数理解析研究所講究録
第 1884 巻 2014 年 61-69

On equivariant homeomorphisms of boundaries of CAT(0) groups and Coxeter groups

静岡大学理学研究科

保坂 哲也 (Tetsuya Hosaka)

1. Introduction

In this note, we introduce on equivariant homeomorphisms of boundaries of CAT(0) groups (and Coxeter groups) and (boundary-)rigidity in [17].

A *geometric* action on a CAT(0) space is an action by isometries which is proper and cocompact. We note that every CAT(0) space X on which some group G acts geometrically is a proper space and we can consider its ideal boundary ∂X (cf. [4], [11]). A group G is called a *CAT(0) group*, if G acts geometrically on some CAT(0) space X.

It is well-known that if a Gromov hyperbolic group G acts geometrically on two negatively curved spaces X and Y, then the natural quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ extends continuously to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries of X and Y (cf. [4], [5], [11], [12], [13]).

M. Gromov [13] asked whether the boundaries of two CAT(0) spaces X and Y are G-equivariant homeomorphic whenever a CAT(0) group G acts geometrically on the two CAT(0) spaces X and Y. P. L. Bowers and K. Ruane [3] have constructed an example that the natural quasi-isometry $Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ does not extend continuously to any map between the boundaries ∂X and ∂Y of X and Y. Also, C. Croke and B. Kleiner [6] have constructed a CAT(0) group G which acts geometrically on two CAT(0) spaces X and Y whose boundaries are not homeomorphic, and J. Wilson [26] has proved that this CAT(0) group has uncountably many boundaries.

In this note, we suppose that a CAT(0) group G acts geometrically on two CAT(0) spaces X and Y. Let $x_0 \in X$ and $y_0 \in Y$.

Then we consider the following question.

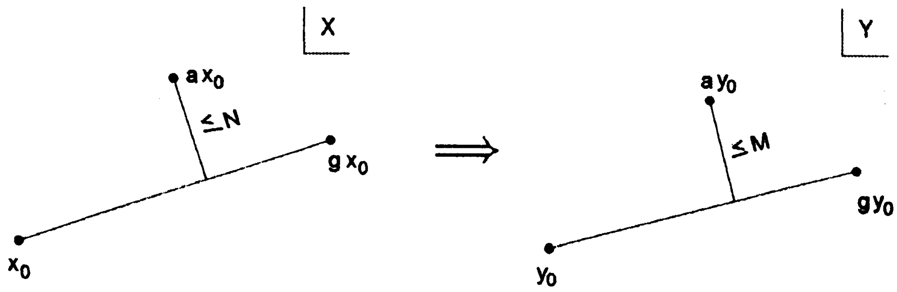
Question. When does the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extend to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries?

$$\begin{array}{ccccccc} & & X & \supset & Gx_0 & \longleftrightarrow & \partial X \\ G & & & & \phi \downarrow & & \downarrow \bar{\phi}\,? \\ & & Y & \supset & Gy_0 & \longleftrightarrow & \partial Y \end{array}$$

2. Main theorems

The following condition $(*)$ comes from observating the Bowers-Ruane's example.

$(*)$ There exist constants $N > 0$ and $M > 0$ such that $GB(x_0, N) = X$, $GB(y_0, M) = Y$ and for any $g, a \in G$, if $[x_0, gx_0] \cap B(ax_0, N) \neq \emptyset$ in X then $[y_0, gy_0] \cap B(ay_0, M) \neq \emptyset$ in Y.



Then we obtain the following theorem.

Theorem 1 ([17]). *If the condition $(*)$ holds, then the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.*

We also consider the following condition $(**)$.

$(**)$ For any sequence $\{g_i \mid i \in \mathbb{N}\} \subset G$, the sequence $\{g_i x_0 \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $X \cup \partial X$ if and only if the sequence $\{g_i y_0 \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $Y \cup \partial Y$.

Then we also obtain the following theorem.

Theorem 2 ([17]). *The condition $(**)$ holds if and only if the quasi-isometry $\phi :$ $Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.*

3. Rigidity of boundaries

In this note, a CAT(0) group G is said to be *(boundary-)rigid*, if G determines its ideal boundary up to homeomorphisms, i.e., all boundaries of CAT(0) spaces on which G acts geometrically are homeomorphic.

Also a CAT(0) group G is said to be *equivariant (boundary) rigid*, if G determines its ideal boundary by the equivariant homeomorphisms as above (i.e., if for any two CAT(0) spaces X and Y on which G acts geometrically the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries).

As an application of Theorem 1, we can obtain examples of equivariant rigid CAT(0) groups.

