New approach to the Hadamard variational formula for the Green function of the Stokes equations.

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Introduction.

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider the stationary Stokes equations with the homogeneous boundary condition, which describes the motion of the incompressible viscous fluid moving infinitesimally in Ω.

$$(0.1) \qquad \begin{cases} \Delta \boldsymbol{v} - \nabla q = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{v} = \boldsymbol{v}(x) = (v^1(x), \cdots, v^d(x))$ and $q = q(x)$ denote the unknown velocity and the pressure at $x = (x^1, \cdots, x^d) \in \Omega$ respectively, while $\boldsymbol{f} = \boldsymbol{f}(x) = (f^1(x), \cdots, f^d(x))$ is the given external force.

Our motivation is to lay the foundation of the analysis of the Hadamard variational formula for the Green functions of the Stokes equations, which may lead us to the generalization to the higher variation $\delta^n G$ and $\delta^n R$. Here G is the Green matrix for the velocity and R is the Green function for the pressure. More precisely, let $\{\Omega_\varepsilon\}_{\varepsilon \geq 0}$ be a family of the domain with a smooth boundary. For a small parameter $\varepsilon \geq 0$, we regard Ω_ε as a perturbed domain Ω with a volume preserving diffeomorphism $\Phi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$. We consider the Green function associated with (0.1) in Ω_ε

$$(0.2) \qquad \begin{cases} \Delta_x \boldsymbol{G}_{\varepsilon,n}(x,y) - \nabla_x R_{\varepsilon,n}(x,y) = e_n\delta(x-y), & (x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \operatorname{div} \boldsymbol{G}_{\varepsilon,n}(x,y) = 0, & (x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \boldsymbol{G}_{\varepsilon,n}(x,y) = 0, & x \in \partial\Omega_\varepsilon, y \in \Omega_\varepsilon, \end{cases}$$

for $n = 1, \cdots, d$, where $\boldsymbol{G}_{\varepsilon,n}(x,y) = \{G^i_{\varepsilon,n}(x,y)\}_{i,n=1,\cdots,d}$ is the Green matrix for the velocity, $R_{\varepsilon,n}(x,y) = \{R_{\varepsilon,n}(x,y)\}_{n=1,\cdots,d}$ is the Green function for the pressure and $\{e_n\}_{n=1,\cdots,d}$ denotes a canonical basis in $\mathbb{R}^d$. In this paper, we construct a new and systematic method and establish the formula for the first variation $\{\delta G, \delta R\}$ defined by $\delta G(x,y) :=$

$\lim_{\varepsilon \to 0} \varepsilon^{-1}(G_\varepsilon(x,y) - G(x,y))$ and $\delta R(x,y) := \lim_{\varepsilon \to 0} \varepsilon^{-1}(R_\varepsilon(x,y) - R(x,y))$, respectively. Moreover, we inductively derive the second variation $\{\delta^2 G, \delta^2 R\}$ defined by

$$\delta^2 G(x,y) := \lim_{\varepsilon \to 0} \frac{G_\varepsilon(x,y) - G(x,y) - \varepsilon\delta G(x,y)}{\varepsilon^2}$$

and

$$\delta^2 R(x,y) := \lim_{\varepsilon \to 0} \frac{R_\varepsilon(x,y) - R(x,y) - \varepsilon\delta R(x,y)}{\varepsilon^2}.$$

For the smooth function ρ defined on $\partial\Omega$, we denote Ω_ε by the interior domain with the boundary $\partial\Omega_\varepsilon := \{x + \varepsilon\rho(x)\nu_x \ ; \ x \in \partial\Omega\}$, where ν_x denotes the unit outer normal to $\partial\Omega$. The variational formula for the usual Laplace operator was first introduced by Hadamard [7] when ρ is a non-negative or a non-positive function on $\partial\Omega$. Later on, Garabedian [5], Garabedian-Schiffer [6] and Aomoto [1] refined the proof of [7], and they gave the formula as

$$(0.3) \qquad \delta G(y,z) = \int_{\partial\Omega} \frac{\partial G}{\partial\nu_x}(x,y)\frac{\partial G}{\partial\nu_x}(x,z)\rho(x)\, d\sigma_x$$

