数理解析研究所講究録
第 1883 巻 2014 年 21-32

STABILITY OF SOLITARY WAVES FOR THE COUPLED BBM EQUATIONS

室蘭工業大学・工学部　加藤正和 (Masakazu Kato)
Department of Engineering, Muroran Institute of Technology

1 Introduction

In this note, we consider large time behavior of the global solutions to the coupled BBM equations:

$$q_t - q_{txx} + r_x + (qr)_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$r_t - r_{txx} + q_x + qq_x + rr_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$q(x,0) = q_0(x), \quad r(x,0) = r_0(x). \tag{1.3}$$

Here, for an integer $s \geq 0$, $H^s(\mathbb{R})$ denotes the space of functions $u = u(x)$ such that $\partial_x^l u$ are L^2-functions on $\mathbb{R}$ for $0 \leq l \leq s$, endowed, with the norm $\|\cdot\|_{H^s}$, while $H_a^1(\mathbb{R})$ is a space of functions whose element satisfy $\|u\|_{H_a^1} \equiv \|e^{ax}u\|_{H^1} < \infty$ for $a \in \mathbb{R}$. We can write the system (1), (2) and (3) in the following system.

$$\partial_t u = Lu + f(u), \tag{1.4}$$

$$u(x,0) = u_0(x) \tag{1.5}$$

where $u = \begin{pmatrix} q \\ r \end{pmatrix}$ and,

$$L = \begin{pmatrix} 0 & -\partial_x(I-\partial_x^2)^{-1} \\ -\partial_x(I-\partial_x^2)^{-1} & 0 \end{pmatrix}, \quad f(u) = \begin{pmatrix} -\partial_x(I-\partial_x^2)^{-1}(qr) \\ -\frac{1}{2}\partial_x(I-\partial_x^2)^{-1}(q^2+r^2) \end{pmatrix}.$$

The BBM equations have two parameter family of solitary wave. In [1], they show that solitary waves $u_{c_0}(x - c_0 t + \gamma_0) = \phi_{c_0}(x - c_0 t + \gamma_0)(1,1)^T$ exist for any speed $c_0 > 1$ and shift $\gamma_0 \in \mathbb{R}$. Explicitly,

$$\begin{aligned} \phi_{c_0}(x) &= \frac{3(c_0-1)}{2}\operatorname{sech}^2\left(\frac{1}{2}\nu x\right) \\ &= O\left(\exp\left(-\nu \mid x \mid\right)\right) \quad (\mid x \mid \to \pm\infty), \end{aligned} \tag{1.6}$$

where $\nu = \sqrt{\frac{c_0-1}{c_0}}$. By using the Lyapunov theory, they derive the following theorem on orbital stability.

Theorem 1.1. *Let $c_0 > 1$. For any $\epsilon > 0$ there exists $\delta > 0$ such that if $u \in C([0,t_0); H^1(\mathbb{R}))$ is a solution to (1.4) and (1.5) with $\|u_0 - u_{c_0}\|_{H^1} \leq \delta$, then $u(t)$ can be extended to a solution in $0 \leq t < +\infty$, and*

$$\sup_{t\geq 0}\inf_{\xi\in\mathbb{R}} \|u(\cdot,t) - u_{c_0}(\cdot - \xi)\|_{H^1} \leq \epsilon. \tag{1.7}$$

However we can't expect that if the initial data $u_0(x)$ is close to some solitary wave $u_{c_0}(x+\gamma_0)$ with speed c_0, then the solution tends to the translate of same solitary wave as t goes to ∞ asymptotically. Our aim is to describe the long-time asymptotic behavior of solutions initially close to a solitary wave. Main result is following. Convergence in H^1_a means local uniform convergence.

Theorem 1.2. *Let $0 < a < \nu$. We assume $u_0(x) = u_{c_0}(x+\gamma_0) + v_0(x) \in H^2 \cap H^1_a$. There exist $\epsilon > 0$, $c_1 > 1$ and $b > 0$ such that if $c_0 \in (1, c_1)$ and $\|v_0\|_{H^1} + \|v_0\|_{H^1_a} < \epsilon$, then*

$$\|u(\cdot,t) - u_{c_+}(\cdot - c_+t + \gamma_+)\|_{H^1} \leq C\epsilon, \tag{1.8}$$

$$\|u(\cdot + c_+t - \gamma_+, t) - u_{c_+}(\cdot)\|_{H^1_a} \leq C\epsilon e^{-bt}, \tag{1.9}$$

for some $c_+ > 1$, $\gamma_+ \in \mathbb{R}$ with $| c_0 - c_+ | < C\epsilon$, $| \gamma_0 - \gamma_+ | < C\epsilon$.