Example ([17]). Any group of the form

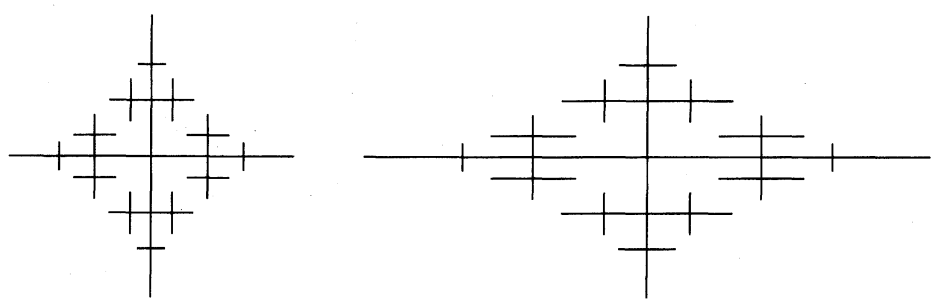
$$\mathbb{Z}^{n_1} * \cdots * \mathbb{Z}^{n_k} * A_1 * \cdots * A_l$$

where $n_i \in \mathbb{N}$ and each A_j is a finite group is an equivariant rigid CAT(0) group.

As an application of Theorem 2, we can also obtain examples of non equivariant rigid CAT(0) groups.

Example ([17]). Let $G = F_2 \times \mathbb{Z}$, where F_2 is the rank 2 free group generated by $\{a, b\}$. Let T and T' be the Cayley graphs of F_2 with respect to the generating set $\{a, b\}$ such that

(1) in T, all edges $[g, ga]$ and $[g, gb]$ $(g \in F_2)$ have the unit length, and
(2) in T', the length of $[g, ga]$ is 2 and the length of $[g, gb]$ is 1 for any $g \in F_2$.

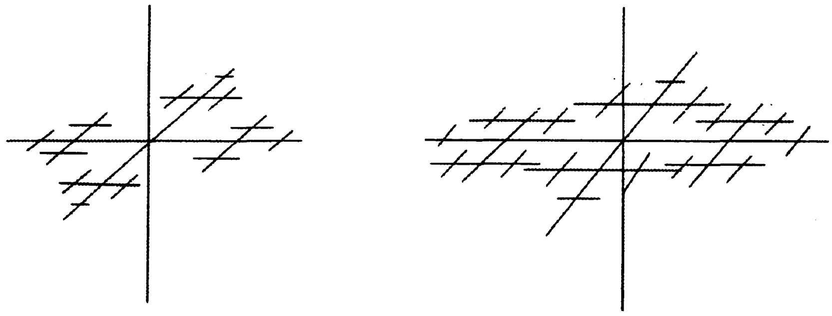


T $\qquad\qquad$ T'

Here we note that F_2 acts naturally and geometrically on T and T'.

Let $X = T \times \mathbb{R}$ and $Y = T' \times \mathbb{R}$.

We consider the natural actions of the group $G = F_2 \times \mathbb{Z}$ on the CAT(0) spaces X and Y. Then the group G acts geometrically on the two CAT(0) spaces X and



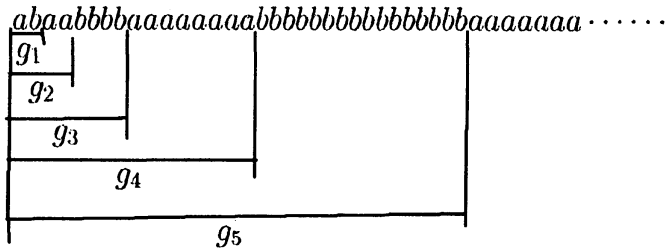
$X = T \times \mathbb{R}$ $\qquad\qquad$ $Y = T' \times \mathbb{R}$

Y, and the quasi-isometry $gx_0 \mapsto gy_0$ (where $x_0 = (1,0) \in X$ and $y_0 = (1,0) \in Y$) does not extend continuously to any map from ∂X to ∂Y.

Indeed, we can consider the sequence $\{g_n \mid n \in \mathbb{N}\} \subset F_2$ such that $g_1 = ab$ and

$$g_n = \begin{cases} g_{n-1}a^{2^{n-1}} & \text{if } n \text{ is even} \\ g_{n-1}b^{2^{n-1}} & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 2$. Here we note that the length of the words of g_n in F_2 is 2^n.



Let $\bar{g}_n = (g_n, 2^n) \in F_2 \times \mathbb{Z}$ for $n \in \mathbb{N}$. Then $\{\bar{g}_n x_0\}$ is a Cauchy sequence in $X \cup \partial X$. On the other hand, $\{\bar{g}_n y_0\}$ is *not* a Cauchy sequence in $Y \cup \partial Y$ (see Figure 1).

Hence, the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ does not continuously extend to any map $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.

Remark ([17]).

