

$$\|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})} \leq \|f\|_{L^\theta_T(B^s_{p,r})} \text{ if } r \geq \theta; \qquad \|f\|_{\widetilde{L}^\theta_T(B^s_{p,r})} \geq \|f\|_{L^\theta_T(B^s_{p,r})} \text{ if } r \leq \theta.$$

First, we give an elementary fact that indicates the connection between the homogeneous and inhomogeneous Chemin-Lerner spaces, which have been recently shown in [19]. Precisely, it reads as follows:

Lemma 2.2. *Let $s > 0, 1 \leq \theta, p, r \leq +\infty$. When $\theta \geq r$, it holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) = \widetilde{L}^\theta_T(B^s_{p,r})$$

for any $T > 0$.

In [20], we develop general commutator estimates, which the derivative information can be excluded from the commutators.

Proposition 2.2. *For $s > -1$, $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, there is a constant $C > 0$ such that*

$$\|[f, \dot{\Delta}_q]g\|_{L^p} \leq C c_q 2^{-q(s+1)} \Big(\|\nabla f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^{p_1}} \|f\|_{\dot{B}^{s+1}_{p_2,r}} \Big)$$

and

$$\begin{aligned}&\|[f, \dot{\Delta}_q]g\|_{L^\theta_T(L^p)} \\ \leq & C c_q 2^{-q(s+1)} \Big(\|\nabla f\|_{L^{\theta_1}_T(L^\infty)} \|g\|_{\widetilde{L}^{\theta_2}_T(\dot{B}^s_{p,r})} + \|g\|_{L^{\theta_3}_T(L^{p_1})} \|f\|_{\widetilde{L}^{\theta_4}_T(\dot{B}^{s+1}_{p_2,r})} \Big),\end{aligned}$$

where $1/p = 1/p_1 + 1/p_2$, $1/\theta = 1/\theta_1 + 1/\theta_2 = 1/\theta_3 + 1/\theta_4$ and c_q denotes a sequence such that $\|(c_q)\|_{\ell^r} \leq 1$.

Finally, we present the proposition which describes the smoothing effect of the solution for the heat equation.

Proposition 2.3. *Let $s \in \mathbb{R}$ and $1 \leq \alpha, p, r \leq \infty$. Let $T > 0, u_0 \in B^s_{p,r}$ and $f \in \widetilde{L}^\alpha_T(B^{s-2+\frac{2}{\alpha}}_{p,r})$. Then the problem of heat equation*

$$\partial_t u - \mu \Delta u = f, \quad u|_{t=0} = u_0$$

has a unique solution $u \in \widetilde{L}^\alpha_T(B^{s+\frac{2}{\alpha}}_{p,r}) \cap \widetilde{L}^\infty_T(B^s_{p,r})$ and there exists a constant C depending only on N and such that for all $\alpha_1 \in [\alpha, +\infty]$, we have

$$\mu^{\frac{1}{\alpha_1}} \|u\|_{\widetilde{L}^{\alpha_1}_T(B^{s+\frac{2}{\alpha}}_{p,r})} \leq C\Big\{(1+T^{\frac{1}{\alpha_1}})\|u_0\|_{B^s_{p,r}} + (1+T^{1+\frac{1}{\alpha_1}-\frac{1}{\alpha}})\mu^{\frac{1}{\alpha}-1}\|f\|_{\widetilde{L}^\alpha_T(B^{s-2+\frac{2}{\alpha}}_{p,r})}\Big\}.$$

In addition, if r is finite then u belongs to $\mathcal{C}([0,T]; B^s_{p,r})$.

3 Global well-posedness

3.1 Reformulation and local well-posedness

Let us introduce the energy function which is just an entropy in the sense of Definition 2.1 of [6]:

$$\eta(\rho, \mathbf{m}) := \frac{|\mathbf{m}|^2}{2\rho} + h(\rho) \quad \text{with} \quad \mathbf{m} = \rho\mathbf{v} \text{ and } h'(\rho) = \int_1^\rho \frac{p'(s)}{s} ds.$$

For the rigorous verification, see [19]. Furthermore, the associated entropy flux is

$$q(\rho, \mathbf{m}) = \Big(\frac{|\mathbf{m}|^2}{2\rho^2} + \rho h'(\rho)\Big)\frac{\mathbf{m}}{\rho}.$$

