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# Asymptotic stability for viscous conservation law on the half line and its application

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**Abstract**

In this talk, we consider two different topics about viscous conservation law. In the first half, we consider the way of analysis for viscous conservation law on the self-similar line, and in the second half, we present the topic of radial symmetric solution of Burgers equation. The first part of this research is a joint work with Professor Heinrich Freistühler in Konstanz university.

## 1 Analysis of viscous conservation law on the self-similar line

We consider the initial-boundary value problem for viscous conservation law:

$$\begin{cases} u_t + f(u)_x = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0, \quad \lim_{x \to \pm\infty} u_0(x) = u_\pm. \end{cases} \tag{1.1}$$

We would like to consider the asymptotic behavior of the solution to (1.1) with non-convex flux. For viscous conservation laws with non-convex flux on the half line, Liu-Nishihara [13] showed the asymptotic stability of shock wave. Matsumura-Nishihara [8] and Freistüher-Serre [1] considered the asymptotic stability of viscous shock wave on $\mathbb{R}$. On the other hand, Weinberger [9] analyzed the solution of (1.1) on self-similar lines ($\xi = x/t$). Hereinafter, $V(\xi)$ stands for Riemann solution of (1.1).

**Theorem 1.1** (Weinberger [9]). *Let $u_0$ is bounded and $f''$ be continuously differentiable. Suppose that*

$$\text{The points } a \text{ such that } f''(a) = 0 \text{ are isolated.} \tag{1.2}$$

*If the closed interval $[r,s]$ contains no point of discontinuity of $V$, then*

$$\lim_{t\to\infty}\max_{r\le\xi\le s}|u(t,\xi t)-V(\xi)|=0.$$

**Theorem 1.2** (Serre-Freistühler [1]). *For every $c\in R$ and any function $u_0(x)\in c+L^1(R)$ with*

$$\int_{-\infty}^{\infty}u_0(x)-c\,dx=0,$$

*the solution of viscous conservation law $u_t+f(u)_x=u_{xx}$ with initial data $u_0(x)$ satisfies*

$$\lim_{t\to\infty}\|u(t,\cdot)-c\|_{L^1}=0.$$

As an application of [1] and [9], the following results are obtained.

**Theorem 1.3.** *Let $u_0$ is bounded, $f\in C^1$ and $(V-u_0)(x)\in L^1$. Then, the solution $u$ of* (1.1) *satisfies:*

$$\lim_{t\to\infty}u(\xi t,t)=V(\xi),\quad \text{in}\quad L^1.$$

In this article, we also consider viscous conservation law on the half line:

$$\begin{cases} u_t+f(u)_x=u_{xx}, & x>0,\ t>0,\\ u(t,0)=u_-, & t>0,\\ \lim\limits_{x\to\infty}u(t,x)=u_+, & t>0,\\ u(0,x)=u_0(x), & x>0.\end{cases}\tag{1.3}$$

For the problem (1.3), we obtain the following result.

**Theorem 1.4.** *Let $u_0$ is bounded, $u_-<0<u_+$ and $u_0-V(\xi)\in L^\infty$. Then, independently of what $u_+$ is, the solution of* (1.3) *satisfies:*

$$\lim_{t\to\infty}u(t,\xi t)=V(\xi),\quad \text{for a.e. } \xi>0.$$

This Section proceeds as follows: The outline of the proof of Theorem 1.3 is carried out in Section 1.1. We describe the outline of the proof of Theorem 1.4 in Section 1.2.

## 1.1 Conservation law on $R$

In this section, we introduce the outline of the proof of Theorem 1.3. The following properties are associated with conservation law $u_t+f(u)_x=u_{xx}$.

**Corollary 1.5.** *Let $\{S(t) : L^\infty(R) \to L^\infty(R); t \geq 0\}$ be semigroup associated with* (1.1)*. Then, the following properties are satisfied:*

1. *(Comparison) If $a \leq b$, then $S(t)a \leq S(t)b$ a.e. for all $t > 0$.*
2. *(Conservation) If $a - b \in L^1(R)$, then for all $t > 0$, $S(t)a - S(t)b \in L^1(R)$ and $\int_{-\infty}^{\infty} S(t)a - S(t)b \, dx = \int_{-\infty}^{\infty} a - b \, dx$.*
3. *(Contraction) If $a - b \in L^1(R)$, then $\int_{-\infty}^{\infty} |S(t)a - S(t)b| dx$ is a non-increasing function of $t \geq 0$.*

We note that the conservation property 2 does not satisfied on the half space.