- $G = F_2 \times \mathbb{Z}$ is a non equivariant rigid CAT(0) group.
- $G = F_2 \times \mathbb{Z}$ is a rigid CAT(0) group whose boundary is the suspension of the Cantor set.

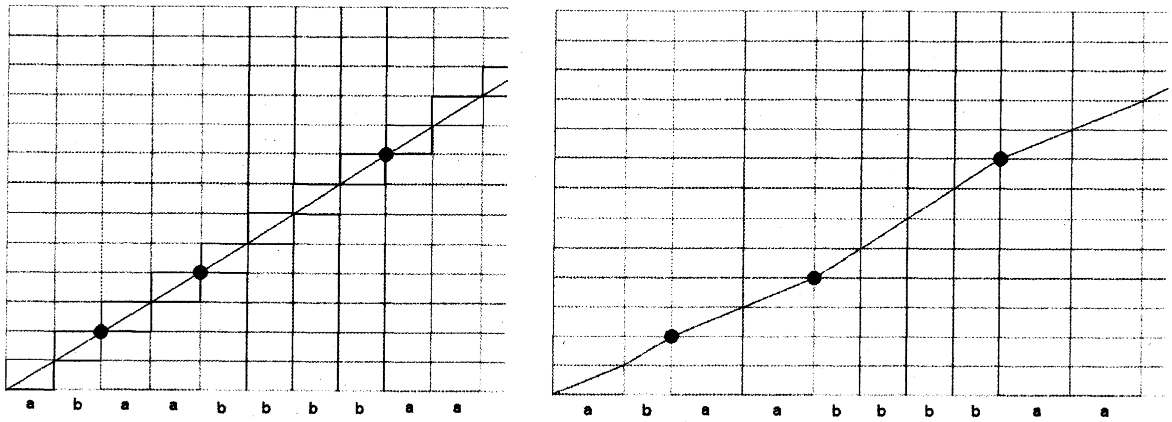


FIGURE 1

- By the same idea, every CAT(0) group of the form $G = F \times H$ where F is a free group of rank $n \geq 2$ and H is an infinite CAT(0) group, is non equivariant rigid.

4. Coxeter groups acting CAT(0) spaces as reflection groups

A Coxeter group W is said to be *equivariant rigid as a reflection group*, if for any two CAT(0) spaces X and Y on which W acts geometrically as reflection groups, the quasi-isometry $\phi : Wx_0 \to Wy_0$ $(wx_0 \mapsto wy_0)$ where $x_0 \in X$ and $y_0 \in Y$ continuously extends to a W-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.

Theorem 3 ([17]). *The following statements hold.*

(i) *If Coxeter groups W_1 and W_2 are equivariant rigid as reflection groups, then so is $W_1 * W_2$.*

(ii) *For a Coxeter group $W = W_A *_{W_{A\cap B}} W_B$ where $W_{A\cap B}$ is finite, if W determines its Coxeter system up to isomorphism, and if W_A and W_B are equivariant rigid as reflection groups then so is W, where W_T is the parabolic subgroup of W generated by T.*

Corollary 4 ([17]). *Any group of the form*

$$W = W_1 * \cdots * W_n$$

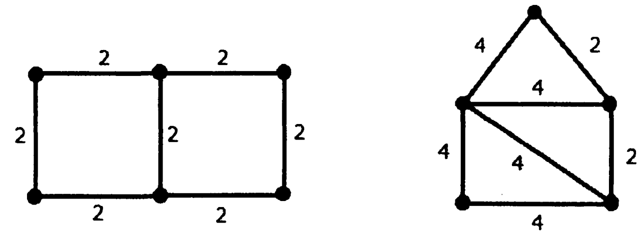
where each W_i is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, is an equivariant rigid as a reflection group.

Corollary 5 ([17]). *Any Coxeter group of the form*

$$W = (\cdots(W_{A_1} *_{W_{B_1}} W_{A_2}) *_{W_{B_2}} W_{A_3}) * \cdots) *_{W_{B_{n-1}}} W_{A_n}$$

where each W_{A_i} is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, each W_{B_i} is finite and W determines its Coxeter system up to isomorphism, is an equivariant rigid as a reflection group.

Example. The Coxeter groups defined by the following diagrams are equivariant rigid as reflection groups.



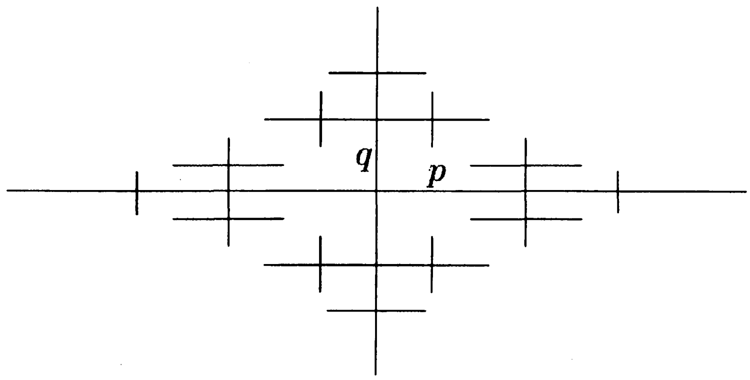
5. Conjecture

Now we introduce a conjecture.