for all $y, z \in \Omega$. Furthermore, Garabedian-Schiffer [6] also gave rigorous proof of the second variation formula under the monotone perturbation. The formula was extended to some normal elliptic boundary value problem by Peetre [15] and Fujiwara-Ozawa [4]. In this paper, we consider the generalization of their results to the elliptic system of equations, and establish the formula for the higher variation. Recently, the author and Kozono [9] established the formula for the Green function of the Stokes equations corresponding to the velocity, and later on, its formula for the pressure was also proved by [17]. More precisely, they transformed (0.2) to the system in Ω by making use of the volume preserving diffeomorphism $\Phi_\varepsilon : \Omega \to \Omega_\varepsilon$ as in Inoue-Wakimoto [8], and gave the representation formula of $\delta G(y,z)$ and $\delta R(y,z)$ as

$$(0.4) \qquad \delta G_m^n(y,z) = \int_{\partial\Omega} \sum_{i=1}^d \frac{\partial G_n^i}{\partial\nu_x}(x,y)\frac{\partial G_m^i}{\partial\nu_x}(x,z)\rho(x)\, d\sigma_x,$$

$$(0.5) \qquad \delta R_m(y,z) = \int_{\partial\Omega} \sum_{i=1}^d \frac{\partial R_i(y,x)}{\partial\nu_x}\frac{\partial G_m^i}{\partial\nu_x}(x,z)\rho(x)\, d\sigma_x$$

for $m, n = 1, \cdots, d$ and for all $y, z \in \Omega$, respectively.

The analysis of the ε-dependence of the Green function G_ε is one of the most essential problem to derive the Hadamard variational formula. The author [17] investigated that by making use of the method developed by Fujiwara-Ozawa [4] and Ozawa [12]. Indeed, she applied the Schauder estimate of the Stokes equations due to Solonnikov [16], and proved that there exist $\delta G_n'(x,y) := \frac{d}{d\varepsilon}(\Phi_\varepsilon^{-1})_* G_{\varepsilon,n}(x,y)|_{\varepsilon=0}$ and $\delta R_n'(x,y) := \frac{d}{d\varepsilon}(\Phi_\varepsilon^{-1})_* R_{\varepsilon,n}(x,y)|_{\varepsilon=0}$. Then, she derived that $\{\delta G', \delta R'\}$ is a solution of the inhomogeneous Stokes equations in Ω with the homogeneous Dirichlet boundary condition on

$\partial\Omega$. Since $\{\delta\boldsymbol{G}', \delta R'\}$ is slightly different from the usual variation $\{\delta\boldsymbol{G}, \delta R\}$, we construct the variational formula for $\{\delta\boldsymbol{G}, \delta R\}$ by integration by parts of the volume integral for $\{\delta\boldsymbol{G}', \delta R'\}$. In such a procedure, we need to restrict ourselves to choose a special diffeomorphism $\Phi_\varepsilon(x)$ for each fixed $y \in \Omega$ in such a way that Φ_ε vanishes identically in the neighborhood of y. Indeed, since the order of the singularity of $R(x, y)$ at $x = y$ is by one degree higher than that of $G(x, y)$, it is necessary to control behavior of $R(x, y)$ near $x = y$ in transforming the volume integral in Ω that of the surface integral on $\partial\Omega$.

In the present paper, we handled the general volume preserving perturbation as in [9]. By making use of the error estimates between $\delta\boldsymbol{G} - \delta\boldsymbol{G}'$ and $\delta R - \delta R'$, we directly prove that $\{\delta\boldsymbol{G}, \delta R\}$ satisfies the homogeneous Stokes equations in Ω with some inhomogeneous boundary condition on $\partial\Omega$. As a result, by the usual integral representation formula we succeed to obtain a satisfactory Hadamard variational formula for both δG and δR like (0.4) and (0.5) without any restriction on choice of the diffeomorphism Φ_ε. Furthermore, we can inductively investigate the second variation of the Green function $\{\delta^2 G, \delta^2 R\}$. Namely, we establish a certain algorithm to get the second variation of the Hadamard variational formula for the velocity and the pressure.

1   Results.

To state our result, we first introduce an assumption on the perturbation $\{\Omega_\varepsilon\}_{\varepsilon \geq 0}$ of domains from Ω.

Assumption. For every $\varepsilon \geq 0$, there is a diffeomorphism $\Phi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ satisfying the following condition.