*Outline of proof of Theorem 1.3*

Let $u$ and $\tilde{u}$ be the solution of (1.1) and $u_0 - \tilde{u}_0 \in L^1$, where $u_0$ and $\tilde{u}_0$ are the initial value of $u$ and $\tilde{u}$, respectively. Then according to the property 3 of Corollary 1.5, the following contraction property is satisfied:

$$\frac{d}{dt}\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1} \leq 0. \tag{1.4}$$

Next we define a new variable $\xi$ by $x = \xi t$ and let $w(t,\xi) := u(t,\xi t)$, $\tilde{w}(t,\xi) := \tilde{u}(t,\xi t)$. Then using the estimate (2), we have

$$\|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{L^1} \leq \frac{1}{t}\|w(1,\cdot) - \tilde{w}(1,\cdot)\|_{L^1}, \tag{1.5}$$

where we derive (1.5) by

$$\begin{aligned}
\|w(t,\cdot) - \tilde{w}(t,\cdot)\|_{L^1} &= \int_{-\infty}^{\infty} |(w - \tilde{w})(t,\xi)| d\xi = \int_{-\infty}^{\infty} |(u - \tilde{u})(t,\xi t)| d\xi \\
&= \frac{1}{t}\int_{-\infty}^{\infty} |(u - \tilde{u})(t,x)| dx \leq \frac{1}{t}\int_{-\infty}^{\infty} |(u - \tilde{u})(1,x)| dx \\
&= \frac{1}{t}\int_{-\infty}^{\infty} |(w - \tilde{w})(1,\xi)| d\xi = \frac{1}{t}\|w(1,\cdot) - \tilde{w}(1,\cdot)\|_{L^1}.
\end{aligned}$$

From (1.5), we can see that $w(t,\xi)$ makes a Cauchy sequence. This shows that there exists a solution $\bar{w}$ which fills the following formula.

$$\lim_{t\to\infty} \|w(t,\cdot) - \bar{w}(t,\cdot)\|_{L^1} = 0. \tag{1.6}$$

At the last, we consider what $\bar{w}$ is. In our case of $L^1$ space, the statement by Serre-Freistühler [1] cover the condition (1.2) in the result of Weinberger [9], then we see that

$$\lim_{t\to\infty} w(t,\xi) = \lim_{t\to\infty} u(t,\xi t) = V(\xi). \tag{1.7}$$

Then, $\bar{w}(\xi) = V(\xi)$ and $\lim_{t\to\infty} u(t,\xi t) = V(\xi)$, for a.e. $\xi \in R$. □

## 1.2 Conservation law on the half space

In this section, we show the outline of the proof of Theorem1.4. For this purpose, we prepare some proposition.

**Proposition 1.6.** *Under the condition of Theorem1.4, the following inequalities are satisfied.*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}_+} u(t,x) \leq u_+, \tag{1.8}$$

$$\lim_{t\to+\infty} \inf_{x\in\mathbb{R}_+} u(t,x) \geq u_-. \tag{1.9}$$

*Outline of proof of Proposition 1.6*

Let $M := \sup_{x>0} u_0(x)$ and choose $\bar{M} > M$ close to $M$. Then there exists $m_1' \in [u_+, \bar{M}]$ such that $f'(m_1') = s_1$, where $s_1$ is the gradient of line segment from $(u_+, f(u_+))$ and $(\bar{M}, f(\bar{M}))$. Choose $m_1 > m_1'$ and close to $m_1'$. Then we can choose $d_0$ sufficient large such that a travelling wave $\phi(x - d_0)$ of the half line is over $u_0(x)$. On the other hand, we consider a function $\psi_0(x)$ which satisfies $m_1$ at $x = 0$ and $\bar{M}$ on $x > 0$. By using the theory of Liu-Nishihara [13], the solution with initial data $\phi_0(x) := \phi(x - d_0)$ tends to travelling wave $\phi$ and

$$u(t,x) \leq S(t)\phi_0, \qquad t > 0.$$

On the other hand, by using the theory of Liu-Matsumura-Nishihara [3], $\psi_0(x)$ tends to a rarefaction wave and

$$u(t,x) \leq S(t)\psi_0, \qquad t > 0.$$

Noting that $f'(m_1) > s_1$ and using maximum principle, we can see that $\limsup_{t\to\infty} \sup_{x>0} u(t,x) \leq m_1$. We repeat this argument, then we have the inequality (1.8).

On the other hand, setting $\bar{u}_0(x) := \min\{u_0(x), u_-\}$ and using the theory of Freistühler and Serre [2] for the half space, we get (1.9). $\square$

We note that together with Proposition1.6 and the following proposition show Theorem 1.4.