We remark that the estimate similar to (6) and (7) were obtained for other types of equation (see [2], [3], [4] and [5]).

2 Spectrum

In order to prove Theorem 1.2, we derive the equation for the perturbation. We put $y = x - \int_0^t c(s)ds + \gamma(t)$ and $u(x,t) = u_{c(t)}(y) + v(y,t)$. Then $v(y,t)$ satisfies

$$\partial_t v = Av + F, \tag{2.1}$$

where $v = \begin{pmatrix} \rho \\ \eta \end{pmatrix}$ and,

$$\begin{aligned} A &= \partial_y(I - \partial_y^2)^{-1}L_{c_0}, \qquad (2.2)\\ L_{c_0} &= \begin{pmatrix} c_0(I-\partial_y^2) - \phi_{c_0} & -(1+\phi_{c_0}) \\ -(1+\phi_{c_0}) & c_0(I-\partial_y^2) - \phi_{c_0} \end{pmatrix}, \\ F &= -(\dot\gamma\partial_y\phi_c + \dot c\partial_c\phi_c)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c - c_0 - \dot\gamma)\partial_y v \\ &\quad -\partial_y(I-\partial_y^2)^{-1}(\phi_c - \phi_{c_0})\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v + \begin{pmatrix} -\partial_y(I-\partial_y^2)^{-1}\rho\eta \\ -\frac{1}{2}\partial_y(I-\partial_y^2)^{-1}(\rho^2+\eta^2) \end{pmatrix}. \end{aligned}$$

To study the weighted perturbation $e^{ay}v$, we deal with the spectrum of the operator $A_a = e^{ay}Ae^{-ay}$. We can see that if $0 < a < \nu$, then the essential spectrum of A_a lie in the open left half plane. We put

$$A_a^{\infty} = D_a(I - D_a^2)^{-1}\begin{pmatrix} c_0(I - D_a^2) & -1 \\ -1 & c_0(I - D_a^2) \end{pmatrix}. \tag{2.3}$$

Since ϕ_{c_0} decays exponentially as $| y | \to \infty$, we obtain from (2.2) and (2.3)

$$\sigma_{ess}(A_a) = \sigma_{ess}(A_a^{\infty}). \tag{2.4}$$

Applying the Fourier transform to A_a^{∞}, we see

$$\sigma_{ess}(A_a^{\infty}) = \left\{ z \in \mathbb{C} \mid z = \frac{(ik - a)(-c_0(ik - a)^2 + c_0 \pm 1)}{1 - (ik - a)^2}, \ k \in \mathbb{R} \right\}. \tag{2.5}$$

Therefore, it follows that

$$\text{Re } \sigma_{ess}(A_a) \leq -ac + \frac{1}{2(1 + a^2)} := -b_* < 0. \tag{2.6}$$

Hence, if $0 < a < \nu$, then the essential spectrum of A_a lie in the open left half plane.

Proposition 2.1. *Let $0 < a < \nu$.*
(1) *There exists $\nu_* \in (0, 1)$ such that for all $\nu \in (0, \nu_*)$, the only eigenvalue λ of A with $Re\ \lambda \geq 0$ is $\lambda = 0$ and $\lambda = 0$ is the eigenvalue with algebraic multiplicity* 2.
(2) *For each $\nu \in (0, \nu_*)$, there exists $\epsilon(\nu) > 0$ such that the only eigenvalue λ of A_a with $Re\lambda \geq -\epsilon(\nu)$ is $\lambda = 0$.*

We put

$$-b_{max} = \inf\{ -b \mid \lambda = 0 \text{ is the only eigenvalue of } A_a \text{ with } \mathrm{Re}\lambda \geq -b > -b_* \}.$$

Then, the decay estimates for the linearized equations (2.7) below play a crucial role in our analysis. By using (2.6) and The Prüss's theorem [6], we obtain the following.

Proposition 2.2. *Assume $0 < a < \nu$ and $\lambda = 0$ is the only eigenvalue of A in the closed right half plane. Then the problem*

$$\begin{cases} w_t = A_a w \\ w(x, 0) = w_0(x) \in \textit{range } Q \end{cases} \tag{2.7}$$

has a solution with

$$\|w(\cdot, t)\|_{H_a^1} \leq Ce^{-bt}\|w_0\|_{H_a^1} \tag{2.8}$$

for some b with $0 < b < b_{\max}$. Here Q is a projection onto $Ker\{A_a^*\}^{\perp}$ and A_a^* is the adjoint operator of A_a.

