数理解析研究所講究録
第 1883 巻 2014 年 60-65


# The Stokes semigroup on non-decaying spaces

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**Abstract**

In this brief note, we review recent results on the analyticity of the Stokes semigroup in spaces of bounded functions. The Stokes equations are well understood on $L^p$ space, $p\in(1,\infty)$, for various kinds of domains such as bounded or exterior domains with smooth boundaries. However, the situation is very different on $L^\infty$ since in this case the Helmholtz projection does not act as a bounded operator on $L^\infty$ anymore. The purpose of this note is to review an approach to prove the analyticity of the semigroup on $L^\infty$, especially, on $L_\sigma^\infty$ (and $BUC_\sigma$) for exterior domains and perturbed half spaces. Note that for merely bounded initial data, even existence of solutions are non-trivial. We approximate merely bounded initial data on $L_\sigma^\infty$ and prove the unique existence of solutions together with the analyticity of the semigroup. This note is based on joint works with Y. Giga [2], [3] and the thesis [1].

## 1 Introduction

We consider the initial-boundary problem for the Stokes equations in the domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$:

$$v_t-\Delta v+\nabla q=0 \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

$$\operatorname{div} v=0 \quad \text{in} \quad \Omega\times(0,T), \tag{1.2}$$

$$v=0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.3}$$

$$v=v_0 \quad \text{on} \quad \Omega\times\{t=0\}. \tag{1.4}$$

It is well known that the solution operator (called the Stokes semigroup)

$$S(t): v_0 \longmapsto v(\cdot,t), \quad t\geq 0,$$

forms an analytic semigroup on the solenoidal $L^p$ space, $L_\sigma^p(\Omega)$, $p\in(1,\infty)$, for various kind of domains $\Omega$, such as bounded and exterior domains with smooth boundaries [25], [13]. However, it had been a long-standing open problem whether or not the Stokes semigroup $\{S(t)\}_{t\geq0}$ is analytic on $L^\infty$-type spaces even if $\Omega$ is bounded. When $\Omega$ is a half space, it is known that the Stokes semigroup $\{S(t)\}_{t\geq0}$ is analytic on $L^\infty$-type spaces since explicit solution formulas are available [6], [19], [26].

In [2], Y. Giga and the author gave an affirmative answer to this open problem at least when $\Omega$ is bounded as a typical example. Later, this approach was extended to exterior domains [3] and perturbed half spaces ($n\geq 3$) [1]. The propose of this note is to review an approach to prove the existence of solutions for merely bounded initial data as well as the analyticity of the semigroup on $L^\infty$-type spaces.

We begin with a typical statement for bounded domains. Let $C_{0,\sigma}(\Omega)$ denote the $L^\infty$-closure of $C_{c,\sigma}^\infty(\Omega)$, the space of all smooth solenoidal vector fields with compact support in $\Omega$. When $\Omega$ is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all solenoidal vector fields continuous in $\bar{\Omega}$ vanishing on $\partial\Omega$ [18]. A typical result proved in [2, Theorem 1.1] is the following:

**Theorem 1.1** (Analyticity on $C_{0,\sigma}$). *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^3$-boundary. Then, the solution operator (the Stokes semigroup) $S(t) : v_0 \mapsto v(\cdot, t)$ is a $C_0$-analytic semigroup on $C_{0,\sigma}(\Omega)$.*

The approach to prove Theorem 1.1 was to establish an priori estimate for

$$N(v,q)(x,t) = \left|v(x,t)\right| + t^{\frac{1}{2}}\left|\nabla v(x,t)\right| + t\left|\nabla^2 v(x,t)\right| + t\left|\partial_t v(x,t)\right| + t\left|\nabla q(x,t)\right| \tag{1.5}$$

of the form

$$\sup_{0<t<T_0} \left\|N(v,q)\right\|_\infty(t) \leq C\|v_0\|_\infty \tag{1.6}$$

for some $T_0 > 0$ and $C$ depending only on the domain $\Omega$, where $\|v_0\|_\infty = \|v_0\|_{L^\infty(\Omega)}$ denotes the sup-norm of $|v_0|$ in $\Omega$. The a priori estimate (1.6) was proved by an indirect argument called *a blow-up argument* which is often used in the study of non-linear elliptic and parabolic equations [12], [14], [21], [20] (see also [17], [16] for the Navier–Stokes equations). Later, a direct approach to prove Theorem 1.1 was found in [4]. The approach in the paper is to derive $L^\infty$-estimates for solutions of the resolvent problem corresponding to (1.1)–(1.4) based on the Masuda-Stewart technique for elliptic operators.