(A.1) $\Phi_\varepsilon = (\phi_\varepsilon^1, \phi_\varepsilon^2, \cdots, \phi_\varepsilon^d) \in C^\infty(\overline{\Omega})^d$;

(A.2) $\Phi_0(x) = x$ for all $x \in \overline{\Omega}$;

(A.3) There exists $S = (S^1, S^2, \cdots, S^d) \in C^\infty(\overline{\Omega})^d$ such that $K(x; \varepsilon) := \Phi_\varepsilon(x) - x - S(x)\varepsilon$ satisfies

$$\sup_{x \in \overline{\Omega}} |K_j(x; \varepsilon)| + \sup_{x \in \overline{\Omega}} |\nabla K_j(x; \varepsilon)| = O(\varepsilon^3), \quad \text{as } \varepsilon \to 0, \quad j = 1, \cdots, d;$$

(A.4) It holds that

$$\det\left(\frac{\partial \phi_\varepsilon^i}{\partial x^j}(x)\right)_{i,j=1,\cdots,d} = 1 \quad \text{for all } x \in \overline{\Omega} \text{ and all } \varepsilon \geq 0.$$

It should be noticed that by (A.4), Φ_ε defines a volume preserving diffeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}_\varepsilon$ i.e., $\text{vol}(\Omega_\varepsilon) = \text{vol}(\Omega)$ for all $\varepsilon \geq 0$. Moreover, the vector function $S \in C^\infty(\overline{\Omega})^d$ defined by (A.3) satisfies the divergence free property. Namely, it holds that

(1.1) $\text{div}\, S(x) = 0$

for all $x \in \bar{\Omega}$. For the proof, see Inoue-Wakimoto[8, Proposition 2.3].

Now we can state our result.

Let $\{G_{\varepsilon,m}, R_{\varepsilon,m}\}_{m=1,\cdots,d}$ be the Green function of the Dirichlet boundary value problem for the Stokes equations, i.e.,

$$(1.2) \quad \begin{cases} \Delta_{\tilde{x}} G_{\varepsilon,m}(\tilde{x}, \tilde{z}) - \nabla_{\tilde{x}} R_{\varepsilon,m}(\tilde{x}, \tilde{z}) = \delta(\tilde{x} - \tilde{z})e_m & (\tilde{x}, \tilde{z}) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ \operatorname{div}_{\tilde{x}} G_{\varepsilon,m}(\tilde{x}, \tilde{z}) = 0 & (\tilde{x}, \tilde{z}) \in \Omega_\varepsilon \times \Omega_\varepsilon, \\ G_{\varepsilon,m}(\tilde{x}, \tilde{z}) = 0 & \tilde{x} \in \partial\Omega_\varepsilon, \ \tilde{z} \in \Omega_\varepsilon \end{cases}$$

for $m = 1, \cdots, d$, where $\{e_m\}_{m=1,\cdots,d}$ denotes a canonical basis in $\mathbb{R}^d$. Clearly, the Green function $\{G_{0,m}, R_{0,m}\}_{m=1,\cdots,d} = \{G_m, R_m\}_{m=1,\cdots,d}$, respectively. Furthermore, we introduce the fundamental solution $\{p_m\}_{m=1,\cdots,d}$ for the pressure of (1.2) as follows;

$$(1.3) \quad p_m(x, z) := -\frac{1}{\omega_d} \frac{(x^m - z^m)}{|x - z|^d}, \quad m = 1, \cdots, d,$$

where ω_d denotes the volume of unit sphere in $\mathbb{R}^d$, i.e., $\omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

Theorem 1.1. *Let* $\{G_{\varepsilon,m}, R_{\varepsilon,m}\}_{m=1,\cdots,d}$ *be the Green matrix of the Dirichlet boundary value problem for (1.2) with* $\int_{\Omega_\varepsilon} (R_{\varepsilon,m}(\tilde{x}, \tilde{z}) - p_m(\tilde{x}, \tilde{z})) d\tilde{x} = 0$ *for all* $\tilde{z} \in \Omega_\varepsilon$. *Then there exist*

$$(1.4) \quad \delta G_m^n(y, z) = \lim_{\varepsilon \to 0} \frac{G_{\varepsilon,m}^n(y, z) - G_m^n(y, z)}{\varepsilon}, \quad \delta R_m(y, z) := \lim_{\varepsilon \to 0} \frac{R_{\varepsilon,m}(y, z) - R_m(y, z)}{\varepsilon}$$

for all $y, z \in \Omega$ *with* $y \neq z$, *which have the explicit representation as*

