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Uniquness of mild solutions bounded on the whole time axis to the Navier-Stokes equations

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1 Introduction

This note is a survey of the work [12] jointly with R. Farwig(Technische Universität Darmstadt) and T. Nakatsuka(Nagoya University). Let Ω be 3-D exterior domain, the half-space $\mathbb{R}_{+}^{3}$, the whole space $\mathbb{R}^{3}$, a perturbed half-space, or an aperture domain with $\partial\Omega\in C^{\infty}$. The motion of a viscous incompressible fluid in Ω is governed by the Navier-Stokes equations:

$$\text{(N-S)}\left\{\begin{array}{rcll} \partial_{t}u-\Delta u+u\cdot\nabla u+\nabla p & = & f, & t\in\mathbb{R},\ \ x\in\Omega,\\ \operatorname{div} u & = & 0, & t\in\mathbb{R},\ \ x\in\Omega,\\ u|_{\partial\Omega} & = & 0, & t\in\mathbb{R}, \end{array}\right.$$

where $u=(u^{1}(x,t),u^{2}(x,t),u^{3}(x,t))$ and $p=p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t)\in\Omega\times\mathbb{R}$. Here f is a given external force. In this paper we consider the uniqueness of mild solutions to (N-S) in *unbounded* domains Ω which are bounded on the whole time axis. Typical examples of such solutions are periodic-in-time and almost periodic-in-time solutions.

In case where $\Omega\subset\mathbb{R}^{3}$ is bounded, the existence and uniqueness of time-periodic solutions were considered by several authors; see e.g. [8] and references therein. Maremonti [31, 32] was the first to prove the existence of unique time-periodic regular solutions to (N-S) in *unbounded* domains, namely for $\Omega=\mathbb{R}^{3}$ and $\Omega=\mathbb{R}_{+}^{3}$. In the case of more general unbounded domains, the existence of time-periodic solutions was proven by Kozono-Nakao [24], Maremonti-Padula [33], Salvi [39], Yamazaki [46], Galdi-Sohr [17], Kubo [28], Crispo-Maremonti [6] and Kang-Miura-Tsai [22]. In particular, Yamazaki [46] proved the existence of time-periodic mild solutions in $L^{3,\infty}(\Omega)$ in the case where Ω is a 3D exterior domain with $\partial\Omega \in C^\infty$. Here $L^{p,q}$ denotes the Lorentz space and $L^{p,\infty}$ is equivalent to the weak-L^p space (L^p_w). Without time-periodic condition on f, the existence of mild solutions bounded on the whole time axis was also shown in [24], [46] and [22]. Furthermore, Kang-Miura-Tsai [22] showed the existence of mild solutions u with the spatial decay

$$\sup_t \sup_{|x|>r} |x|^\alpha |u(x,t) - U(x)| < \infty \tag{1.1}$$

for some $\alpha > 1$, $r > 0$ and some function $U(x)$ with $\sup_{|x|>r} |x||U(x)| < \infty$, if $\Omega \subset \mathbb{R}^3$ is an exterior domain and if f satisfies adequate conditions. They also dealt with the inhomogeneous boundary value problem. Concerning the uniqueness of solutions bounded on the whole time-axis, roughly speaking, it was shown in [31, 32, 24, 33, 46, 28, 6] that a small solution in some function spaces (e.g. $BC(\mathbb{R}; L^{3,\infty}(\Omega))$) is unique within the class of solutions which are sufficiently small; i.e., if u and v are solutions for the same force f and if *both of them* are small, then $u = v$. In [17], Galdi-Sohr showed that a small time-periodic solution is unique within the larger class of all periodic weak solutions v with $\nabla v \in L^2(0,T;L^2)$, satisfying the energy inequality $\int_0^T \|\nabla v\|_{L^2}^2 \, d\tau \le -\int_0^T (F, \nabla v)\, d\tau$ and mild integrability conditions on the corresponding pressure; here T is a period of F and $f = \nabla \cdot F$.

Another type of uniqueness theorem was proven in [44, 13, 14] without assuming the energy inequality. In the case of an exterior domain $\Omega \subset \mathbb{R}^3$, the whole space $\mathbb{R}^3$, the halfspace $\mathbb{R}^3_+$, a perturbed halfspace, or an aperture domain, it was shown in [44, 13, 14] that if u and v are periodic-in-time, almost periodic-in-time or *backward asymptotically almost* periodic-in-time solutions in

$$BC(\mathbb{R}; L^{3,\infty}) \cap L^2_{uloc}(\mathbb{R}; L^{6,2}) \tag{1.2}$$

for the same force f, and if *one of them* is small in $L^{3,\infty}$, then $u = v$. In [37, 38], similar uniqueness theorems for stationary solutions were proven. In [38], it was shown that if u and v are stationary solutions in $L^{3,\infty}$ with $\nabla u, \nabla v \in L^{3/2,\infty}$ for the same force f, and if u is small in $L^{3,\infty}$ and $v \in L^3 + L^\infty$, then $u = v$.