**Proposition 1.7.** *Under the condition of Theorem1.4, the following inequalities are satisfied.*

$$\limsup_{t\to+\infty} \max_{x\leq\xi t} u(t,x) \leq V(\xi), \tag{1.10}$$

$$\liminf_{t\to+\infty} \min_{x\geq\xi t} u(t,x) \geq V(\xi). \tag{1.11}$$

*Outline of proof of Proposition 1.7*

First we prove the inequality (1.10). Let $u$ be the solution of (1.4) with an initial data 0 at $x = 0$, and $u_+$ for $x > 0$. Using Proposition 1.6, comparison principle and the theory of rarefaction wave by Liu-Matsumura-Nishihara [3], we have

$$\limsup_{t \to +\infty} u(t,x) \leq \psi^R(\xi), \quad \text{where} \quad \xi = \frac{x}{t}. \tag{1.12}$$

As $\psi^R(\xi)$ is Riemann solution and monotone increasing, we can rewrite (1.12) as

$$\limsup_{t \to +\infty} \max_{x \leq \xi t} u(t,x) \leq V(\xi). \tag{1.13}$$

Next we proceed to prove inequality (1.11). For $\xi \in [0, (f')^{-1}(u_+)]$, there exists $\tilde{v} > 0$ such that $f'(\tilde{v}) = \xi$. Choose $v_0$ close to $\tilde{v}$ and make line segment $l$ connect from $(M_-, f(M_-))$ and $(v_0, f(v_0))$ with a gradient $f'(\tilde{v})$, where $M_-$ is intersect point of $l$ and $f$ under $u_-$. Then there exists $d_0 > 0$ such that $u_0(x) \geq \phi(x - d_0)$, where $\phi$ is travelling wave connect from $M_-$ and $v_0$. By comparison principle, we see that

$$u(t,x) \geq \phi(x - ct - d_0(t)), \qquad t > 0.$$

From the monotonicity of $\phi$, we see that

$$\begin{aligned} \min_{x \geq \xi t} u(t,x) \geq \min_{x \geq \xi t} \phi(x - ct - d_t) &\geq \phi(\xi t - ct - d_t) \\ &= \phi\left((\xi - c)t - d_t\right). \end{aligned} \tag{1.14}$$

Noting that $f'(\tilde{v}) = \xi \ > c$, the right most term of (1.14) tends to $\phi(+\infty) = v_0$ as $t \to \infty$. Since $v_0$ can taken suitably close to $\tilde{v}$, we have

$$\liminf_{t \to +\infty} \min_{x \geq \xi t} u(t,x) \geq \tilde{v} = (f')^{-1}(\xi) = V(\xi). \tag{1.15}$$

For $\xi \geq (f')^{-1}(u_+)$, choose $v_0'$ close to $u_+$ and make line segment $l'$ connect $(M_-', f(M_-'))$ and $(v_0', f(v_0'))$ with a gradient $f'(u_+)$, where $M_-'$ is intersect point of $l'$ and $f$ under $u_-$. Then there exists $d_0 > 0$ such that $u_0(x) \geq \phi(x - d_0)$, where $\phi$ is travelling wave connect from $M_-'$ and $v_0'$. By the same strategy of (1.14), we see that

$$\liminf_{t \to +\infty} \min_{x \geq \xi t} u(t,x) \geq v_+. \tag{1.16}$$

By (1.15) and (1.16), we get (1.11). □

*Proof of Theorem 1.4*

By the equality (1.10) and (1.11) of proposition 1.7, we see that

$$\lim_{t \to \infty} u(t, \xi t) = V(\xi). \qquad \square$$

# 2 Radially symmetric solutions for Burgers equation

We consider Burgers equation on multi-dimensional space,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \mu \Delta u, \quad (t > 0, x \in \mathbb{R}^n), \tag{2.1}$$

where $\mu$ is positive constant. In this paper, we study radially symmetric solution for (2.1) on the exterior domain $|x| > r_0$ for some positive constant $r_0$. For this purpose, we transform the unknown function $u(t,x)$ in (2.1) to $v(t,r)$ by means of $u \equiv (x/r)v(t,r)$, where $r$ is defined by $r := |x|$. Then we have the initial boundary value problem for Burgers equation:

$$\begin{cases} v_t + v v_r = \mu(v_{rr} + (n-1)(\frac{v}{r})_r), & r > r_0, \quad t > 0, \\ v(t, r_0) = v_-, \quad \lim\limits_{r \to +\infty} v(t,r) = v_+, & t > 0, \\ v(0,r) = v_0(r), & r > r_0, \end{cases} \tag{2.2}$$

where the initial data $v_0$ is assumed to satisfy $v_0(r_0) = v_-$ and $\lim\limits_{r \to +\infty} v_0(r) = v_+$ as the compatibility conditions.

For viscous conservation laws on the half line, T.-P. Liu, A. Matsumura and K. Nishihara [12] investigated the case where the flux is convex and the corresponding Riemann problem for the hyperbolic part admits a transonic rarefaction wave. It was shown in [12] that depending on the signs of the characteristic speeds, asymptotic states of the solutions are classified into three cases, that is, stationary wave, rarefaction wave and composite wave. In this article, we show that even for the solution of (2.2), the asymptotic behavior is similar to that of [12]. We consider the following three cases: (a) $v_- < v_+ \leq 0$, (b) $0 = v_- < v_+$ and (c) $v_- < 0 < v_+$. In case (a), as all characteristic speeds of corresponding Riemann problem are negative, we can expect that the solution tend to a stationary wave $\phi$ which is defined through the stationary problem corresponding to (2.2),

$$\begin{cases} \left(\frac{1}{2}\phi^2\right)_r = \mu\left(\phi_{rr} + (n-1)\left(\frac{\phi}{r}\right)_r\right), & r > r_0, \\ \phi(r_0) = v_-, \quad \lim\limits_{r \to +\infty} \phi(r) = v_+, \end{cases} \tag{2.3}$$

and we have the following theorem.