$$\delta G_m^n(y, z) = \int_{\partial\Omega} \sum_{i=1}^d \frac{\partial G_n^i}{\partial \nu_x}(x, y) \frac{\partial G_m^i}{\partial \nu_x}(x, z) S(x) \cdot \nu_x \, d\sigma_x,$$

$$\delta R_m(y, z) = \int_{\partial\Omega} \sum_{i=1}^d \frac{\partial R_i(y, x)}{\partial \nu_x} \frac{\partial G_m^i}{\partial \nu_x}(x, z) S(x) \cdot \nu_x \, d\sigma_x$$

for $m, n = 1, \cdots, d$, *where* $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ *is the unit outer normal to* $\partial\Omega$ *at* $x \in \partial\Omega$ *and* σ_x *denotes the surface element of* $\partial\Omega$.

We next mention the Hadamard variational formula for the second variation.

Theorem 1.2. *Let* $\{G_{\varepsilon,m}, R_{\varepsilon,m}\}_{m=1,\cdots,d}$ *be the Green function of the Dirichlet boundary value problem for (1.2) with* $\int_{\Omega_\varepsilon} (R_{\varepsilon,m}(\tilde{x}, \tilde{z}) - p_m(\tilde{x}, \tilde{z})) d\tilde{x} = 0$ *for all* $\tilde{z} \in \Omega_\varepsilon$. *Then there exist*

$$(1.5) \quad \delta^2 G_m^n(y, z) := \lim_{\varepsilon \to 0} \frac{G_{\varepsilon,m}^n(y, z) - G_m^n(y, z) - \varepsilon \delta G_m^n(y, z)}{\varepsilon^2},$$

$$(1.6) \quad \delta^2 R_m(y, z) := \lim_{\varepsilon \to 0} \frac{R_{\varepsilon,m}(y, z) - R_m(y, z) - \varepsilon \delta R_m(y, z)}{\varepsilon^2}$$

for all $y, z \in \Omega$ with $y \neq z$, which have the explicit representation as

$$\delta^2 G_m^n(y, z) = \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial G_n^i}{\partial \nu_x}(x, y) \left(\nabla_x \delta G_m^i(x, z) \cdot S(x) + 2\frac{\partial^2 G_m^i}{\partial \nu_x^2}(x, z)(S(x) \cdot \nu_x)^2 \right) d\sigma_x,$$

$$\delta^2 R_m(y, z) = \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial R_i(y, x)}{\partial \nu_x} \left(\nabla_x \delta G_m^i(x, z) \cdot S(x) + 2\frac{\partial^2 G_m^i}{\partial \nu_x^2}(x, z)(S(x) \cdot \nu_x)^2 \right) d\sigma_x$$

for $m, n = 1, \cdots, d$, where $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Remark 1.1. (1) Theorem 1.1 are the same formulas given by [17]. However, we establish a new approach to prove them, which is different from the previous one. This method enables us to remove a restriction about the diffeomorphism Φ_ε and to simplify the computation in [17].

(2) We establish the method to construct the second variation of the Green function $\{G_\varepsilon, R_\varepsilon\}$ in Theorem 1.2. This approach seems to be also applicable to the case of the Laplace equation under the general perturbation of domains.

2   Preliminaries.

Although our main result Theorem 1.1 and 1.2 hold for all $d \geq 2$, for the sake of simplicity, we restrict ourselves to the case $d \geq 3$. Indeed, it is easy to see that our argument in the subsequent sections works even in the case $d = 2$.

Firstly, we introduce the Green matrix of the Stokes equations derived by Odqvist [11]. The Green matrix $\{G_n^i\}_{n=1,\cdots,d}$ for the velocity and the Green function $\{R_n\}_{n=1,\cdots,d}$ for the pressure can be represented by the fundamental tensor $\{u_n, p_n\}_{n=1,\cdots,d}$ of the Stokes equations (1.2) with the compensation term $\{q_n, q_n\}_{n=1,\cdots,d}$ as follows.

$$(2.1) \qquad \begin{cases} G_n^i(x, y) = u_n^i(x, y) - q_n^i(x, y), \\ R_n(x, y) = p_n(x, y) - q_n(x, y), \end{cases}$$

where

$$(2.2) \qquad u_n^i(x, y) := -\frac{1}{2\omega_d(d-2)} \left[\frac{\delta^{in}}{|x - y|^{d-2}} + (d - 2)\frac{(x^i - y^i)(x^n - z^n)}{|x - y|^d} \right],$$