Next, we study a basis for the generalized zero eigen space $\mathrm{Ker}_g(A_a)$. We can see

$$\mathrm{Ker}_g(A_a) = \mathrm{span}\{\partial_y u_{c_0}, \partial_c u_{c_0}\}. \tag{2.9}$$

The solitary wave u_{c_0} satisfies the following.

$$\begin{pmatrix} c_0(I-\partial_y^2)+\frac{1}{2}\phi_{c_0} & -1+\frac{1}{2}\phi_{c_0} \\ -1+\frac{1}{2}\phi_{c_0} & c_0(I-\partial_y^2)+\frac{1}{2}\phi_{c_0} \end{pmatrix} u_{c_0} = 0. \tag{2.10}$$

Differentiating (2.10) with y and c, we obtain

$$L_{c_0}\partial_y u_{c_0} = 0, \tag{2.11}$$

and

$$L_{c_0}\partial_c u_{c_0} = -(1-\partial_y^2)u_{c_0}. \tag{2.12}$$

From (2.11), we get

$$A\partial_y u_{c_0} = 0. \tag{2.13}$$

Hence, we obtain $\partial_y u_{c_0} \in \mathrm{Ker}_g(A)$. From (2.12), we have

$$A\partial_c u_{c_0} = -\partial_y u_{c_0}. \tag{2.14}$$

It follows that

$$A^2\partial_c u_{c_0} = 0. \tag{2.15}$$

Hence, we obtain (2.9). Let us introduce $\tilde{\xi}_1$, $\tilde{\xi}_2$ by

$$\tilde{\xi}_1 = \partial_y u_{c_0}, \quad \tilde{\xi}_2 = \partial_c u_{c_0}.$$

We take biorthogonal bases $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ and $\{\tilde{\eta}_1, \tilde{\eta}_2\}$ for $\mathrm{Ker}_g(A)$ and $\mathrm{Ker}_g(A^*)$.
Let

$$\xi_i = e^{ay}\tilde{\xi}_i, \quad \eta_i = e^{-ay}\tilde{\eta}_i,$$

for $i = 1, 2$. Then $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ are biorthogonal bases for $\mathrm{Ker}_g(A_a)$ and $\mathrm{Ker}_g(A_a^*)$.

3 Modulation Equation

To obtain the decay estimate of perturbation $w = e^{ay}v$, we need to let w belong to orthogonal to $\mathrm{Ker}_g(A_a^*)$. The following Proposition 3.1 concerned the existence of a decomposition $u(x,t) \mapsto (v(y,t), \gamma(t), c(t))$ satisfies the condition (3.3) below.

Proposition 3.1. *Let $0 < a < \nu$ and $T \geq 0$. Then there exists δ_0, $\delta_1 > 0$ such that for any $\gamma_0 \in \mathbb{R}$, if $e^{ax}u(x) \in C([0,T], H^1)$ and*

$$\sup_{0\leq t\leq T} \|e^{a(\cdot+\gamma_0)}(u(\cdot,t) - u_{c_0}(\cdot - c_0 t + \gamma_0))\|_{H^1} < \delta_0 \tag{3.1}$$

holds, then there exists a unique function $t \to (\gamma(t), c(t)) \in C([0,t_1], \mathbb{R}^2)$ with

$$\sup_{0\leq t\leq t_1} \mid \gamma(t) - \gamma_0 \mid + \mid c(t) - c_0 \mid < \delta_1 \tag{3.2}$$

such that

$$\int_{\mathbb{R}} [u(x,t) - u_{c(t)}(y)]e^{ay}\eta_k(y)dy = 0, \tag{3.3}$$

for $k = 1, 2$ and $0 \leq t \leq T$.

Proposition 3.2. *There exist δ_2, $\delta_3 > 0$ such that for any $T > 0$, if $e^{ax}u(x) \in C([0,T], H^1)$ and*

$$\sup_{0\leq t\leq T} \|e^{ay}v(y,t)\|_{H^1} \leq \delta_2, \tag{3.4}$$

and

$$\sup_{0\leq t\leq T} \mid c(t) - c_0 \mid \leq \delta_3, \tag{3.5}$$

hold, then a unique extension of $(\gamma, c) \in C([0, T + t_*], \mathbb{R}^2)$ exists for some $t_* > 0$ with

$$\int_{\mathbb{R}} [u(x,t) - u_{c(t)}(y)]e^{ay}\eta_k(y)dy = 0, \tag{3.6}$$

for $k = 1, 2$ and $0 \leq t \leq T + t_*$.