**Theorem 2.1** ([4]). *Suppose* (a) *holds. Assume that* $v_0 \in H^1$. *Let* $\phi(r)$ *be the stationary solution satisfying the problem* (2.3). *Then the initial-boundary* *value problem* (2.2) *has a unique solution* $v$ *globally in time satisfying*

$$v-\phi \in C^0([r_0,\infty);H^1), \qquad (v-\phi)_r \in L^2(r_0,T;H^1), \quad T>0,$$

*and the asymptotic behavior*

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t)-\phi(r)\right| = 0.$$

*In addition to the above, assume* $v_+ = 0$ *and* $v_0 \in L^1$*. Then the solution satisfies the following quantitative estimate:*

$$\|(v-\phi)(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}}.$$

In case (b), as all characteristic speeds of corresponding Riemann problem are positive, then we can expect that the solution tend to a rarefaction wave $\psi^R$ which is defined by $\psi^R((r-r_0)/t) = \psi^R(s)$ for $t>0$, where $\psi^R(s)$ is defined by

$$\psi^R(s) \;=\; \begin{cases} 0, & s \leq 0(=v_-), \\ s, & 0 \leq s \leq v_+, \\ v_+, & v_+ \leq s. \end{cases} \tag{2.4}$$

We have the following theorem.

**Theorem 2.2** ([4]). *Suppose* (b) *holds. Assume that* $v_0 - v_+ \in H^1$*. Let* $\psi^R$ *be the rarefaction wave satisfying* (2.4)*. Then the initial-boundary value problem* (2.2) *has a unique solution* $v$ *globally in time satisfying*

$$v-v_+ \in C^0([r_0,\infty);H^1), \qquad (v-v_+)_r \in L^2(r_0,T;H^1), \quad T>0,$$

*and the asymptotic behavior*

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t)-\psi^R\left(\frac{r-r_0}{t}\right)\right| = 0.$$

*In addition to the above, assume* $v_0 - v_+ \in L^1$*. Then the solution satisfies the following quantitative estimate:*

$$\left\|v-\psi^R\left(\frac{\cdot}{t}\right)\right\|_{H^1} \leq C(1+t)^{-\frac{1}{4}}\log^2(2+t).$$

We consider case (c). As the characteristic speed of the corresponding Riemann problem changes from negative to positive, we can expect that the solution tends to a superposition of a stationary wave and a rarefaction wave. The statement of the theorem for the case (c) is as follows.

**Theorem 2.3** ([4]). *Suppose* (c) *holds. Assume that* $v_0 - v_+ \in H^1$. *Let* $\phi(r)$ *be the stationary wave satisfying problem* (2.3) *and* $\psi^R((r-r_0)/t)$ *be rarefaction wave defined by* (2.4). *Then the initial-boundary value problem* (2.2) *has a unique solution* $v$ *globally in time satisfying*

$$v - v_+ \in C^0([r_0,\infty); H^1), \qquad (v-v_+)_r \in L^2(r_0,T;H^1), \quad T>0,$$

*and the asymptotic behavior*

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(r,t) - \phi(r) - \psi^R\left(\frac{r-r_0}{t}\right)\right| = 0.$$

*In addition to the above, assume* $v_0 - v_+ \in L^1$. *Then the solution satisfies the following quantitative estimate:*

$$\|(v-\phi-\psi^R)(t)\|_{H^1} \le C(1+t)^{-\frac{1}{4}}\log^2(2+t).$$

Case (a), (b) and (c) are investigated in Section 2.1, 2.2 and 2.3, respectively. We also give the outline of the proof of Theorem 2.1, 2.2 and 2.3.

## 2.1 Asymptotic stability of Stationary wave

**Reformulation of the problem** As the properties of the stationary solution $\phi$ which is given by (2.3), we have the following lemma.