$\{p_n\}_{n=1,\cdots,d}$ is as in (1.3) for $i, n = 1, \cdots, d$. The compensation term $\{q_n, q_n\}_{n=1,\cdots,d}$ is chosen so that (1.2) is satisfied, that is,

$$(2.3) \qquad \begin{cases} \Delta_x q_n^i(x, y) - \nabla_{x^i} q_n(x, y) = 0, & x \in \Omega, \\ \displaystyle\sum_{i=1}^{d} \nabla_{x^i} q_n^i(x, y) = 0, & x \in \Omega, \\ q_n^i(x, y) = u_n^i(x.y), & x \in \partial\Omega, \quad i, n = 1, \cdots, d, \end{cases}$$

where $\nabla_{x^i} := \frac{\partial}{\partial x^i}$ for $i = 1, \cdots, d$. For any fixed $y \in \Omega$, $q_n^i(\cdot, y)$ and $q_n(\cdot, y)$ are analytic functions in Ω and continuous in $\overline{\Omega}$.

Finally, we recall the Green integral formula for the Stokes operator $\mathcal{L}$ as in [11], [10],

$$
\begin{aligned}
(2.4) \quad & \int_\Omega \sum_{i=1}^d \left\{ \mathcal{L}^i(\boldsymbol{v}, \pi)(x) w^i(x) - \mathcal{L}^i(\boldsymbol{w}, -\tilde{\pi})(x) v^i(x) \right\} dx \\
& = \int_{\partial\Omega} \sum_{i,j=1}^d \left\{ T^{ij}(\boldsymbol{v}, \pi)(x) w^i(x) - T^{ij}(\boldsymbol{w}, -\tilde{\pi})(x) v^i(x) \right\} \nu_x^j \, d\sigma_x,
\end{aligned}
$$

where $\{T^{ij}\}_{i,j=1,\cdots,d}$ is the stress tensor for the velocity $\boldsymbol{v}$ and the pressure π defined by

$$
(2.5) \qquad T^{ij}(\boldsymbol{v}, \pi)(x) := -\delta^{ij}\pi(x) + \left(\frac{\partial v^i}{\partial x^j}(x) + \frac{\partial v^j}{\partial x^i}(x) \right), \qquad i, j = 1, \cdots, d
$$

for the vector functions $\boldsymbol{v}, \boldsymbol{w} \in C^2(\overline{\Omega})^d$ with $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{w} = 0$ in Ω and scalar functions $\pi, \tilde{\pi} \in C^1(\overline{\Omega})$, $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Lemma 2.1. *Let* $\{\boldsymbol{G}_n, R_n\}_{n=1,\cdots,d}$ *be the Green function for the Stokes equations as in (2.1). Then it holds that*

$$
(2.6) \qquad \lim_{\rho \to 0} \int_{\partial B_\rho(y)} \sum_{i,j=1}^d T^{ij}(\boldsymbol{G}_n, R_n)(x, y) v^i(x) \nu_x^j \, d\sigma_x = -v^n(y), \quad n = 1, \cdots, d
$$

for all $y \in \Omega$ *and all smooth vector functions* $\boldsymbol{v} = (v^1, \cdots, v^d)$ *near* y, *where* $\{T^{ij}\}_{i,j=1,\cdots,d}$ *is the stress tensor defined by (2.5), $\partial B_\rho(y)$ denotes the surface centered at y with the radius ρ, ν_x is the unit inner normal vector to $\partial B_\rho(y)$ at x and σ_x denotes the surface element of $\partial B_\rho(y)$.*

3 Analysis of the ε-dependence of the Green function.

In this section, we consider the ε-dependence of the Green function $\{\boldsymbol{G}_{\varepsilon,m}, R_{\varepsilon,m}\}_{m=1,\cdots,d}$. Namely, we investigate the differentiability of the Green function with respect to ε. Indeed, it holds that;