To estimate the weighted perturbation, we need to estimate $\dot{\gamma}$ and $\dot{c}$. We shall derive the modulation equations (3.11) below. We put $\tau = \int_0^t c(s)ds - \gamma(t)$. Then, w satisfies

$$w_\tau = \frac{1}{c_0}A_a w + J \tag{3.7}$$

where

$$J = -\frac{1}{c - \dot{\gamma}}e^{ay}(\dot{\gamma}\partial_y u_c + \dot{c}\partial_c u_c) + \tilde{J}, \tag{3.8}$$

where $\tilde{J} = J_1 + J_2 + J_3$, $D_a = \partial_y - a$ and

$$\begin{aligned}
J_1 &= \frac{1}{c_0}D_a(I - D_a^2)^{-1}(\phi_{c_0} - \phi_c)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} w, \\
J_2 &= \frac{1}{c_0}\frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}D_a(I - D_a^2)^{-1}\left\{\phi_c\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} w, \\
J_3 &= -\frac{1}{c - \dot{\gamma}}D_a(I - D_a^2)^{-1}\begin{pmatrix} e^{ay}qr \\ -\frac{1}{2}e^{ay}(q^2 + r^2) \end{pmatrix}.
\end{aligned} \tag{3.9}$$

P denotes the projection onto the zero eigenspace of A_a. In order to prove the decay estimate (1.9), we require $PJ = 0$. Then, we obtain

$$< \eta_i, J >= 0. \tag{3.10}$$

for $i = 1, 2$, where $< \cdot, \cdot >$ denotes the L^2 inner product. It follows from (3.8) that

$$-\frac{1}{c-\dot{\gamma}}\{< \eta_i, e^{ay}\partial_y u_c > \dot{\gamma} + < \eta_i, e^{ay}\partial_c u_c > \dot{c}\} + < \eta_i, \tilde{J} >= 0.$$

We put $e_1 = \partial_y u_c - \partial_y u_{c_0}$, $e_2 = \partial_c u_c - \partial_c u_{c_0}$. Since $< \tilde{\eta_i}, \tilde{\xi_j} >= \delta_{ij}$, we derive

$$\begin{pmatrix} 1+ < \tilde{\eta_1}, e_1 > & < \tilde{\eta_1}, e_2 > \\ < \tilde{\eta_2}, e_1 > & 1+ < \tilde{\eta_2}, e_2 > \end{pmatrix} \begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = c_0 \left(1 - \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}\right)^{-1} \begin{pmatrix} < \eta_1, \tilde{J} > \\ < \eta_2, \tilde{J} > \end{pmatrix}.$$

Hence, we obtain

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(| c(t) - c_0 |)\right\} \begin{pmatrix} \dot{\gamma} \\ \dot{c} \end{pmatrix} = c_0 \left(1 - \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}\right)^{-1} \begin{pmatrix} < \eta_1, \tilde{J} > \\ < \eta_2, \tilde{J} > \end{pmatrix}.$$

Therefore, if $| c(t) - c_0 | + | \dot{\gamma}(t) |$ is small, then it follows that

$$\begin{aligned} | \dot{\gamma}(t) | + | \dot{c}(t) | &\leq C\|\tilde{J}\|_{L^2} \\ &\leq C(\|\tilde{J}_1\|_{L^2} + \|\tilde{J}_2\|_{L^2} + \|\tilde{J}_3\|_{L^2}). \end{aligned} \tag{3.11}$$

4 Energy estimate

In order to prove the decay estimate given by Proposition 5.1, we prepare the following Proposition 4.1. The proposition is concerned with the energy estimate for the BBM equations (1.1), (1.2) and (1.3). A solitary wave u_{c_0} is a critical point of energy functional

$$E[u; c] = H[u] - c_0 N[u] \tag{4.1}$$

where $u = (q, r)^T$ and

$$H[u] = \int_{\mathbb{R}} (qr + \frac{1}{2}q^2 r + \frac{1}{6}r^3)dx, \tag{4.2}$$

$$N[u] = \frac{1}{2}\int_{\mathbb{R}} (q^2 + r^2 + q_x^2 + r_x^2)dx. \tag{4.3}$$

We denote $H[u]$ and $N[u]$ are conserved integrals of BBM equations.