Define

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := \nabla\eta(\rho, \mathbf{m}) = \begin{pmatrix} -\frac{|\mathbf{m}|^2}{2\rho^2} + h'(\rho) \\ \mathbf{m}/\rho \end{pmatrix}.$$

Clearly, the mapping $U \to W$ is a diffeomorphism from $\mathcal{O}_{(\rho,\mathbf{m})} := \mathbb{R}^+ \times \mathbb{R}^d$ onto its range $\mathcal{O}_W$, and for classical solutions $(\rho, \mathbf{v})$ away from vacuum, (1.1) is equivalent to the symmetric system

$$A^0(W)W_t + \sum_{j=1}^d A^j(W)W_{x_j} = H(W) \tag{3.1}$$

with

$$A^0(W) = \begin{pmatrix} 1 & W_2^\top \\ W_2 & W_2 \otimes W_2 + p'(\rho) I_d \end{pmatrix},$$

$$A^j(W)$$
$$= \begin{pmatrix} W_{2j} & W_2^\top W_{2j} + p'(\rho) e_j^\top \\ W_2 W_{2j} + p'(\rho) e_j & W_{2j}(W_2 \otimes W_2 + p'(\rho) I_d) + p'(\rho)(W_2 \otimes e_j + e_j \otimes W_2) \end{pmatrix},$$

$$H(W) = G(U(W)) = p'(\rho) \begin{pmatrix} 0 \\ -\frac{W_2}{\tau} \end{pmatrix},$$

where I_d stands for the $d \times d$ unit matrix, and e_j is d-dimensional vector where the jth component is one, others are zero. It follows from the definition of entropy variable W that $h'(\rho) = W_1 + |W_2|^2/2$, so $p'(\rho)$ can be viewed as a function of W, since ρ is the function of $W_1 + |W_2|^2/2$, i.e. of W.

The corresponding initial data become into

$$W(0,x) := W_0 = \Big(-\frac{|\mathbf{v}_0|^2}{2} + h'(\rho_0), \mathbf{v}_0 \Big). \tag{3.2}$$

Remark 3.1. *From the symmetric system (3.1), it is easy to see that the matrices of coefficients A^j $(j = 0, 1, 2 \cdots, d)$ truly depend on the total variable W rather than W_2 only. In [2], Beauchard and Zuazua investigated the general partially dissipative hyperbolic system with the dependence of the matrices A^j with respect to W_2, and achieved the uniform global well-posedness with respect to the parameter τ. They proposed a open question (see Remark 15, [2]) whether the corresponding result with $A^j = A^j(W)$ still hold or not, since the partial dependence is crucial in their analysis. Our results can be regarded as the partial effort to the open question. However, to the best of our knowledge, it is still unknown for generally hyperbolic systems.*

Based on the recent work [19], we can obtain the following local existence for the concrete problem (3.1)-(3.2).

Proposition 3.1. *For any fixed relaxation time $\tau > 0$, assume that the initial data W_0 satisfy $W_0 - \bar{W} \in B^{\alpha}_{2,r}(\bar{W} := (h'(\bar{\rho}), \mathbf{0}))$ and take values in a compact subset of $\mathcal{O}_W$, then, there exists a time $T_0 > 0$ such that:*

(i) Existence: the Cauchy problem (3.1)-(3.2) has a unique classical solution $W \in \mathcal{C}^1([0,T_0] \times \mathbb{R}^d)$ satisfying

$$W - \bar{W} \in \widetilde{\mathcal{C}}_{T_0}(B^{\alpha}_{2,r}) \cap \widetilde{\mathcal{C}}^1_{T_0}(B^{\alpha-1}_{2,r});$$

(ii) *Blow-up criterion: there exists a constant $C_0 > 0$ such that the maximal time T^* of existence of such a solution can be bounded from below by $T^* \geq \frac{C_0}{\|W_0 - \bar{W}\|_{B^\alpha_{2,r}}}$. Moreover, if T^* is finite, then*

$$\limsup_{t \to T^*} \|W - \bar{W}\|_{B^\alpha_{2,r}} = \infty$$

if and only if

$$\int_0^{T^*} \|\nabla W\|_{L^\infty} dt = \infty.$$

3.2 Global a priori estimate

This section is devoted to the global existence in Theorem 1.1. To show that the solutions of (3.1)-(3.2), are globally defined, we need further a priori estimate.