Lemma 3.1. *For any fixed z in Ω, there exist* $\{\delta \boldsymbol{G}_m'(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ *and* $\{\delta R_m'(\cdot, z)\}_{m=1,\cdots,d} \in C^{1+\theta}(\overline{\Omega} \setminus \{z\})$ *with $0 < \theta < 1$, defined by*

$$
(3.1) \qquad \delta \boldsymbol{G}_m'(x, z) := \lim_{\varepsilon \to 0} \frac{\boldsymbol{g}_{\varepsilon,m}(x, z) - \boldsymbol{G}_m(x, z)}{\varepsilon},
$$

$$
(3.2) \qquad \delta R_m'(x, z) := \lim_{\varepsilon \to 0} \frac{r_{\varepsilon,m}(x, z) - R_m(x, z)}{\varepsilon}, \qquad\qquad m = 1, \cdots, d
$$

for all $x \in \overline{\Omega}$ with $x \neq z$, where $\{g_{\varepsilon,m}, r_{\varepsilon,m}\}_{m=1,\cdots,d}$ is respectively defined by

$$(3.3) \qquad g^i_{\varepsilon,m}(x,z) := \sum_{j=1}^{d} \frac{\partial x^i}{\partial \tilde{x}^j} G^j_{\varepsilon.m}(\tilde{x}, \tilde{z}),$$

$$(3.4) \qquad r_{\varepsilon,m}(x,z) := R_{\varepsilon,m}(\tilde{x}, \tilde{z}), \qquad i, m = 1, \cdots, d.$$

Similarly to Lemma 3.1, we can also assure the existence of the second variation $\{\delta^2 G'_m, \delta^2 R'_m\}_{m=1,\cdots,d}$ defined by

$$(3.5) \qquad \delta^2 G'_m(x,z) := \lim_{\varepsilon \to 0} \frac{g_{\varepsilon,m}(x,z) - G_m(x,z) - \varepsilon \delta G'_m(x,z)}{\varepsilon^2},$$

$$(3.6) \qquad \delta^2 R'_m(x,z) := \lim_{\varepsilon \to 0} \frac{r_{\varepsilon,m}(x,z) - R_m(x,z) - \varepsilon \delta R'_m(x,z)}{\varepsilon^2}, \qquad m = 1, \cdots, d,$$

where $\{g_{\varepsilon,m}, r_{\varepsilon,m}\}_{m=1,\cdots,d}$ and $\{\delta G'_m, \delta R'_m\}_{m=1,\cdots,d}$ are as in (3.3), (3.4), (3.1) and (3.2), respectively.

Corollary 3.1. *For any fixed z in Ω, there exist $\{\delta^2 G'_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ and $\{\delta^2 R'_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ with $0 < \theta < 1$.*

Remark 3.1. *The existence of δG defined by (1.4) is assured by the following identity, which is derived by Lemma 3.1. Indeed, it holds that*

$$(3.7)$$

$$\delta G^n_m(x,z) = \delta G'^n_m(x,z) - \nabla_x G^n_m(x,z) \cdot S(x) - \nabla_z G^n_m(x,z) \cdot S(z) + \sum_{i=1}^{d} \frac{\partial S^n}{\partial x^i}(x) G^i_m(x,z)$$

for $m, n = 1, \cdots, d$ and for all $x \in \overline{\Omega} \setminus \{z\}$, where $\{\delta G'_m, \delta R'_m\}_{m=1,\cdots,d}$ are as in (3.1) and (3.2), respectively, and $\{S^i\}_{i=1,\cdots,d}$ is the vector function introduced by (A.3). Furthermore, we also have by Corollary 3.1 that

$$(3.8)$$

$$\delta^2 G^n_m(x,z)$$
$$= \delta^2 G'^n_m(x,z) - \nabla_x \delta G^n_m(x,z) \cdot S(x) - \nabla_z \delta G^n_m(x,z) \cdot S(z) + \delta G_m(x,z) \cdot \nabla_x S^n(x)$$
$$+ \sum_{i=1}^{d} \left(\nabla_z G^i_m(x,z) \cdot S(z) \frac{\partial S^n}{\partial x^i}(x) - G_m(x,z) \cdot \nabla_x S^i(x) \frac{\partial S^n}{\partial x^i}(x) + \nabla_x G^i_m(x,z) \cdot S(x) \frac{\partial S^n}{\partial x^i}(x) \right)$$
$$- \nabla_x \nabla_z G^n_m(x,z) S(x) \cdot S(z) - \frac{1}{2} \nabla_z^2 G^n_m(x,z) S(z) \cdot S(z) \quad - \frac{1}{2} \nabla_x^2 G^n_m(x,z) S(x) \cdot S(x)$$

for $m, n = 1, \cdots, d$ and for all $x \in \overline{\Omega} \setminus \{z\}$, where $\{\delta^2 G'_m, \delta^2 R'_m\}_{m=1,\cdots,d}$ are as in (3.5) and (3.6), respectively.