Note that stationary as well as continuous time-periodic and almost periodic-in-time $L^{3,\infty}$-solutions u have a *precompact* range $R(u) = \{u(t); t \in \mathbb{R}\}$ in $L^{3,\infty}$, see [5, Theorem 6.5]. Furthermore, there exist many functions which have a precompact range and are not almost periodic, e.g. $a\sin(t^2)$ for $a \ne 0$. Hence, the set of all functions having precompact range is much larger than the set of all almost periodic functions. In this article, we establish new uniqueness theorems for bounded continuous solutions having precompact range on the whole time axis, which improve our previous results in [44, 13, 14, 37, 38]. We also consider the uniqueness of solutions with (1.1) and solutions in weighted L^∞ spaces.

Our proof is based on an idea given by Lions-Masmoudi [30]. They proved the uniqueness of L^n-solutions to the initial-boundary value problem of (N-S) by using the backward initial-boundary value problem of *dual* equations. Of course, in the initial-boundary value problem of (N-S), the initial condition $u(0) = v(0)$ plays an important role in proving $w(t) := u(t) - v(t) = 0$ for $t > 0$. In our problem, however, we cannot assume $u(0) = v(0)$, and hence, it is difficult to prove $w \equiv 0$ directly. A key point of our proof is to show $\lim_{j\to\infty} j^{-1} \int_{-j}^{0} \|w(t)\|_{L^2(B)}^2 dt = 0$ for any ball B, by using the method of dual equations. Then, applying some uniqueness theorems on mild solutions, we can conclude $w \equiv 0$, under some hypotheses.

Throughout this paper we impose the following assumption on the domain.

Assumption 1 *$\Omega \subset \mathbb{R}^3$ is an exterior domain, the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^\infty$.*

For the definitions of perturbed half-spaces and aperture domains, see Kubo-Shibata [29] and Farwig-Sohr [9, 10]. Let $BC(I;X)$ denote the set of all bounded continuous functions on an interval I with values in a Banach space X. The open ball in X with center 0 and radius $R > 0$ will be denoted by $B_R(0) = B_R$.

Now our main results on uniqueness of mild $L^{3,\infty}$-solutions, to be defined in the next section, read as follows:

Theorem 1. *Let Ω satisfy Assumption 1. There exists a constant $\delta(\Omega) > 0$ such that if $T \le \infty$, u and v are mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty,T); \tilde{L}^{3,\infty}_\sigma), \tag{1.3}$$

$$\text{the range } \mathcal{R}(v) := \{v(t); t \in (-\infty,T)\} \text{ is precompact in } L^{3,\infty} \tag{1.4}$$

and if

$$\limsup_{t\to-\infty} \|u(t)\|_{L^{3,\infty}} < \delta, \tag{1.5}$$

then $u \equiv v$ on $(-\infty,T)$. Here $\tilde{L}^{3,\infty}_\sigma = \overline{L^{3,\infty}_\sigma \cap L^\infty}^{\|\cdot\|_{L^{3,\infty}}}$.

Remark 1. (i) Yamazaki [46] proved the existence of bounded continuous mild $L^{3,\infty}$-solutions u on the whole time axis, if f can be written in the form $f = \nabla \cdot F$, $F \in BUC(\mathbb{R}; L^{3/2,\infty})$ and F is sufficiently small. We note that, in addition to this smallness condition on F, if we assume $f \in BC(\mathbb{R}; L^{3,\infty})$, then standard arguments easily prove that Yamazaki's small solution u belongs to $L^\infty(\mathbb{R}; L^9) \cap BC(\mathbb{R}; L^{3,\infty}_\sigma)$; see [13, Remark 2]. Then, u belongs $BC(\mathbb{R}; \tilde{L}^{3,\infty}_\sigma)$, since $L^{3,\infty}_\sigma \cap L^9$ is dense in $\tilde{L}^{3,\infty}_\sigma$. Moreover, Yamazaki showed that if F is almost periodic in $L^{3/2,\infty}$, then u is almost periodic in $L^{3,\infty}$. Since an almost periodic function in $L^{3,\infty}_\sigma$ has a precompact range in $L^{3,\infty}_\sigma$, Theorem 1 is applicable to his solution. For the definition and properties of almost periodic functions in a Banach space, see [5].