To do this, for any time $T > 0$ and for any solution $W - \bar{W} \in \widetilde{\mathcal{C}}_T(B^\alpha_{2,r}) \cap \widetilde{\mathcal{C}}^1_T(B^{\alpha-1}_{2,r})$, we define by $E(T)$ the energy functional and by $D_\tau(T)$ the corresponding dissipation functional:

$$E(T) := \|W - \bar{W}\|_{\widetilde{L}^\infty_T(B^\alpha_{2,r})},$$

$$D_\tau(T) := \frac{1}{\sqrt{\tau}} \|W_2\|_{\widetilde{L}^2_T(B^\alpha_{2,r})} + \sqrt{\tau} \|\nabla W\|_{\widetilde{L}^2_T(B^{\alpha-1}_{2,r})},$$

and $E(0) := \|W_0 - \bar{W}\|_{B^\alpha_{2,r}}$. We also define

$$S(T) := \|W\|_{L^\infty([0,T] \times \mathbb{R}^d)} + \|\nabla W\|_{L^\infty([0,T] \times \mathbb{R}^d)}.$$

Note that the embedding in Lemma 2.1 and Remark 2.1, we have $S(T) \leq CE(T)$ for some generic constant $C > 0$.

The next central task is to construct the desired a priori estimate, which is included in the following proposition.

Proposition 3.2. *Let W be the solution of (3.1)-(3.2) satisfying $W - \bar{W} \in \widetilde{\mathcal{C}}_T(B^\alpha_{2,r}) \cap \widetilde{\mathcal{C}}^1_T(B^{\alpha-1}_{2,r})$. If $W(t, x)$ takes values in a neighborhood of $\bar{W}$, then there exists a non-decreasing continuous function $C : \mathbb{R}^+ \to \mathbb{R}^+$ which is independent of τ, such that the following nonlinear inequality holds:*

$$E(T) + D_\tau(T) \leq C(S(T)) \Big(E(0) + E(T)^{1/2} D_\tau(T) + E(T) D_\tau(T) \Big). \tag{3.3}$$

Furthermore, there exist some positive constants δ_1, μ_1 and C_1 independent of τ, if $E(T) \leq \delta_1$, then

$$E(T) + \mu_1 D_\tau(T) \leq C_1 E(0). \tag{3.4}$$

The proof of Proposition 3.2, in fact, is to capture the dissipation rates from contributions of $W = (W_1, W_2)$ in turn, where the Lemma 2.2 and Proposition 2.2 play a key role in the subsequent analysis. Here we omit it, see [20] for details.

Thanks to the standard boot-strap argument (see [[14], Theorem 7.1]), we extend the local-in-time solutions in Proposition 3.1 to the global-in-time classical solutions of (3.1)-(3.2). Furthermore, we arrive at Theorem 1.1.

4 Diffusive relaxation limit

Proof of Theorem 1.2. From (1.8) and Remark 2.1, we deduce that quantities $\sup_{s\geq 0} \|\rho^\tau - \bar{\rho}\|_{B^\alpha_{2,r}}$ and

$$\frac{1}{\tau}\int_0^\infty \|\rho\mathbf{v}(t)\|^2_{B^\alpha_{2,r}} dt = \frac{1}{\tau^2}\int_0^\infty \|\rho^\tau \mathbf{v}^\tau(s)\|^2_{B^\alpha_{2,r}} ds$$

are bounded uniformly with respect to τ. Therefore, the left-hand side of (1.4) reads as $\tau^2\times$ the time derivative of a quantity which is bounded in $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$, plus $\tau^2\times$ the space derivative of a quantity which is bounded in $L^1(\mathbb{R}^+ \times \mathbb{R}^d)$. This allows us to pass to the limit $\tau \to 0$ in the sense of distributions, and we arrive at

$$-\frac{\rho^\tau \mathbf{v}^\tau}{\tau} - \nabla P(\rho^\tau) \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d).$$

Inserting the weak convergence property into the first equation of (1.4), we have

$$\partial_s \rho^\tau - \Delta P(\rho^\tau) \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d), \tag{4.5}$$

as $\tau \to 0$.