4 Construction of the Hadamard variational formula.

In this section, we shall introduce the key lemmas to establish the representation formula for both of them. The following lemma is essential to construct the Hadamard variational formula for the first variation $\{\delta G_m, \delta R_m\}_{m=1,\cdots,d}$.

Lemma 4.1. *For any fixed* $z \in \Omega$, *there exist* $\{\delta G_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ *and* $\{\delta R_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ *with* $0 < \theta < 1$, *defined by (1.4) respectively, and they satisfy the Stokes equations with the inhomogeneous Dirichlet boundary condition, i.e.,*

$$(4.1) \quad \begin{cases} \Delta_x \delta G_m(x, z) - \nabla_x \delta R_m(x, z) = 0, & x \in \Omega, \\ \operatorname{div} \delta G_m(x, z) = 0, & x \in \Omega, \\ \delta G_m^n(x, z) = -\nabla_x G_m^n(x, z) \cdot S(x), & x \in \partial\Omega \end{cases}$$

for $m, n = 1, \cdots, d$, *where* $\{S^i\}_{i=1,\cdots,d}$ *is the vector function introduced by (A.3).*

We have similarly to Lemma 4.1 that $\{\delta^2 G_m, \delta^2 R_m\}_{m=1,\cdots,d}$ as in (1.5) and (1.6) also satisfies the homogeneous Stokes equations with the inhomogeneous Dirichlet boundary condition. Indeed, it holds;

Corollary 4.1. *For any fixed* $z \in \Omega$, *there exist* $\{\delta^2 G_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{2+\theta}(\overline{\Omega} \setminus \{z\})$ *and* $\{\delta^2 R_m(\cdot, z)\}_{m=1,\cdots,d} \in C^{1+\theta}(\overline{\Omega} \setminus \{z\})$ *defined by (1.5) and (1.6) with* $0 < \theta < 1$, *and they satisfy the Stokes equations with the inhomogeneous Dirichlet boundary condition, i.e.,*

$$(4.2) \quad \begin{cases} \Delta_x \delta^2 G_m(x, z) - \nabla_x \delta^2 R_m(x, z) = 0, & x \in \Omega, \\ \operatorname{div} \delta^2 G_m(x, z) = 0, & x \in \Omega, \\ \delta^2 G_m^n(x, z) = -2\nabla_x G_m^n(x, z) \cdot S(x) - \nabla_x^2 G_m^n(x, z)S(x) \cdot S(x), & x \in \partial\Omega \end{cases}$$

for $m, n = 1, \cdots, d$, *where* $\{S^i\}_{i=1,\cdots,d}$ *is the vector function introduced by (A.3).*

5 Proof of the main theorems.

In this section, we complete the proofs of Theorem 1.1 and 1.2.

5.1 Proof of Theorem 1.1.

We apply the Green integral formula for the Stokes operator $\mathcal{L}$ introduced by (2.4) in $\Omega \setminus B_\rho(y)$ to the functions

$$v = \delta G_m(x, z), \qquad \pi = \delta R_m(x, z),$$
$$w = G_n(x, y), \qquad \tilde{\pi} = -R_n(x, y)$$

for $m, n = 1, \cdots, d$. Then since $\mathcal{L}(\boldsymbol{G}_n, R_n)(x, y) = 0$ in $\Omega \setminus B_\rho(y)$, we have by Lemma 4.1 that

(5.1)
$$\int_{\partial\{\Omega \setminus B_\rho(y)\}} \sum_{i,j=1}^d \left\{ T^{ij}(\delta\boldsymbol{G}_m, \delta R_m)(x, z)G_n^i(x, y) - T^{ij}(\boldsymbol{G}_n, R_n)(x, y)\delta G_m^i(x, z) \right\} d\sigma_x = 0$$

for $m, n = 1, \cdots, d$. We next consider the limit $\rho \to 0$ at the point of y. It follows from Lemma 2.1 that

(5.2)
$$\lim_{\rho\to 0} \int_{\partial B_\rho(y)} \sum_{i,j=1}^d \left(T^{ij}(\delta\boldsymbol{G}_m, \delta R_m)(x, z)G_n^i(x, y) - T^{ij}(\boldsymbol{G}_n, R_n)(x, y)\delta G_m^i(x, z) \right) \nu_x^j \, d\sigma_x$$
$$= \delta G_m^n(y, z), \qquad m, n = 1, \cdots, d.$$