(ii) In [13], a similar uniqueness theorem was proven for almost periodic mild $L^{3,\infty}$-solutions. Since it was assumed that both of u and v are almost periodic and belong to (1.2) and since the class (1.3) is strictly larger than (1.2), Theorem 1 improves the result given in [13].

(iii) The condition (1.3) can be replaced by some condition more general than (1.3). For details, see [12]

Theorem 2. *Let Ω satisfy Assumption 1. There exists a constant $\delta(\Omega) > 0$ with the following property: Let $R > 0$, $p > 3$, $T \leq \infty$, u and v be mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty, T); \tilde{L}^{3,\infty}_\sigma(\Omega) \cap L^p(\Omega \cap B_R)),$$

and let

$$\limsup_{t \to -\infty} \|u(t)\|_{L^{3,\infty}} < \delta.$$

Assume that either

(i) The range

$$\{v(t)\big|_{\Omega \setminus B_R}\ ; t \in (-\infty, T)\} \text{ is precompact in } L^{3,\infty}(\Omega \setminus B_R), \tag{1.6}$$

or

(ii) there exists a function $V(x) \in L^{3,\infty}(\Omega \setminus B_R)$ such that

$$\limsup_{t \to -\infty} \big\|v(t) - V\big\|_{L^{3,\infty}(\Omega \setminus B_R)} < \delta. \tag{1.7}$$

Then $u \equiv v$ on $(-\infty, T)$.

The following corollaries are direct consequences of Theorem 2.

Corollary 1. *Let $\Omega = \mathbb{R}^3$, $T \le \infty$ and $\alpha > 1$. If u, v are mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty, T); X_\alpha), \quad \limsup_{t \to -\infty} \|u(t)\|_{L^{3,\infty}} < \delta,$$

then $u \equiv v$ on $(-\infty, T)$. Here $X_\alpha := \{f \in L^\infty \ ; \ \|(1+|x|)^\alpha f(x)\|_{L^\infty} < \infty\}$.

It is straightforward to see that if $v \in BC((-\infty,T); X_\alpha)$ for some $\alpha > 1$, then v belongs to $BC((-\infty,T); L^{3,\infty} \cap L^\infty)$ and satisfies (1.7) with $V \equiv 0$ for large $R > 0$.

Corollary 2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^\infty$, $T \le \infty$, $\alpha > 1$ and $p > 3$. If u, v are mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty, T); L^{3,\infty}_\sigma \cap L^p(\Omega)), \quad \limsup_{t \to -\infty} \|u(t)\|_{L^{3,\infty}} < \delta,$$

and if there exist $r > 0$, $s \in (-\infty, T)$ and $V \in L^{3,\infty}(\Omega \setminus B_r)$ such that

$$\sup_{t<s} \sup_{|x|>r} |x|^\alpha |v(x,t) - V(x)| < \infty, \tag{1.8}$$

then $u \equiv v$ on $(-\infty, T)$.

For the proof note that $L^{3,\infty}_\sigma \cap L^p \subset \tilde{L}^{3,\infty}$. Moreover, we see easily that if v satisfies (1.8) for some $\alpha > 1$, then (1.7) holds for sufficiently large $R > r$.

Remark 2. The existence of small mild solutions with property (1.8) was proven by Kang-Miura-Tsai [22] if Ω is a 3D exterior domain with $\partial\Omega \in C^\infty$ and under adequate conditons on f. Moreover, if $\Omega = \mathbb{R}^3$, the existence of small mild solutions in $BC(\mathbb{R}; X_\alpha)$ was also proven in [22] for $1 \le \alpha < 2$.

2 Preliminaries

In this section, we introduce some notation, function spaces and key lemmata. Let $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector fields $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that div $\phi = 0$. Then L^r_σ, $1 < r < \infty$, is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$. Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. Let $L^{p,q}(\Omega)$, $1 \le p, q \le \infty$, denote the Lorentz spaces and $\|\cdot\|_{p,q}$ the norm (not quasi-norm) of $L^{p,q}(\Omega)$; for the definition and properties of $L^{p,q}(\Omega)$, see e.g. [1]. The symbol $(\cdot,\cdot)$ denotes the L^2- inner product and the duality pairing between $L^{p,q}$ and $L^{p',q'}$, where $1/p+1/p'=1$ and $1/q+1/q'=1$. We note that $L^{p,\infty}$ is norm equivalent to the weak-L^p space (L^p_w) and $L^{p,p}$ is norm equivalent to L^p. Moreover, when $1<p<\infty$ and $1\le q<\infty$, then the dual space of $L^{p,q}$ is isometrically isomorphic to $L^{p',q'}$.