On the other hand, by (1.4), we see that $\partial_s \rho^\tau$ is bounded in $L^2(\mathbb{R}^+, B^{\alpha-1}_{2,r})$. For any $T > 0$, there exists a function $\mathcal{N} \in \mathcal{C}([0,T], \bar{\rho} + B^{\alpha-1}_{2,1}) \cap L^\infty(0,T; \bar{\rho} + B^\alpha_{2,1})$ such that, up to subsequences, it holds that

$$\rho^\tau - \bar{\rho} \rightharpoonup \mathcal{N} - \bar{\rho} \quad \text{weakly in} \quad H^1(0,T; B^{\alpha-1}_{2,r}).$$

Of course,

$$\rho^\tau \rightharpoonup \mathcal{N} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d).$$

From the same argument in [3], we get $\mathcal{N}(x,0) = \rho_0$, since $\rho^\tau(x,0) = \rho_0$.

Furthermore, it follows from Proposition 2.1 and the Aubin-Lions compactness lemma in [16] that

$$\rho^\tau \to \mathcal{N} \quad \text{strongly in} \quad \mathcal{C}([0,T],(B^{\alpha-\delta}_{2,r})(B_R)),$$

for $\delta \in (0,1)$ and $0 < R < \infty$, as $\tau \to 0$, where B_R denotes the ball of radius R in $\mathbb{R}^d$. The strong convergence property enable us to obtain $P(\rho^\tau)$ converges towards $P(\mathcal{N})$ in the space $\mathcal{C}([0,T],(B^{\alpha-\delta}_{2,1})(B_R))$. Together with (4.5), we deduce that $\mathcal{N}$ is a solution to the system (1.6) satisfying (1.9).

Next, we check the solution $\mathcal{N}$ has the desired regularity. Thanks to $\mathcal{N} \in L^\infty(0,T;\bar{\rho}+B^\alpha_{2,r})$, the series $\sum_{q\geq -1}\left(2^{q\alpha}\|\Delta_q(\mathcal{N}(t)-\bar{\rho})\|_{L^2}\right)^r$ converges uniformly on $[0,T]$. In addition, the map $t \mapsto \|\Delta_q(\mathcal{N}(t)-\bar{\rho})\|_{L^2}$ is continuous on $[0,T]$, since $\mathcal{N} \in \mathcal{C}([0,T];\bar{\rho}+B^{\alpha-1}_{2,r})$. Then $\Delta_q(\mathcal{N}(t)-\bar{\rho}) \in \mathcal{C}([0,T];B^\alpha_{2,r})$ for all $q \geq -1$, which yields $\mathcal{N} \in \mathcal{C}([0,T];\bar{\rho}+B^\alpha_{2,r})$.

Finally, let us show the uniqueness of solution to (1.6).

Set $\tilde{\mathcal{N}} = \mathcal{N}_1 - \mathcal{N}_2$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two solutions to the system (1.6) with the same initial data, respectively. Then the error $\tilde{\mathcal{N}}$ satisfies

$$\partial_s\tilde{\mathcal{N}} - P'(\bar{\rho})\Delta\tilde{\mathcal{N}} = \Delta[P(\mathcal{N}_1) - P(\mathcal{N}_2) - P'(\bar{\rho})\tilde{\mathcal{N}}]. \tag{4.6}$$

Then taking $\alpha_1 = \infty, \alpha = 1,\ s = \alpha - 2,\ p = 2$ and $1 \leq r \leq 2$ in Proposition 2.3, and applying the resulting inequality to (4.6), with the aid of Remark 2.1, we arrive at

$$\begin{aligned}
\|\tilde{\mathcal{N}}\|_{L^\infty_T(B^\alpha_{2,r})} &\leq C\Big(\|\Delta[P(\mathcal{N}_1) - P(\mathcal{N}_2) - P'(\bar{\rho})\tilde{\mathcal{N}}]\|_{\tilde{L}^1_T(B^{\alpha-2}_{2,r})}\Big), \\
&\leq C\Big(\|\Delta[P(\mathcal{N}_1) - P(\mathcal{N}_2) - P'(\bar{\rho})\tilde{\mathcal{N}}]\|_{L^1_T(B^{\alpha-2}_{2,r})}\Big) \\
&\leq C\int_0^T \|\tilde{\mathcal{N}}\|_{L^\infty_\tau(B^\alpha_{2,r})}d\tau.
\end{aligned} \tag{4.7}$$

Gronwall's inequality gives $\tilde{\mathcal{N}} \equiv 0$ immediately. □

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