In both approaches, a key is to estimate pressure gradient in terms of velocity, i.e.,

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x,\cdot)| \leq C\|w\|_{L^\infty(\partial\Omega)}, \tag{1.7}$$

where

$$w(v) = -(\nabla v - \nabla^T v)n_\Omega. \tag{1.8}$$

Here, $d_\Omega$ denotes the distance from $x \in \Omega$ to $\partial\Omega$, i.e., $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ and $n_\Omega$ denotes the unit outward normal vector field on $\partial\Omega$. For $n = 3$, $w(v)$ is nothing but a tangential component of vorticity, i.e., $-\mathrm{curl}\, v \times n_\Omega$. For $n = 2$, $w(v)$ agrees with $-\mathrm{curl}\, v n_\Omega^\perp$, where $n_\Omega^\perp = (n_\Omega^2, -n_\Omega^1)$. The estimate (1.7) plays an important role for estimating pressure gradient $\nabla q = (I - \mathbb{P})\Delta v$ by the velocity $v$ on $L^\infty$ since the Helmholtz projection $\mathbb{P}$ does not act as a bounded operator on $L^\infty$. Actually, the estimate (1.7) is a special case of the estimate for the homogeneous Neumann problem of the form

$$\Delta q = 0 \quad \text{in } \Omega, \quad \frac{\partial q}{\partial n_\Omega} = \mathrm{div}_{\partial\Omega}\, w \quad \partial\Omega, \tag{1.9}$$

where $\mathrm{div}_{\partial\Omega}$ denotes the surface divergence on $\partial\Omega$. Since the divergence-free condition for velocity implies

$$\Delta v \cdot n_\Omega = \mathrm{div}_{\partial\Omega}\, w(v) \quad \text{on } \partial\Omega,$$

the pressure $q$ solves the Neumann problem (1.9) for $w = w(v)$. The estimate (1.7) is valid for various domains, but it may not be true for general domains so we call $\Omega$ *strictly admissible* if the a priori estimate (1.7) holds for all solutions of the Neumann problem (1.9). Of course, a half space is strictly admissible. Moreover, it was proved that bounded domains [2, Theorem 2.5] and exterior domains [3, Theorem 3.1] of class $C^3$ are strictly admissible. However, layer domains are not strictly admissible. In fact, in a layer domain, $\Omega = \{x = (x', x_n) \in \mathbb{R}^n \mid 0 < x_n < 1\}$, $P = x_1$ does not satisfy the estimate (1.9) for $w = 0$. We conjecture that quasi-cylindrical domains, i.e., $\overline{\lim}_{|x|\to\infty} d_\Omega(x) < \infty$, are not strictly admissible.

Actually, it is possible to extend Theorem 1.1 for general strictly admissible, uniformly $C^3$-domains [2, Theorem 1.3] by using the $\tilde{L}^p$-theory developed in [8], [9], [10] since the space $C_{0,\sigma}$ is the $L^\infty$-closure of $C^\infty_{c,\sigma}$. Once we have the a priori estimate (1.6) for $v_0 \in C^\infty_{c,\sigma}$, it is extendable for $v_0 \in C_{0,\sigma}$. Note that the $L^p$-theory is also available for uniformly $C^3$-domains for which the Helmholtz projection is bounded on $L^p$ [11] so we are able to extend Theorem 1.1 through the $L^p$-theory for domains such as exterior domains or perturbed half spaces.

## 2 Non-decaying solenoidal spaces

It is natural to extend Theorem 1.1 for the larger space than $C_{0,\sigma}$,

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f \cdot \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space $\hat{W}^{1,1}(\Omega) = \{ \varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi \in L^1(\Omega) \}$. Since the space $L^\infty_\sigma$ includes discontinuous functions, we approximate $v_0 \in L^\infty_\sigma(\Omega)$ by elements of $C^\infty_{c,\sigma}$ by the pointwise convergence in $\Omega$. We extend the Stokes semigroup $S(t)$ to $L^\infty_\sigma$ by the following approximation [2, Lemma 6.3].