Proposition 4.1. *If* $| c(t) - c_0 | + \|v(\cdot, t)\|_{H^1}$ *is sufficiently small for* $0 \leq t \leq T$, *then we have*

$$\|v(\cdot, t)\|_{H^1} \leq C(\sqrt{| \delta E |} + | c(t) - c_0 | + \|w(\cdot, t)\|_{L^2}) \tag{4.4}$$

for $0 \leq t \leq T$, *where* $\delta E = E[u(\cdot, 0)] - E[u_{c_0}(\cdot)]$.

PROOF. Put $z(y,t) = (z_1, z_2)^T = u(x,t) - u_{c_0}(y)$. Since δE is constant in time, it follows that

$$\begin{aligned}\delta E &= E[z(y,t) + u_{c_0}(y)] - E[u_{c_0}(y)] \\ &= -\frac{1}{2}\int_{\mathbb{R}} z^T L_{c_0} z dy + \frac{1}{2}\int_{\mathbb{R}} z_1^2 z_2 dy + \frac{1}{6}\int_{\mathbb{R}} z_2^3 dy. \qquad (4.5)\end{aligned}$$

From the Sobolev inequality, we have

$$\begin{aligned}\left| \frac{1}{2}\int_{\mathbb{R}} z_1^2 z_2 dy + \frac{1}{6}\int_{\mathbb{R}} z_2^3 dy \right| &\leq C\|z\|_{H^1}^3 \\ &\leq C(| c(t) - c_0 | + \|v\|_{H^1})^3, \qquad (4.6)\end{aligned}$$

where $z(y,t) = v(y,t) + u_{c(t)}(y) - u_{c_0}(y)$.
We obtain

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}} z^T L_{c_0} z dy &= -\frac{1}{2}\int_{\mathbb{R}} v^T L_{c_0} v dy - \int_{\mathbb{R}} v^T L_{c_0}(u_c(y) - u_{c_0}(y)) dy \\ &\quad -\frac{1}{2}\int_{\mathbb{R}} (u_c(y) - u_{c_0}(y))^T L_{c_0}(u_c(y) - u_{c_0}(y)) dy \\ &\leq -\frac{1}{2}\int_{\mathbb{R}} v^T L_{c_0} v dy + \left|\int_{\mathbb{R}} e^{-ay} w^T L_{c_0}(u_c(y) - u_{c_0}(y)) dy\right| \\ &\quad +\frac{1}{2}\left|\int_{\mathbb{R}} (u_c(y) - u_{c_0}(y))^T L_{c_0}(u_c(y) - u_{c_0}(y)) dy\right| \\ &\leq -\frac{1}{2}\int_{\mathbb{R}} v^T L_{c_0} v dy + C(| c(t) - c_0 |^2 + \|w\|_{L^2}^2), \qquad (4.7)\end{aligned}$$

and

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}} v^T L_{c_0} v dy &= -\frac{1}{2}c_0(\|v(\cdot,t)\|_{L^2} + \|v_y(\cdot,t)\|_{L^2}) - \int_{\mathbb{R}} (1 + \phi_{c_0}) v_1 v_2 dy \\ &\quad - \int_{\mathbb{R}} \phi_{c_0} v^T v dy \\ &\leq -\frac{1}{2}c_0(\|v(\cdot,t)\|_{L^2} + \|v_y(\cdot,t)\|_{L^2}) + \frac{1}{2}\|v(\cdot,t)\|_{L^2} \\ &\quad -\frac{1}{2}\int_{\mathbb{R}} e^{-ay}\phi_{c_0} w^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v dy \\ &\leq -\frac{1}{2}\{(c_0 - 1)\|v(\cdot,t)\|_{L^2}^2 + c_0\|v_y(\cdot,t)\|_{L^2}^2\} + C\|v\|_{L^2}\|w\|_{L^2} \\ &\leq -\frac{1}{2}\{\frac{1}{2}(c_0 - 1)\|v(\cdot,t)\|_{L^2}^2 + c_0\|v_y(\cdot,t)\|_{L^2}^2\} + C\|w(\cdot,t)\|_{L^2}^2. \qquad (4.8)\end{aligned}$$

Summarizing (4.5), (4.6), (4.7) and (4.8), we get

$$\frac{1}{2}\{(c_0 - 1)\|v(\cdot,t)\|_{L^2}^2 + c_0\|v_y(\cdot,t)\|_{L^2}^2\} \leq C(| \delta E | + | c(t) - c_0 |^2 + \|w(\cdot,t)\|_{L^2}^2). \qquad (4.9)$$