**Lemma 2.4.** *The stationary problem* (2.3) *has a unique smooth solution* $\phi(r)$ *satisfying* $v_- \le \phi(r) < v_+ \le 0$ *and* $\phi_r(r) > 0$ *for* $r > r_0$. *Moreover,* $\phi$ *satisfies the following properties as* $r \to \infty$*:*
(i) *If* $v_+ = 0$, *then we have*

$$|\phi(r)| \sim \begin{cases} (r\log r)^{-1}, & n=2, \\ r^{-n+1}, & n\ge 3. \end{cases} \tag{2.5}$$

(ii) *If* $v_+ < 0$, *then we have*

$$|\phi(r) - v_+| \sim e^{\frac{|v_+|}{\mu}(r-r_0)}. \tag{2.6}$$

Let $\phi$ be the stationary solution satisfying (2.3). We introduce perturbation $w(r,t)$ by

$$v(t,r) = \phi(r) + w(t,r).$$

Then we rewrite our original problem (2.2) in terms of $w(t,r)$ as

$$\begin{cases} w_t + \frac{1}{2}(w^2 + 2\phi w)_r = \mu \left( w_{rr} + (n-1) \left(\frac{w}{r}\right)_r \right), & r > r_0, \quad t > 0, \\ w(r_0, t) = 0, & t > 0, \\ w(r, 0) = w_0(r). & r > r_0. \end{cases} \tag{2.7}$$

The theorem for the reformulated problem (2.7) we shall prove is

**Theorem 2.5.** *Assume $v_- < v_+ \le 0$ and $v_0 \in H^1$. Then, initial boundary value problem (2.7) has a unique solution $w$ globally in time*

$$w \in C([r_0, \infty); H^1), \quad w_x \in L^2(r_0, \infty; H^1), \quad T > 0,$$

*and the asymptotic behavior*

$$\lim_{t \to \infty} \sup_{r > r_0} |w(r,t)| = 0.$$

Main Theorem 2.1 is a direct consequence of Theorem 2.5. Theorem 2.5 itself is proved by combining the local existence theorem together with an *a priori* estimate.

To state the local existence theorem precisely, we define the solution set for any interval $I \subset R$ and constant $M > 0$ by

$$X_M(I) = \{ w \in C(I; H_0^1); w_r \in L^2(0, T; H^1), \sup_{t \in I} ||w(t)||_{H^1} \le M \}, \tag{2.8}$$

then we state the local existence theorem.

**Proposition 2.6** (local existence). *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ such that if $||w_0||_{H^1} \le M$, the initial boundary value problem (2.7) has a unique solution $w \in X_{2M}([0, t_0])$.*

Since we can prove Proposition 2.6 by a standard iterative method, we omit the proof. Next, let us state the a priori estimate.

**Proposition 2.7** (*a priori* estimate).

*Suppose that the same assumptions as in Theorem 2.1 hold true. Then if $w \in X_\infty([0, T])$ is the solution of the problem (2.7) for some $T > 0$, it holds*

$$\|w(t)\|_{H^1}^2 + \int_0^t \|\sqrt{\phi_r} w(\tau)\|_{L^2}^2 + \mu \|w_r(\tau)\|_{H^1}^2 + \mu \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \le C(\|w_0\|_{H^1}^2 + 1). \tag{2.9}$$

<u>*Outline of proof of Proposition 2.7*</u>

Multiplying (2.7) by $w$ and integrate the resultant equality over $[r_0, \infty]$, we get

$$\left(\int_{r_0}^{\infty} \frac{1}{2} w^2 dr\right)_t + \frac{1}{2}\int_{r_0}^{\infty} \phi_r w^2 dr + \mu \int_{r_0}^{\infty} w_r^2 \, dr + \mu(n-1)\int_{r_0}^{\infty} \frac{w^2}{2r^2} dr = 0, \tag{2.10}$$

where we use the relation

$$\int_{r_0}^{\infty} \left(\frac{w}{r}\right)_r w dr = -\int_{r_0}^{\infty} \left(\frac{w}{r}\right) w_r dr = -\int_{r_0}^{\infty} \left(\int_{r_0}^{w} \frac{\eta}{r} d\eta\right)_r + \frac{w^2}{2r^2} dr.$$

Integrating (2.10) in terms of $[0, t]$, we obtain

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\phi_r} w(\tau)\|_{L^2}^2 + \mu\|w_r(\tau)\|_{L^2}^2 + \mu(n-1)\left\|\frac{w(\tau)}{r}\right\|_{L^2}^2 d\tau = \|w_0\|_{L^2}^2. \tag{2.11}$$

The higher order estimate is proved in the similar fashion. We omit the proof here. □

Next we present the decay rate of the solution of the reformulated problem (2.7).

**Proposition 2.8** (Decay rate). *Suppose that $v_+ = 0$ and the same conditions as in Proposition 2.7 hold true. Then, if $w_0 \in L^1$, we have*

$$\|w(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}}, \tag{2.12}$$

*for $t > 0$.*

The stability of degenerate stationary wave for viscous conservation laws was investigated by Ueda-Nakamura-Kawashima in [16]. We use the time weighted energy method developed in [10] and omit the proof here.