In this paper, we denote by C various constants. In particular, $C=C(*,\cdots,*)$ denotes a constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition: $L^r(\Omega)=L^r_\sigma\oplus G_r$ $(1<r<\infty)$, where $G_r=\{\nabla p\in L^r; p\in L^r_{loc}(\overline{\Omega})\}$, see Fujiwara-Morimoto [15], Miyakawa [35], Simader-Sohr [42], Borchers-Miyakawa [2], and Farwig-Sohr [9, 11]; P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is defined by $A_r=-P_r\Delta$ with domain $D(A_r)=W^{2,r}\cap W^{1,r}_0\cap L^r_\sigma$. It is known that $(L^r_\sigma)^*$ (the dual space of L^r_σ) $=L^{r'}_\sigma$ and A^*_r (the adjoint operator of A_r) $=A_{r'}$, where $1/r+1/r'=1$. It is shown by Giga [18], Giga-Sohr [19], Borchers-Miyakawa [2] and Farwig-Sohr [9, 11] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t\ge 0\}$ of class C_0 in L^r_σ. Since $P_ru=P_qu$ for all $u\in L^r\cap L^q$ $(1<r,q<\infty)$ and since $A_ru=A_qu$ for all $u\in D(A_r)\cap D(A_q)$, for simplicity, we shall abbreviate P_ru, P_qu as Pu for $u\in L^r\cap L^q$ and A_ru, A_qu as Au for $u\in D(A_r)\cap D(A_q)$, respectively. By real interpolation, we define $L^{p,q}_\sigma$ by

$$L^{p,q}_\sigma := [L^{p_0}_\sigma, L^{p_1}_\sigma]_{\theta,q}$$

where $1<p_0<p<p_1<\infty$, $\theta\in(0,1)$, $q\in[1,\infty]$ satisfy $1/p=(1-\theta)/p_0+\theta/p_1$.

Now, we define mild $L^{3,\infty}$-solutions to (N-S), following [25].

Definition 1 ([25]). *Let $T\le\infty$ and $f\in L^1_{loc}(-\infty,T;D(A_p)^*+D(A_q)^*)$ for some $1<p,q<\infty$. A function $v\in C((-\infty,T);L^{3,\infty}_\sigma)$ is called a mild $L^{3,\infty}$-solution to (N-S) on $(-\infty,T)$ if v satisfies*

(2.1)
$$(v(t),\phi)=\left(e^{-(t-s)A}v(s),\phi\right)+\int_s^t \left(\left(v(\tau)\cdot\nabla e^{-(t-\tau)A}\phi, v(\tau)\right)+<f(\tau),e^{-(t-\tau)A}\phi>\right)d\tau$$

for all $\phi\in L^{3/2,1}_\sigma$ and all $-\infty<s<t<T$.

In order to prove our main results, we recall properties of the Lorentz spaces, estimates of the Stokes semigroup and several uniqueness theorems for mild solutions.

Lemma 2.1 (Shibata [40, 41]). *For all $t > 0$ and $\phi \in L^{q,s}_\sigma$, the following inequalities are satisfied:*

$$\|e^{-tA}\phi\|_{p,r} \le Ct^{-3/2(1/q-1/p)}\|\phi\|_{q,s} \quad \text{when} \quad \begin{cases} 1 < q \le p < \infty, \ r = s \in [1,\infty], \\ 1 < q < p < \infty, \ r = 1, s = \infty, \end{cases} \tag{2.2}$$

$$\|\nabla e^{-tA}\phi\|_{p,r} \le Ct^{-1/2-3/2(1/q-1/p)}\|\phi\|_{q,s} \quad \text{when} \quad \begin{cases} 1 < q \le p \le 3, \ r = s \in [1,\infty], \\ 1 < q < p \le 3, \ r = 1, s = \infty. \end{cases} \tag{2.3}$$

In the case where Ω is an exterior domain, Shibata [40, 41] proved (2.2) and (2.3) for all $r = s$. If $q < p$, his estimates (2.2)-(2.3) with $r = s$ and real interpolation yield (2.2)-(2.3) even for $r = 1, s = \infty$. In the restricted case $r = 1$, Yamazaki [46] obtained (2.3) also by a method different from [40, 41]. In the case where Ω is $\mathbb{R}^3$, $\mathbb{R}^3_+$, a perturbed halfspace or an aperture domain, the usual L^q-L^p estimates for the Stokes semigroup and real interpolation directly yield (2.2)-(2.3), since in this case the L^q-L^p estimates hold for all $1 < q \le p < \infty$. For details of L^q-L^p estimates for the Stokes semigroup, see [45, 19, 21, 2, 3, 23, 40, 20, 29, 27].