This completes the proof. □

5 Decay estimate

Proposition 5.1. *There exist $\delta_4 > 0$ and $\epsilon_* > 0$ such that if the decomposition exists for $t \in [0,T]$ and the following conditions hold:*

$$(i)\ \sqrt{|\delta E|} + \|w(\cdot,t)\|_{H^1} + |c(t) - c_0| + \left|\frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}\right| + \|v(\cdot,t)\|_{H^1} \le \delta_4, \quad (5.1)$$

$$(ii)\ |c(0) - c_0| + \sqrt{|\delta E|} + \|w(\cdot .0)\|_{H^1} \le \epsilon \le \epsilon_*, \quad (5.2)$$

then we have

$$e^{\kappa bt}\|w(\cdot,t)\|_{H^1} + |c(t) - c_0| + \left|\frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}\right| + \|v(\cdot,t)\|_{H^1} \le C\epsilon, \quad (5.3)$$

where $\kappa = 1 + 2\delta_4/c_0$.

Proof. First, we evaluate $|\dot{\gamma}(t)| + |\dot{c}(t)|$. If $|c(t) - c_0| + |\frac{c-c_0-\dot{\gamma}}{c-\dot{\gamma}}|$ is small, then we have from (3.11) and (3.9)

$$\begin{aligned} |\dot{\gamma}(t)| + |\dot{c}(t)| &\le C(\|J_1\|_{L^2} + \|J_2\|_{L^2} + \|J_3\|_{L^2}) \\ &\le C(|c(t) - c_0| + |\dot{\gamma}(t)| + \|v(\cdot,t)\|_{H^1})\|w(\cdot,t)\|_{H^1}. \quad (5.4) \end{aligned}$$

Therefore, We get from (5.4) and (5.1)

$$|\dot{\gamma}(t)| + |\dot{c}(t)| \le C\delta_4^2. \quad (5.5)$$

Next, we evaluate $\|w(\cdot,t)\|_{H^1}$. By the Duhamel principle, we see from (3.7)

$$w(\tau) = e^{\frac{1}{c_0}A_a\tau}w(0) + \int_0^\tau e^{\frac{1}{c_0}A_a(\tau-s)}QJ(s)ds. \quad (5.6)$$

Let $b < b' < b_{max}$. For $0 \le \tau \le \tau(T)$, we find from Proposition 2.2

$$\|w(\cdot,\tau)\|_{H^1} \le Ce^{-\frac{b'}{c_0}\tau}\|w(\cdot,0)\|_{H^1} + \int_0^\tau e^{-\frac{b'}{c_0}(\tau-s)}\|QJ\|_{H^1}ds. \quad (5.7)$$

We obtain from (3.8), (3.9) and (5.4)

$$\begin{aligned} \|J\|_{H^1} &\le C\left(1 + \frac{c - c_0 - \dot{\gamma}}{c - \dot{\gamma}}\right)(|\dot{\gamma}(t)| + |\dot{c}(t)|) + C\delta_4\|w\|_{H^1} \\ &\le C\delta_4\|w\|_{H^1}. \quad (5.8) \end{aligned}$$

Now, we define $M(T)$ by

$$M(T) = \sup_{0\le\tau\le\tau(T)} e^{\frac{b}{c_0}\tau}\|w(\cdot,\tau)\|_{H^1}.$$

It follows from (5.7) and (5.8) that

$$\begin{aligned} e^{\frac{b}{c_0}\tau}\|w(\cdot,\tau)\|_{H^1} &\leq C\|w(\cdot,0)\|_{H^1} + C\delta_4 M(T)\int_0^\tau e^{-\frac{b'-b}{c_0}(\tau-s)}ds \\ &\leq C\|w(\cdot,0)\|_{H^1} + C\delta_4 M(T). \end{aligned} \tag{5.9}$$

Therefore, if δ_4 is small, then (5.9) gives the desired estimate

$$M(T) \leq C\|w(\cdot,0)\|_{H^1}. \tag{5.10}$$

In order to evaluate $\mid c(\tau) - c_0 \mid$, we shall use

$$\begin{aligned} c(\tau) - c(0) &= \int_0^\tau \frac{dc}{ds}(s)ds \\ &= \int_0^\tau \dot{c}(s)\frac{dt}{ds}(s)ds. \end{aligned} \tag{5.11}$$