## 2.2 Asymptotic stability of the Rarefaction wave

**Reformulation of the problem.** We reformulate our problem (2.2) by introducing the perturbation $w(t, r)$ by

$$v(t, r) = \psi(t, r) + w(t, r),$$

where $\psi(t,r)$ is the smooth approximation of the rarefaction wave defined by the problem

$$\begin{cases} \psi_t + \psi\psi_r = \psi_{rr}, \quad r \in \mathbb{R}, \quad t > -1, \\ \psi(r,-1) = \begin{cases} -v_+, & r < r_0, \\ v_+, & r > r_0. \end{cases} \end{cases} \tag{2.13}$$

The idea of approximation of rarefaction wave is introduced by Kawashima and Tanaka (2004) [11]. For its proof, we refer the reader to [11].

Using (2.2) and (2.13), we rewrite our original problem (2.2) as

$$\begin{cases} w_t + \frac{1}{2}(w^2 + 2\psi w)_r = \mu\left(w_{rr} + (n-1)\left(\frac{w+\psi}{r}\right)_r\right), & r > r_0, \quad t > 0, \\ w(t, r_0) = 0, & t > 0, \\ w(0,r) = w_0(r). & r > r_0. \end{cases} \tag{2.14}$$

The theorem for the reformulated problem (2.14) we shall prove is

**Theorem 2.9.** *Assume $0 = v_- < v_+$ and $v_0 - v_+ \in H^1$. Then initial boundary value problem (2.14) has a unique solution $w$ globally in time*

$$w \in C([r_0, \infty); H^1), \quad w_x \in L^2(r_0, \infty; H^1), \quad T > 0,$$

*and the asymptotic behavior*

$$\lim_{t\to\infty} \sup_{r > r_0} |w(r,t)| = 0.$$

Main Theorem 2.2 is a direct consequence of Theorem 2.9. Theorem 2.9 itself is proved by combining the local existence theorem together with the *a priori* estimate. We define the solution set $X_M(I)$ as in (2.8), and the statement of the local existence theorem is the same as that of previous section. The statement for an *a priori* estimate is as follows.

**Proposition 2.10** (*a priori* estimate).

*Suppose that the same assumptions as in Theorem 2.2 hold true. Then, if $w \in X_\infty([0,T])$ is the solution of the problem (2.14) for some $T > 0$, it holds*

$$\|w\|_{H^1}^2 + \int_0^t \|\sqrt{\psi_r} w(\tau)\|_{L^2}^2 + \mu \|w_r(\tau)\|_{H^1}^2 + \mu \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \le C(\|w_0\|_{H^1}^2 + 1). \tag{2.15}$$

*for $t \in [0,T]$, where $C$ is a positive constant independent of $T$.*

Outline of proof of Proposition 2.10

Multiplying (2.14) by $w$, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}w^2\right)+\bar{F}_r+\frac{1}{2}\psi_r w^2+\mu w_r^2+\mu(n-1)\frac{w^2}{2r^2}=\mu(n-1)\left(\frac{\psi_r w}{r}-\frac{\psi w}{r^2}\right), \tag{2.16}$$

where $\bar{F}$ is defined by

$$\bar{F}:=\frac{1}{2}\psi w^2+\frac{1}{3}w^3-\mu w_r w-\frac{w^2}{2r}.$$

Integrating (2.16) in terms of $r$ over $[r_0,\infty]$, we obtain

$$\begin{aligned}&\left(\int_{r_0}^{\infty}\frac{1}{2}w^2dr\right)_t+\frac{1}{2}\int_{r_0}^{\infty}\psi_r w^2dr+\mu\int_{r_0}^{\infty}w_r^2dr+\mu(n-1)\int_{r_0}^{\infty}\frac{w^2}{2r^2}dr\\ &=\mu(n-1)\int_{r_0}^{\infty}\frac{\psi_r w}{r}-\frac{\psi w}{r^2}dr.\end{aligned} \tag{2.17}$$

Now, we estimate the right hand side of (2.17). By using the Young's inequality, the first term of the right hand side of (2.17) is estimated as

$$\begin{aligned}\int_{r_0}^{\infty}\frac{\psi_r w}{r}dr&\le\|w\|_{L^\infty}\int_{r_0}^{\infty}\frac{\psi_r}{r}dr\\ &\le\epsilon\|w_r\|_{L^2}^2+C\|w\|_{L^2}^{2/3}\left(\int_{r_0}^{t}dr+\int_{t}^{\infty}dr\right)^{4/3}\\ &=:\epsilon\|w_r\|_{L^2}^2+C\|w\|_{L^2}^{2/3}(I_1+I_2)^{4/3}.\end{aligned} \tag{2.18}$$

Using the estimate of the rarefaction wave which is derived by Kawashima and Tanaka [11], we can estimate $I_1$ as

$$I_1:=\int_{r_0}^{t}\frac{\psi_r}{r}dr\le C(1+t)^{-1}\log(2+t). \tag{2.19}$$

On the other hand $I_2$ is estimated as

$$I_2:=\int_{t}^{\infty}\frac{\psi_r}{r}dr\le\left[\frac{\psi}{r}\right]_t^{\infty}+\int_{t}^{\infty}\frac{\psi}{r^2}dr\le(1+t)^{-1}. \tag{2.20}$$