Since $\boldsymbol{G}_n(x, y) = 0$ for $x \in \partial\Omega$, and by (3.7) we have that

$$\delta G_m^n(x, z) = -\nabla_x G_m^n(x, z) \cdot S(x), \qquad m, n = 1, \cdots, d$$

for $x \in \partial\Omega$, it follows from (5.1) and (5.2) that

(5.3)
$$\delta G_m^n(y, z) = \int_{\partial\Omega} \sum_{i,j,k=1}^d T^{ij}(\boldsymbol{G}_n, R_n)(x, y)\frac{\partial G_m^i}{\partial x^k}(x, z)S^k(x)\nu_x^j \, d\sigma_x$$

for $m, n = 1, \cdots, d$ and for $y, z \in \Omega$. Furthermore, since $\boldsymbol{G}_m(x, z) = 0$ for $x \in \partial\Omega$, we have that

(5.4)
$$\frac{\partial G_m^i}{\partial x^k}(x, z) = \frac{\partial G_m^i}{\partial \nu_x}(x, z)\nu_x^k, \qquad i, k, m = 1, \cdots, d$$

for $x \in \partial\Omega$. Applying (5.4) to the right hand side of (5.3), it holds that

(5.5) $\delta G_m^n(y, z)$
$$= \int_{\partial\Omega} \sum_{i,j=1}^d \left(\frac{\partial G_n^i}{\partial \nu_x}(x, y)\frac{\partial G_m^i}{\partial \nu_x}(x, z) \right.$$
$$\left. + \operatorname{div} \boldsymbol{G}_n(x, y)\operatorname{div} \boldsymbol{G}_m(x, z) - R_n(x, y)\operatorname{div} \boldsymbol{G}_m(x, z) \right) S(x) \cdot \nu_x \, d\sigma_x$$
$$= \int_{\partial\Omega} \sum_{i=1}^d \frac{\partial G_n^i}{\partial \nu_x}(x, y)\frac{\partial G_m^i}{\partial \nu_x}(x, z)S(x) \cdot \nu_x \, d\sigma_x, \qquad m, n = 1, \cdots, d$$

for $y, z \in \Omega$.

Furthermore, it follows from (4.1) at the point $x = y$ and (5.5) that

$$(5.6) \qquad \nabla_{y^n} \delta R_m(y, z) = \Delta_y \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial G_n^i}{\partial \nu_x}(x, y) \frac{\partial G_m^i}{\partial \nu_x}(x, z) S(x) \cdot \nu_x \, d\sigma_x$$

for $m, n = 1, \cdots, d$. Since $\partial\Omega$ is sufficiently smooth, we have that for each $x \in \partial\Omega$,

$$(5.7) \qquad \Delta_y G_m^n(y, x) = \nabla_{y^n} R_m(y.x), \quad m, n = 1, \cdots, d$$

for $y \in \Omega$. Since $G_n^i(x, y) = G_i^n(y, x), i, n = 1. \cdots, d$ for $(x, y) \in \overline{\Omega} \times \Omega$, we apply (5.7) to the right hand side of (5.6), which yields Theorem 1.1.

5.2 Proof of Theorem 1.2.

In the same manner as Theorem 1.1, it follows from the Green integral formula (2.4) and (3.8) that

$$\delta^2 G_m^n(y, z)$$

$$= -\int_{\partial\Omega} \sum_{i,j=1}^{d} T^{ij}(G_n, R_n)(x, y) \left(2\nabla_x \delta G_m^i(x, z) \cdot S(x) + \sum_{k,l=1}^{d} \frac{\partial^2 G_m^i}{\partial x^l \partial x^k}(x, z) S^k(x) S^l(x) \right) \nu_x^j \, d\sigma_x$$

for $m, n = 1, \cdots, d$. Then by (5.4), we thus have that

$$(5.8)$$

$$\delta^2 G_m^n(y, z) = \int_{\partial\Omega} \sum_{i=1}^{d} \frac{\partial G_n^i}{\partial \nu_x}(x, y) \left(2\nabla_x \delta G_m^i(x, z) \cdot S(x) + \frac{\partial^2 G_m^i}{\partial \nu_x^2}(x, z)(S(x) \cdot \nu_x)^2 \right) d\sigma_x$$

for $m, n = 1, \cdots, d$. Furthermore, similarly to the proof of Lemma 1.2, by (4.2) and (5.8) we thus complete Theorem1.2. $\qquad \square$

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