By using (5.2) and (5.4), we have

$$\begin{aligned} \mid c(\tau) - c_0 \mid &\leq \mid c(0) - c_0 \mid + \sup_{0\leq\tau\leq\tau(T)}\left|\frac{1}{c-\dot{\gamma}}\right|\int_0^\tau \mid \dot{c}(s) \mid d\tau \\ &\leq \mid c(0) - c_0 \mid + C\delta_4 M(T)\int_0^\tau e^{-bs}ds \\ &\leq C(\mid c(0) - c_0 \mid + \|w(\cdot,0)\|_{H^1}) \\ &\leq C\epsilon. \end{aligned} \tag{5.12}$$

Next, we consider $\|v(\cdot,t)\|_{H^1}$. From Proposition 4.1, we get

$$\begin{aligned} \|v(\cdot,t)\|_{H^1} &\leq C(\sqrt{\mid \delta E \mid} + \mid c(t) - c_0 \mid + \|w(\cdot,t)\|_{H^1}) \\ &\leq C\epsilon. \end{aligned} \tag{5.13}$$

Finally, we deal with $\left|\frac{c-c_0-\dot{\gamma}}{c-\dot{\gamma}}\right|$. From (5.12) and (5.4), it follows thata

$$\begin{aligned} \left|\frac{c-c_0-\dot{\gamma}}{c-\dot{\gamma}}\right| &\leq \frac{\mid c - c_0 \mid + \mid \dot{\gamma} \mid}{\mid c \mid - \mid \dot{\gamma} \mid} \\ &\leq C\epsilon + C\|w(\cdot,t)\|_{H^1} \\ &\leq C\epsilon. \end{aligned} \tag{5.14}$$

Summarizing (5.10), (5.12), (5.13) and (5.14), we obtain (5.3). This completes the proof. □

6 Proof of Theorem 1.2

PROOF. We assume that

$$u_0(x) = u(x,0) = u_{c_0}(x+\gamma_0) + v_0(x) \in H^2 \cap H^1_a. \tag{6.1}$$

Since the map $t \mapsto \|u(\cdot,t)\|_{H^1} + \|u(\cdot,t)\|_{H^1_a}$ is continuous for $t \in [0,\infty)$, there exist $\epsilon_1 > 0$ and $t_1 > 0$ such that if $\|u(\cdot,0) - u_{c_0}(\cdot+\gamma_0)\|_{H^1_a} \le \epsilon_1$, then

$$\sup_{0\le t\le t_1} \|e^{a(\cdot+\gamma_0)}(u(\cdot,t) - u_{c_0}(\cdot - c_0 t + \gamma_0))\|_{H^1} \le \delta_0, \tag{6.2}$$

where δ_0 is as in Proposition 3.1. Hence, there exists the decomposition $u(x,t) \mapsto (v(y,t), \gamma(t), c(t))$ exists for $0 \le t \le t_1$.
Next, we shall prove (5.2). If $\|e^{ax}v_0\|_{H^1}$ is small, then the map $u(\cdot,0) \mapsto (\gamma(0), c(0))$ is locally Lipschitz on H^1_a and $u_{c_0}(\cdot+\gamma_0) \mapsto (\gamma_0, c_0)$. Therefore, we obtain

$$\begin{aligned} |\,\gamma(0) - \gamma_0\,| + |\,c(0) - c_0\,| &\le C\|u(\cdot,0) - u_{c_0}(\cdot+\gamma_0)\|_{H^1_a} \\ &= C\|v_0\|_{H^1_a}. \end{aligned} \tag{6.3}$$

Also it follows from (6.3) that

$$\begin{aligned} \|w(\cdot,0)\|_{H^1} &= \|u(\cdot,t) - u_{c(0)}(\cdot+\gamma(0))\|_{H^1_a} \\ &\le \|u(\cdot,t) - u_{c_0}(\cdot+\gamma_0)\|_{H^1_a} + \|u_{c_0}(\cdot+\gamma_0) - u_{c(0)}(\cdot+\gamma(0))\|_{H^1_a} \\ &\le C\|v_0\|_{H^1_a}, \end{aligned} \tag{6.4}$$

and we get from (4.5)

$$|\,\delta E\,| \le C\|v_0\|^2_{H_1}. \tag{6.5}$$

Hence, if $\|v_0\|_{H^1} + \|v_0\|_{H^1_a}$ is small, then (5.2) holds.
Next, we derive (5.1). From (5.14), (5.12) and (5.4), we obtain

$$\begin{aligned} \left|\frac{c - c_0 - \dot\gamma}{c - \dot\gamma}\right| &\le C(\sqrt{\delta E} + |\,c(0) - c_0\,| + \|w(\cdot,0)\|_{H^1}) \\ &\le C\|v_0\|_{H^1_a}. \end{aligned} \tag{6.6}$$