By virtue of these two estimates, we rewrite the inequality (2.18) as

$$\int_{r_0}^{\infty}\frac{\psi_r w}{r}dr\le\epsilon\|w_r\|_{L^2}^2+C\|w\|_{L^2}^{2/3}(1+t)^{-4/3}\log^{4/3}(2+t). \tag{2.21}$$

Applying the inequality (2.21), the second term on the right hand side of (2.17) is estimated as

$$\begin{aligned}\int_{r_0}^{\infty} \frac{\psi w}{r^2} dr &\le \|w\|_{L^\infty} \int_{r_0}^{\infty} \frac{\psi}{r^2} dr \\ &\le \epsilon \|w_r\|_{L^2}^2 + C\|w\|_{L^2}^{2/3} (1+t)^{-4/3} \log^{4/3}(2+t).\end{aligned} \tag{2.22}$$

Put (2.21) and (2.22) into (2.17), and integrate the resultant inequality in terms of $t$ over $[0, t]$ to get

$$\begin{aligned}&\|w\|_{L^2}^2 + \int_0^t \|\sqrt{\psi_r} w(\tau)\|_{L^2}^2 + \mu \|w_r(\tau)\|_{L^2}^2 + \mu(n-1) \left\| \frac{w(\tau)}{r^2} \right\|_{L^2}^2 d\tau \\ &\le C(\|w_0\|_{L^2}^2 + 1),\end{aligned} \tag{2.23}$$

here we have also used Gronwall's inequality. The higher order estimate is proved in the similar fashion, we omit here. □

We also derive the decay rate of the solution of the reformulated problem (2.14). The statement is the following.

**Proposition 2.11** (Decay rate). *Suppose that the same conditions as in Proposition 2.10 hold true. Moreover, if $w_0 \in L^1$, we have*

$$\|w(t)\|_{H^1} \le C(1+t)^{-\frac{1}{4}} \log^2(2+t), \quad t > 0. \tag{2.24}$$

## 2.3 Asymptotic stability of Superpositions of Stationary wave and Rarefaction wave

**Reformulation of the problem.** Let $\phi$ and $\psi$ are the stationary wave satisfying (2.3) and smoothed rarefaction wave defined by (2.13), respectively. Now, we define $\Phi(t, r)$ as the superposition of stationary wave and rarefaction wave as

$$\Phi(t, r) := \phi(r) + \psi(t, r),$$

which is an approximation of our solution. By using (2.3) and (2.13), we derive that

$$\begin{cases} \Phi_t + (\frac{1}{2}\Phi^2)_r = \mu \Phi_{rr} + \bar{R}, & r > r_0, \quad t > 0, \\ \Phi(t, r_0) = v_-. & t > 0, \end{cases}$$

where $\bar{R}$ is defined by

$$\bar{R} := (\phi\psi)_r + \mu(n-1) \left(\frac{\phi}{r}\right)_r.$$

Then as in the former section, we reformulate our problem (2.2) by introducing the perturbation $w(t,r)$ by

$$v(t,r) = \Phi(t,r) + w(t,r).$$

Then, we rewrite our original problem (2.2) as

$$\begin{cases} w_t + \frac{1}{2}(w^2 + 2\Phi w)_r = \mu \left( w_{rr} + (n-1) \left(\frac{w+\psi}{r}\right)_r \right) - (\psi\phi)_r, & \\ w(t,r_0) = 0, & t > 0, \\ w(0,r) = w_0(r), & r > r_0. \end{cases} \tag{2.25}$$

The theorem for the reformulated problem (2.25) we shall prove is

**Theorem 2.12.** *Assume $v_- < 0 < v_+$ and $v_0 - v_+ \in H^1$. Then the initial boundary value problem* (2.25) *has a unique solution $w$ globally in time*

$$w \in C([r_0,\infty); H^1), \quad w_x \in L^2(r_0,\infty; H^1), \quad T > 0,$$

*and the asymptotic behavior*

$$\lim_{t\to\infty} \sup_{r>r_0} |w(t,r)| = 0.$$

Main Theorem 2.3 is a direct consequence of Theorem 2.12. Theorem 2.12 itself is proved by combining the local existence theorem together with the *a priori* estimate. Local existence theorem and the solution set $X_M(I)$ is stated as the same as previous section. The statemant of *a priori* estimate and decay rate are as following.