Lemma 2.2 (Meyer [34], Yamazaki [46]). *The following estimates*

$$\int_s^t \left|(F(\tau), \nabla e^{-(t-\tau)A}\phi)\right| d\tau \le C(\operatorname*{ess\,sup}_{s<\tau<t} \|F\|_{3/2,\infty})\|\phi\|_{3/2,1}, \tag{2.4}$$

$$\int_s^t \left|(u \cdot \nabla e^{-(t-\tau)A}\phi, w)(\tau)\right| d\tau \le C(\operatorname*{ess\,sup}_{s<\tau<t} \|u\|_{3,\infty})(\operatorname*{ess\,sup}_{s<\tau<t} \|w\|_{3,\infty})\|\phi\|_{3/2,1} \tag{2.5}$$

hold for all $F \in L^\infty(s,t; L^{3/2,\infty})$, $u, w \in L^\infty(s,t; L^{3,\infty})$, $\phi \in L^{3/2,1}_\sigma(\Omega)$ and all $-\infty \le s < t$, where the constant C depends only on Ω.

In the case where Ω is an exterior domain, the whole space or halfspace, Yamazaki [46] proved Lemma 2.2 by real interpolation. His proof is also valid in the case where Ω is a perturbed halfspace or an aperture domain. In the case where $\Omega = \mathbb{R}^3$ Meyer [34] obtained Lemma 2.2 by a method different from [46].

The following lemma is direct consequence of Lemma 2.2 using the duality $L^{3,\infty}_\sigma = (L^{3/2,1}_\sigma)^*$.

Lemma 2.3 ([46]). *There exists a constant $\epsilon_0 = \epsilon_0(\Omega)$ with the following property: Let $T \le \infty$, $u, v, w \in BC((-\infty,T); L^{3,\infty}_\sigma)$ and let w satisfy*

$$(w(t), \phi) = \int_{-\infty}^t \Big(\big(w \cdot \nabla e^{-(t-\tau)A}\phi, u\big)(\tau) + \big(v \cdot \nabla e^{-(t-\tau)A}\phi, w\big)(\tau) \Big) d\tau \tag{2.6}$$

for all $\phi \in L^{3/2,1}_\sigma$ *and all* $-\infty < t < T$. *Assume that*

$$\sup_{-\infty<t<T} \|u\|_{3,\infty} + \sup_{-\infty<t<T} \|v\|_{3,\infty} < \epsilon_0.$$

Then, $w(t) = 0$ *for all* $t \in (-\infty, T)$.

Lemma 2.4. *Let* $T \leq \infty$. *If* u, v *are mild* $L^{3,\infty}$*-solutions to (N-S) on* $(0,T)$ *for the same force* f, $u(0) = v(0)$ *and*

$$u, v \in BC([0,T); \tilde{L}^{3,\infty}_\sigma), \tag{2.7}$$

then

$$u = v \text{ on } [0,T).$$

Lemma 2.4 was essentially proven by Meyer [34], Yamazaki [46] and Lions-Masmoudi [30]. See also Furioli, Lemarié-Rieusset and Terraneo [16], Cannone-Planchon [4], Monniaux [36]. We note that Lemma 2.4 can be proven by using Lemma 2.2, cf. [14, Lemma 2.5].

Lemma 2.5. *There exists a constant* $\epsilon_1(\Omega) > 0$ *such that if* $T \leq \infty$, u, v *are mild* $L^{3,\infty}$*-solutions to (N-S) on* $(-\infty, T)$ *for the same force* f,

$$u, v \in BC((-\infty,T); \tilde{L}^{3,\infty}_\sigma),$$

$$\limsup_{t\to-\infty} \|u(t)\|_{3,\infty} < \epsilon_1 \text{ and } \liminf_{t\to-\infty} \|u(t) - v(t)\|_{3,\infty} < \epsilon_1,$$

then

$$u = v \text{ on } (-\infty, T).$$

We can prove Lemma 2.5 by using Lemmata 2.3 and 2.4.

Finally, we come to the key lemma of the proof of uniqueness. If u and v are solutions to the Navier-Stokes equations, then $w := u - v$ satisfies

$$(U) \qquad \left\{ \begin{array}{rcl} \partial_t w - \Delta w + w\cdot\nabla u + v\cdot\nabla w + \nabla p' &=& 0, \ \ t \in (-\infty,T),\ x \in \Omega, \\ \operatorname{div} w &=& 0, \ \ t \in (-\infty,T),\ x \in \Omega, \\ w|_{\partial\Omega} &=& 0. \end{array} \right.$$

Hence, if Ω is a bounded domain and if u, v belong to the Leray-Hopf class, under the hypotheses of Theorem 1, the usual energy method and the Poincaré inequality yield $\|w(t)\|_2^2 \leq e^{-c(t-s)}\|w(s)\|_2^2$ for $t > s$. Letting $s \to -\infty$, we get $w(t) = 0$ for all t.