Since the left-hand side of (5.1) is continuous in t, there exist $\epsilon_2 > 0$ and $t_2 > 0$ ($t_2 < t_1$) such that if $\|v_0\|_{H^1} + \|v_0\|_{H^1_a} < \epsilon_2$, then (5.1) holds for $0 \le t \le t_2$.
We put

$$T_{\max} = \sup\{\, T > 0 \;\mid\; \text{the decomposition (3.3) and (5.1) hold for } 0 \le t \le T\}.$$

If $T_{\max} = +\infty$, then we can show Theorem 1.2. If $T_{max} < +\infty$, then we let $C\epsilon < \min\{\delta_2, \delta_3, \delta_4/2\}$ where C is as in Proposition 5.1. From proposition 5.1, if $| c(0) - c_0 | + \sqrt{| \delta E |} + \|w(\cdot, t)\|_{H^1} < \epsilon$, then we have

$$\|w(\cdot, t)\|_{H^1} < C\epsilon < \delta_2,$$
$$| c(t) - c_0 | < C\epsilon < \delta_3,$$

for $0 \leq t \leq T$. From Proposition 3.2, the decomposition can be extended. Then, it follows that there exists t_3 such that the decomposition and the estimate (5.1) hold for $0 \leq t \leq T_{\max} + t_3$. This contradicts the definition of $T_{\max}$. Hence $T_{\max} = +\infty$.
Finally, we shall prove (1.9). From (5.4), we have

$$\begin{aligned} | \dot{c} | + | \dot{\gamma} | &\leq C\|w(\cdot, t)\|_{H^1} \\ &\leq C\epsilon e^{-bt}. \end{aligned} \tag{6.7}$$

Therefore, there exists $c_+ = \lim_{t\to\infty} c(t)$ and $| c(t) - c_+ | \leq C\epsilon e^{-bt}$. Also, there exists

$$\gamma_+ = \lim_{t\to\infty} (\gamma(t) - \int_0^t (c(s) - c_+) ds). \tag{6.8}$$

We put

$$\tilde{\gamma}(t) = - \int_0^t (c(s) - c_+) ds + \gamma(t) - \gamma_+. \tag{6.9}$$

Then, it follows from (5.3)

$$\begin{aligned} & \|u(\cdot + c_+ t - \gamma_+, t) - u_{c_+}(\cdot)\|_{H^1_a} \\ \leq\ & \|u(\cdot + c_+ t - \gamma_+, t) - u_{c(t)}(\cdot + \tilde{\gamma}(t))\|_{H^1_a} + \|u_{c(t)}(\cdot + \tilde{\gamma}(t)) - u_{c_+}(\cdot)\|_{H^1_a} \\ \leq\ & \|v(\cdot + \tilde{\gamma}(t), t)\|_{H^1_a} + C(| c(t) - c_+ | + | \tilde{\gamma}(t) |) \\ \leq\ & C\|w(\cdot, t)\|_{H^1} + C(| c(t) - c_+ | + | \tilde{\gamma}(t) |) \\ \leq\ & C\epsilon e^{-bt}. \end{aligned} \tag{6.10}$$

This completes the proof □

References

[1] L. Cui and Y. Zhao: *Orbital stability of solitary waves for coupled BBM equations*, *Adv. Math.* (*China*), **41** (2012), 341–346.

[2] T. Mizumachi, R.L. Pego and J.R. Quintero: *Asymptotic stability of solitary waves in the Benney-Luke model of water waves*, *Differential Integral Equations* , **26** (2013), 253–301.

[3] J.R. Miller and M.I. Weinstein: *Asymptotic stability of solitary waves for the regularized long-wave equation*, *Comm. Pure Appl. Math.*, **49** (1996), 399–441.

[4] R.L. Pego and M.I. Weinstein: *Asymptotic stability of solitary waves*, *Comm. Math. Phys.*, **164** (1994), 305–349.

[5] R.L. Pego and M.I. Weinstein: *Convective linear stability of solitary waves for Boussinesq equations*, *Stud. Appl. Math.*, **99** (1997), 311–375.

[6] J. Prüss: *On the spectrum of C_0-semigroups*, *Trans. Amer. Math. Soc.*, **284** (1984), 847–857.