**Proposition 2.13** (*a priori* estimate).
*Suppose that the same assumptions as in Theorem 2.3 hold true. Then, if $w \in X_\infty([0,T])$ is the solution of the problem* (2.25) *for some $T > 0$, it holds*

$$\|w\|_{H^1}^2 + \int_0^t \|\sqrt{\Phi_r} w(\tau)\|_{L^2}^2 + \mu\|w_r(\tau)\|_{H^1}^2 + \mu \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \le C(\|w_0\|_{H^1}^2 + 1). \tag{2.26}$$

*for $t \in [0,T]$, where $C$ is a positive constant independent of $T$.*

**Proposition 2.14** (Decay rate). *Suppose that the same conditions as in Proposition 2.13 hold true. Then if $w_0 \in L^1$, we have*

$$\|w(t)\|_{H^1} \le C(1+t)^{-\frac{1}{4}} \log^2(2+t), \quad t > 0. \tag{2.27}$$

*Outline of proof of Proposition 2.13*

Multiplying (2.25) by $w$, we get

$$\begin{aligned}\left(\frac{1}{2}w^2\right)_t + F_r + \frac{1}{2}\Phi_r w^2 + \mu w_r^2 + \mu(n-1)\frac{w^2}{2r^2}\\ = \mu(n-1)\left(\frac{\psi_r w}{r} - \frac{\psi w}{r^2}\right) - (\phi\psi)_r w,\end{aligned} \tag{2.28}$$

where

$$F := \frac{1}{3}w^3 + \frac{1}{2}\Phi w^2 - \mu w w_r - \mu(n-1)\left(\frac{2\psi w}{r} + \frac{w^2}{2r}\right).$$

Integrate (2.28) over $[r_0, \infty]$ in terms of $r$, we have

$$\begin{aligned}&\left(\int_{r_0}^{\infty} \frac{1}{2}w^2\right)_t + \frac{1}{2}\int_{r_0}^{\infty} \Phi_r w^2 dr + \mu \int_{r_0}^{\infty} w_r^2 dr + \mu(n-1)\int_{r_0}^{\infty} \frac{w^2}{2r^2}dr\\ &= \mu(n-1)\int_{r_0}^{\infty} \frac{\psi_r w}{r} - \frac{\psi w}{r^2} - (\phi\psi)_r w \ dr.\end{aligned} \tag{2.29}$$

Similar to (2.18)-(2.22) in the previous section, we estimate the first and second term of the right hand side of (2.29) as

$$\left|\int_{r_0}^{\infty} \frac{\psi_r w}{r} - \frac{\psi w}{r^2}dr\right| \le \epsilon\|w_r\|_{L^2}^2 + C\|w\|_{L^2}^{2/3}(1+t)^{-4/3}\log^{4/3}(2+t).$$

Now we estimate the rightmost term of (2.29).

$$\int_{r_0}^{\infty} -\phi\psi_r w dr \le \|w\|_{L^\infty}\int_{r_0}^{\infty} -\phi\psi_r dr = \|w\|_{L^\infty}\left(\int_{r_0}^{t} + \int_{t}^{\infty}\right) = I_1 + I_2. \tag{2.30}$$

By using the estimate of the stationary wave $\phi$ derived in Lemma 2.4 and the property of rarefaction wave $\psi$, we estimate $I_1$ and $I_2$ as

$$\begin{aligned}I_1 &\le \epsilon\|w_r\|_{L^2}^2 + C\|w\|_{L^2}^{2/3}(1+t)^{-4/3}\log^{4/3}(2+t),\\ I_2 &\le \epsilon\|w_r\|_{L^2}^2 + C\|w\|_{L^2}^{2/3}(1+t)^{-4/3}.\end{aligned} \tag{2.31}$$

On the other hand, by using the integration by parts, we estimate

$$\int_{r_0}^{\infty} -\phi_r\psi w \ dr \le \|w\|_{L^\infty}\int_{r_0}^{\infty} \phi_r\psi dr = \|w\|_\infty \int_{r_0}^{\infty} -\phi\psi_r dr, \tag{2.32}$$

and the rightmost term is the same as (2.30). Then the right hand side of (2.29) is estimated as

$$\left| \int_{r_0}^{\infty} \frac{\psi_r w}{r} - \frac{\psi w}{r^2} - (\phi\psi)_r w \, dr \right| \leq \epsilon \|w_r\|_{L^2}^2 + C\|w\|_{L^2}^{2/3}(1+t)^{-4/3}\log^{4/3}(2+t). \tag{2.33}$$

Putting (2.33) into (2.29) and integrating it in terms of $t$ over $[0,t]$ and using Gronwall's inequality, we have

$$\|w\|_{L^2}^2 + \int_0^t \|\sqrt{\Phi_r} w(\tau)\|_{L^2}^2 + \mu\|w_r(\tau)\|_{L^2}^2 + \mu \left\| \frac{w(\tau)}{r} \right\|_{L^2}^2 d\tau \leq C(\|w_0\|_{L^2}^2 + 1). \tag{2.34}$$

The higher order estimate is derived in the similar fashion, we omit the proof here. □

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