**Lemma 2.1** (Approximation). *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. There exists a constant $C = C_\Omega$ such that for $v_0 \in L^\infty_\sigma(\Omega)$, there exists a sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$ such that*

$$\begin{aligned} &\|v_{0,m}\|_{L^\infty(\Omega)} \leq C\|v_0\|_{L^\infty(\Omega)}, \\ &v_{0,m} \to v_0 \quad \text{a.e. in } \Omega \quad as\ m \to \infty. \end{aligned} \tag{2.1}$$

If we do not care about the divergence-free condition for the sequence $\{v_{0,m}\}_{m=1}^\infty$, it is easy to construct the sequence satisfying (2.1). Lemma 2.1 says that we are able to approximate $v_0 \in L^\infty_\sigma$ by solenoidal vector fields $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}$ keeping the sup-norm, i.e., $\|v_{0,m}\|_\infty \leq C\|v_0\|_\infty$. If $\Omega$ is star-shaped, i.e., $\lambda\overline{\Omega} \subset \Omega$, $\lambda < 1$, it is easy to construct the sequence satisfying (2.1). In fact, for $v_0 \in L^\infty_\sigma(\Omega)$, set $v_{0,\lambda}(x) = v_0(\lambda x)$ for $x \in \lambda\Omega$ and $v_{0,\lambda}(x) = 0$ for $x \in \Omega\backslash\lambda\overline{\Omega}$ so that $v_{0,\lambda}$ is a compactly supported solenoidal vector field in $\Omega$. Then, we get the desired sequence with $C = 1$ in (2.1) by multiplying the mollifier $\eta_\varepsilon$ to $v_{0,\lambda}$, i.e., $v_{0,m} = \eta_{1/m} * v_{0,\lambda_m}$. For general bounded domains, we are able to prove Lemma 2.1 by decomposing $\Omega$ into star-shaped domains.

By the above approximation, we are able to prove that the Stokes semigroup $S(t)$ is a (non-$C_0$-)analytic semigroup on $L^\infty_\sigma(\Omega)$ [2, Theorem 1.5]. Note that the semigroup $S(t)$ is not type $C_0$ since $S(t)v_0$ is smooth for $t > 0$ so $S(t)v_0 \to v_0$ on $L^\infty$ as $t \downarrow 0$ may not hold for general $v_0 \in L^\infty_\sigma$. This means that $S(t)$ is a non-$C_0$-analytic semigroup.

Now, we observe the extension of $S(t)$ to $L^\infty_\sigma(\Omega)$ for unbounded domains $\Omega$. Note that the space $L^\infty_\sigma$ includes non-decaying functions as $|x| \to \infty$ so the existence of solutions for $v_0 \in L^\infty_\sigma(\Omega)$ are non-trivial problem. However, if Lemma 2.1 is valid for the unbounded domain

$\Omega$ (satisfying the strictly admissibility), we are able to prove the existence of solutions for $v_0 \in L^\infty_\sigma(\Omega)$ satisfying the estimate (1.6) (called *$L^\infty$-solutions*). Although the approximation (2.1) is unknown in general, it is known to hold for exterior domains [3, Lemma 5.1] and perturbed half space [1, Lemma 4.3.10]. Let us sketch the approach to prove the existence of solutions for $v_0 \in L^\infty_\sigma$ based on [3] (and [1]) for exterior domains and perturbed half spaces.

Our approach is by the $L^\infty$-estimate (1.6) and the approximation (2.1). We find a solution $(v, q)$ for $v_0 \in L^\infty_\sigma$ by a sequence of $L^p$-solutions $\{(v_m, q_m)\}_{m=1}^\infty$ for $v_{0,m} \in C^\infty_{c,\sigma}$. By the estimates (1.6) and (2.1), the sequence $(v_m, q_m)$ is uniformly bounded, i.e.,

$$\sup_{0<t<T_0} \left\| N(v_m, q_m) \right\|_\infty (t) \leq C \|v_0\|_\infty. \tag{2.2}$$