Consequently, in the case of *bounded* domains, Theorem 1 is obvious. In the case where Ω is an *unbounded* domain, u and v do not belong to the energy class in general and the Poincaré inequality does not hold in general. Hence, since we cannot use the energy method, we will use the argument of Lions-Masmoudi [30].

We recall the dual equations of the above system (U), namely,

$$
\text{(D)} \left\{ \begin{array}{rcl} -\partial_t \Psi - \triangle \Psi - \sum_{i=1}^{3} u^i \nabla \Psi^i - v \cdot \nabla \Psi + \nabla \pi & = & h, \quad t \in (-\infty, 0), \ x \in \Omega, \\ \nabla \cdot \Psi & = & 0, \quad t \in (-\infty, 0), \ x \in \Omega, \\ \Psi|_{\partial\Omega} & = & 0, \\ \Psi(0) & = & 0. \end{array} \right.
$$

Lemma 2.6. *There exists an absolute constant $\delta_0 > 0$ with the following property: Let $u, v \in BC((-\infty, 0]; \tilde{L}^{3,\infty}_\sigma)$, $h \in BC((-\infty, 0]; L^{6/5} \cap L^2)$ and*

$$\sup_{t \le 0} \|u(t)\|_{3,\infty} \le \delta_0.$$

Then there exists a unique solution $\Psi \in L^2_{\mathrm{loc}}((-\infty, 0]; D(A_2)) \cap W^{1,2}_{\mathrm{loc}}((-\infty, 0]; L^2_\sigma)$ to (D) such that

$$\|\Psi(t)\|_2^2 + \int_t^0 \|\nabla \Psi\|_2^2 \, d\tau \le C \int_t^0 \|h\|_{6/5}^2 \, d\tau \tag{2.8}$$

for all $t < 0$. Here C is an absolute constant.

Remark 3. Lemma 2.6 is valid for a general unbounded uniform C^2-domain $\Omega \subset \mathbb{R}^3$. For the properties of the Stokes operator A_2 in a uniform C^2-domain, see [43, 7].

3 Outline of the proof of Main Theorems

In this section, we prove Theorems 1 and 2. As in section 2 let $w = u - v$ for two given mild solutions u and v of (N-S). We first prove the following theorem:

Theorem 3. *Let $T \le \infty$, u and v be mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, T)$ for the same force f,*

$$u, v \in BC((-\infty, T); \tilde{L}^{3,\infty}_\sigma),$$

and let

$$\limsup_{t \to -\infty} \|u(t)\|_{3,\infty} < \delta_0, \tag{3.1}$$

where δ_0 is an absolute constant given in Lemma 2.6. Then there exists $s_0 < T$ such that

$$\lim_{j \to \infty} \frac{1}{j} \int_{-j+s_0}^{s_0} \|w(\tau)\|^2_{L^2(\Omega \cap B_r)} \, d\tau = 0 \ \ \textit{for all } r > 0. \tag{3.2}$$

Moreover, there exists a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} t_n = -\infty \ \textit{and} \ \lim_{n \to \infty} \|w(t_n)\|_{L^2(\Omega \cap B_r)} = 0 \ \ \textit{for all } r > 0. \tag{3.3}$$

Remark 4. (i) Since $\sup_{t<T} \|w(t)\|_{3,\infty} < \infty$ and since $C_0(\Omega)$ is dense in $L^{3/2,1}(\Omega)$, it is straightforward to see that (3.3) implies

$$w(t_n) \rightharpoonup 0 \text{ weakly-}^* \text{ in } L^{3,\infty}(\Omega) \text{ as } n \to \infty. \tag{3.4}$$

(ii) If we assume that both of u and v are stationary or time-periodic in $L^{3,\infty}$, then (3.2) directly yields $w \equiv 0$.