Since $v_{0,m} \to v_0$, it is natural to expect that $(v_m, q_m)$ converges to a solution $(v, q)$ for $v_0 \in L^\infty_\sigma$. In fact, by (1.6) and (2.1), we are able to estimate the Hölder semi-norms of $\nabla q$ in the interior of $\Omega \times (0, T]$ both in space and time variables. Thus, from the parabolic regularity theory, $\{(v_m, q_m)\}_{m=1}^\infty$ (subsequently) converges to a limit $(v, q)$ locally uniformly in $\Omega \times (0, T]$ up to second orders. Actually, the limit $(v, q)$ is continuous in $\overline{\Omega} \times (0, T]$ up to second derivatives since we have local Hölder estimates up to the boundary based on the Solonnikov's Hölder estimate for (1.1)–(1.4) [25], [28], [29] (see [2, Theorem 3.5]). The uniqueness of $L^\infty$-solutions follows from the a priori estimate (1.6) for $v_0 = 0$ so the limit $(v, q)$ is independent of a choice of the sequence $\{v_{0,m}\}_{m=1}^\infty \subset C^\infty_{c,\sigma}$.

To state a result, let us define solutions of (1.1)–(1.4) for $v_0 \in L^\infty_\sigma(\Omega)$ [3, Definition 2.7].

**Definition 2.2** ($L^\infty$-solutions). Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial\Omega \neq \emptyset$. Let $(v, \nabla q) \in C^{2,1}(\bar{\Omega} \times (0, T]) \times C(\bar{\Omega} \times (0, T])$ satisfy (1.1)–(1.3) and (1.4) for $v_0 \in L^\infty_\sigma(\Omega)$ in the sense that $v(\cdot, t) \to v_0$ weakly-$*$ on $L^\infty(\Omega)$ as $t \downarrow 0$. We call $(v, q)$ *an $L^\infty$-solution* if (1.5) and

$$t^{1/2} d_\Omega(x) |\nabla q(x, t)| \tag{2.3}$$

are bounded in $\Omega \times (0, T)$.

Once we know the unique existence of $L^\infty$-solutions, we are able to extend the Stokes semigroup $S(t) : v_0 \mapsto v(\cdot, t), t \geq 0$, for $v_0 \in L^\infty_\sigma$ together with the estimate (1.6). The following statement was proved in [3, Theorem 3.2] for exterior domains and [1, Theorem 4.1.2] for perturbed half spaces.

**Theorem 2.3.** *Let $\Omega$ be an exterior domain in $\mathbb{R}^n$, $n \geq 2$, or a perturbed half space in $\mathbb{R}^n$, $n \geq 3$, with $C^3$-boundary.*
*(i) (Unique existence of $L^\infty$-solutions)*
*For $v_0 \in L^\infty_\sigma(\Omega)$, there exists a unique $L^\infty$-solution $(v, \nabla q)$ satisfying (1.6) for any fixed $T_0$ with some constant $C$ depending only on $T_0$ and $\Omega$.*
*(ii) (Analyticity on $L^\infty_\sigma$)*
*The Stokes semigroup $S(t)$ is uniquely extendable to a (non-$C_0$-)analytic semigroup on $L^\infty_\sigma(\Omega)$.*

***Remark*** 2.4 (Continuity at time zero). It is natural to restrict $S(t)$ to the space of uniformly continuous functions $BUC_\sigma(\Omega)$ so that $S(t)$ is a $C_0$-analytic semigroup on $BUC_\sigma(\Omega)$. Let $BUC(\Omega)$ be the space of all uniformly continuous functions in $\overline{\Omega}$. Define the space $BUC_\sigma(\Omega)$ by

$$BUC_\sigma(\Omega) = \left\{ f \in BUC(\Omega) \,\middle|\, \operatorname{div} f = 0 \text{ in } \Omega,\ f = 0 \text{ on } \partial\Omega \right\}.$$

Then, $S(t)$ is a $C_0$-(analytic) semigroup on $BUC_\sigma(\Omega)$ at least when $\Omega$ is an exterior domain. Note that $C_{0,\sigma}(\Omega) \subset BUC_\sigma(\Omega) \subset L^\infty_\sigma(\Omega)$. When $\Omega$ is bounded, the space $BUC_\sigma(\Omega)$ agrees with $C_{0,\sigma}(\Omega)$ [18], [2, Lemma 6.3].

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