Outline of the proof of Theorem 3. By (3.1), there exists $s_0 < T$ such that $\sup\limits_{t \le s_0} \|u(t)\|_{3,\infty} \le \delta_0$. Without loss of generality, we may assume $0 < T$ and $s_0 = 0$. Let $j \in \mathbb{N}$. For $-3j < t < T$, let

$$\begin{aligned} w_0(t) &:= e^{-(t+3j)A} w(-3j) \\ w_1(t) &:= w(t) - w_0(t). \end{aligned} \tag{3.5}$$

Then, it holds that

$$(w_1(t), \phi) = \int_{-3j}^{t} \big((w \cdot \nabla e^{-(t-s)A} \phi, u) + (v \cdot \nabla e^{-(t-s)A} \phi, w) \big) \, ds$$

for all $\phi \in L^{3/2,1}_\sigma$. By Lemma 2.1, we have for $\varphi \in L^{3/2,1} \cap L^2$

$$\begin{aligned} |(w_1(t), \varphi)| &= |(w_1(t), P\varphi)| \\ &\le C(t+3j)^{\frac{1}{4}} \sup_{-\infty < s < T} \|w(s)\|_{3,\infty} (\|u(s)\|_{3,\infty} + \|v(s)\|_{3,\infty}) \|\varphi\|_2, \end{aligned} \tag{3.6}$$

which implies $w_1(t) \in L^2$ for $-3j < t < T$ and

$$\|w_1(t)\|_2 \le C(t+3j)^{\frac{1}{4}} \sup_{-\infty < s < T} \|w\|_{3,\infty} \sup_{-\infty < s < T} (\|u\|_{3,\infty} + \|v\|_{3,\infty}). \tag{3.7}$$

Furthermore we can observe that w_1 satisfies

$$\begin{aligned} &\int_{-j}^{0} \big((w_1, -\partial_t \psi - \Delta \psi) - (w \cdot \nabla \psi, u) - (v \cdot \nabla \psi, w) \big) \, ds \\ &= (w_1(-j), \psi(-j)) - (w_1(0), \psi(0)) \end{aligned} \tag{3.8}$$

for all $\psi \in W^{1,2}(-j,0;L^2_\sigma) \cap L^2(-j,0;D(A_2))$.

Let $\Omega_r := \Omega \cap B(0,r)$ for fixed $r>0$ and

$$h(x,t) := w(x,t) \cdot 1_{\Omega_r}.$$

In order to show (3.2), we decompose $⨍_{-j}^{0} \|w(\tau)\|^2_{L^2(\Omega_r)}\,d\tau$, the integral mean of $\|w(\tau)\|^2_{L^2(\Omega_r)}$ over the interval $(-j,0)$, into two terms as follows:

$$\begin{aligned} ⨍_{-j}^{0} \|w(\tau)\|^2_{L^2(\Omega_r)}\,d\tau &= ⨍_{-j}^{0} (w(\tau),h(\tau))\,d\tau \\ &= ⨍_{-j}^{0} (w_0(\tau),h(\tau))\,d\tau + ⨍_{-j}^{0} (w_1(\tau),h(\tau))\,d\tau =: I_0 + I_1. \end{aligned}$$

We estimate I_0 and I_1 separately. Since

$$\|h\|_{6/5} = \|w\cdot 1_{\Omega_r}\|_{L^{6/5}} \le C\|w\|_{3,\infty}\|1_{\Omega_r}\|_{2,6/5} \le C\|w\|_{3,\infty}|\Omega_r|^{1/2}, \tag{3.9}$$

from Lemma 2.1 we obtain

$$\begin{aligned} |I_0| &\le ⨍_{-j}^{0} \|w_0(\tau)\|_6\|h\|_{6/5}\,d\tau \le C⨍_{-j}^{0} \|e^{-(\tau+3j)A}w(-3j)\|_6\|w(\tau)\|_{3,\infty}|\Omega_r|^{1/2}\,d\tau \\ &\le C⨍_{-j}^{0} (\tau+3j)^{-\frac{1}{4}}\|w(-3j)\|_{3,\infty}\|w(\tau)\|_{3,\infty}|\Omega_r|^{1/2}\,d\tau \le Cj^{-1/4} \to 0 \end{aligned} \tag{3.10}$$

as $j\to\infty$.

Let Ψ be the solution to (D) with right-hand side $h = w\cdot 1_{\Omega_r}$ and initial value $\Psi(0)=0$, cf. Lemma 2.6. Then, we can observe

$$\begin{aligned} I_1 &= \frac{1}{j}(w_1(-j),\Psi(-j)) + ⨍_{-j}^{0}(w_0\cdot\nabla\Psi,u)\,d\tau + ⨍_{-j}^{0}(v\cdot\nabla\Psi,w_0)\,d\tau \\ &=: J_0+J_1+J_2. \end{aligned}$$

By using (2.8), (3.7), (3.9) and Lemma 2.1 we can show that J_0, J_1 and J_2 converge to 0 as $j\to\infty$. Hence, by (3.10) we have

$$⨍_{-j}^{0} \|w\|^2_{L^2(\Omega_r)}\,d\tau = I_0+I_1 \to 0 \text{ as } j\to\infty,$$

which proves (3.2). It is straightforward to see that (3.2) implies

$$\liminf_{t\to-\infty}\|w(t)\|_{L^2(\Omega_r)} = 0 \text{ for all } r>0.$$

Therefore, with $r=n$, we see that for all $n=1,2,\cdots$, there exists t_n such that

$$t_n < -n, \quad \|w(t_n)\|_{L^2(\Omega_n)} \le 1/n,$$

which implies (3.3). $\square$

Proof of Theorem 1. Let $\delta < \epsilon_1/4$, where ϵ_1 is a constant given in Lemma 2.5. In view of Lemma 2.5, it suffices to show

$$\liminf_{t\to-\infty} \|w(t)\|_{3,\infty} < \epsilon_1. \tag{3.11}$$

Let $\{t_n\}$ be the sequence given in Theorem 3. Due to the precompact range condition on v, i.e., $\mathcal{R}(v) = \{v(t)\ ;\ t < T\}$ is precompact in $L^{3,\infty}(\Omega)$, there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a function $V(x) \in L^{3,\infty}(\Omega)$ such that

$$\lim_{k\to\infty} \|v(t_{n_k}) - V\|_{3,\infty} = 0. \tag{3.12}$$

Since (3.4) implies $w(t_{n_k}) + V \to V$ weakly-* in $L^{3,\infty}(\Omega)$, by (3.12) and the assumption $\limsup_{t\to-\infty} \|u\|_{3,\infty} < \delta$ we have

$$\|V\|_{3,\infty} \le \liminf_{k\to\infty} \|w(t_{n_k}) + V\|_{3,\infty} \le \limsup_{k\to\infty} \|u(t_{n_k}) - (v(t_{n_k}) - V)\|_{3,\infty} < \delta. \tag{3.13}$$

Therefore, since $w = u - (v - V) - V$, we obtain

$$\limsup_{k\to\infty} \|w(t_{n_k})\|_{3,\infty} \le \limsup_{k\to\infty} (\|u(t_{n_k})\|_{3,\infty} + \|v(t_{n_k}) - V\|_{3,\infty} + \|V\|_{3,\infty}) < 2\delta,$$

which proves (3.11). □

Proof of Theorem 2. Let δ be the constant given in Proof of Theorem 1 and let $\{t_n\}$ be the sequence given in Theorem 3. Since, with $\Omega_R = \Omega \cap B_R$,

$$\|w(t_n)\|_{L^{3,\infty}(\Omega_R)} \le C \|w(t_n)\|^{\theta}_{L^2(\Omega_R)} \|w(t_n)\|^{1-\theta}_{L^p(\Omega_R)}$$

holds for $\frac{1}{3} = \frac{\theta}{2} + \frac{1-\theta}{p}$, by (3.3) and the assumption $u, v \in BC((-\infty, T; L^p(\Omega_R))$, we have

$$\lim_{n\to\infty} \|w(t_n)\|_{L^{3,\infty}(\Omega_R)} = 0. \tag{3.14}$$

Let $E := \Omega \setminus B_R$.

(i) Assume that (1.6) holds. In the same way as in (3.12)-(3.13), from (3.4) and (1.6), we observe that there exist a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a function $V(x) \in L^{3,\infty}(E)$ such that $\lim_{k\to\infty} \|v(t_{n_k}) - V\|_{L^{3,\infty}(E)} = 0$ and consequently also that $\|V\|_{L^{3,\infty}(E)} < \delta$. Then we conclude that

$$\limsup_{k\to\infty} \|w(t_{n_k})\|_{L^{3,\infty}(E)} \le \limsup_{k\to\infty} (\|u(t_{n_k})\|_{L^{3,\infty}(E)} + \|v(t_{n_k}) - V\|_{L^{3,\infty}(E)} + \|V\|_{L^{3,\infty}(E)}) < 2\delta.$$

This and (3.14) prove (3.11) and hence the first part of the theorem.

(ii) Assume that (1.7) holds. Since $\limsup_{n\to\infty} \|v(t_n) - V\|_{L^{3,\infty}(E)} < \delta$ and since (3.4) implies $w(t_n) + V \rightharpoonup V$ weakly-* in $L^{3,\infty}(E)$, in the same way as in the proof of (3.13), we obtain $\|V\|_{L^{3,\infty}(E)} < 2\delta$ and

$$\limsup_{n\to\infty} \|w(t_n)\|_{L^{3,\infty}(E)} \leq \limsup_{n\to\infty}(\|u(t_n)\|_{L^{3,\infty}(E)} + \|v(t_n) - V\|_{L^{3,\infty}(E)} + \|V\|_{L^{3,\infty}(E)}) < 4\delta.$$

This and (3.14) prove (3.11). □

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