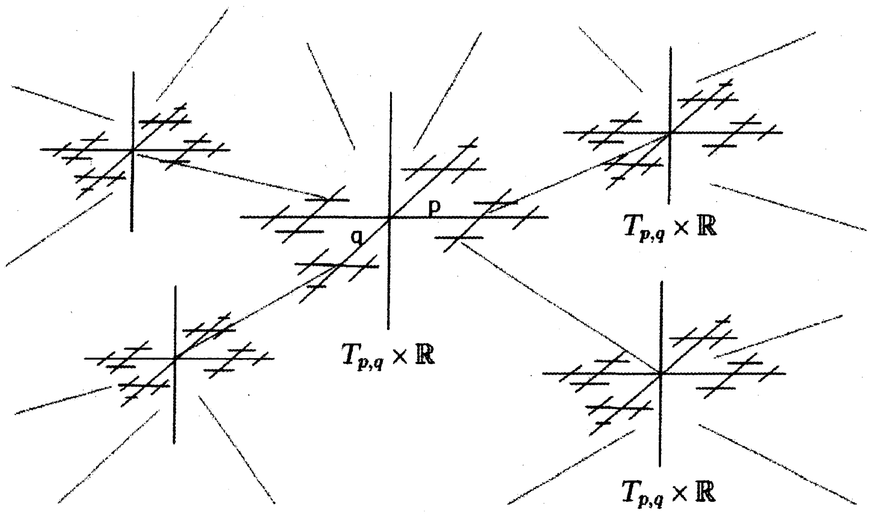
Conjecture ([17]). The group $G = (F_2 \times \mathbb{Z}) * \mathbb{Z}_2$ will be a non-rigid CAT(0) group with uncountably many boundaries.

For $p \geq q \geq 1$, let $T_{p,q}$ be the Cayley graph of the free group F_2 with the generating set $\{a, b\}$ such that

- the length of $[g, ga]$ is p and the length of $[g, gb]$ is q for any $g \in F$.



$T_{p,q}$



$\Sigma_{p,q}$

Then $F_2 \times \mathbb{Z}$ acts naturally on $T_{p,q} \times \mathbb{R}$. We can construct a *cuboidal* cell complex $\Sigma_{p,q}$ on which $G = (F_2 \times \mathbb{Z}) * \mathbb{Z}_2$ acts geometrically, where the 1-skeleton of $\Sigma_{p,q}$ is the Cayley graph of G and $T_{p,q} \subset \Sigma^{(1)}_{p,q}$.

Then, the author thinks that if $\frac{p}{q} \neq \frac{p'}{q'}$ then the boundaries $\partial\Sigma_{p,q}$ and $\partial\Sigma_{p',q'}$ will be not homeomorphic.

6. On rigidity

Finally, we introduce problems of rigidity in group actions.

Let G and H be groups acting geometrically (i.e. properly and cocompactly by isometries) on metric spaces (X, d_X) and (Y, d_Y) respectively. We consider orbits $Gx_0 \subset X$ and $Hy_0 \subset Y$ where $x_0 \in X$ and $y_0 \in Y$.

Let $\phi : G \to H$ be a map and let $\phi' : Gx_0 \to Hy_0$ $(gx_0 \mapsto \phi(g)y_0)$.

Here if X and Y are Gromov hyperbolic spaces, CAT(0) spaces or Busemann spaces, then we can define the boundaries ∂X and ∂Y.

Then it is well-known that if $\phi : G \to H$ is an isomorphism then $\phi' : Gx_0 \to Hy_0$ is a quasi-isometry and moreover if G is Gromov hyperbolic then ϕ' induces an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$.

Theorem 2 implies that if $\phi : G \to H$ is an isomorphism and the map $\phi' : Gx_0 \to Hy_0$ satisfies the condition $(**)$ then ϕ' induces an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$.

$$\begin{array}{ccccccc} G & \overset{\cdot}{\curvearrowright} & X & \supset & Gx_0 & \longleftrightarrow & \partial X \\ \downarrow \phi & & & & \downarrow \phi' & & \downarrow \bar{\phi} \\ H & \overset{\cdot}{\curvearrowright} & Y & \supset & Hy_0 & \longleftrightarrow & \partial Y \end{array}$$

Then there are problems of rigidity.

(I) If $\phi : G \to H$ is an isomorphism then when does there exist an homeomorphism $\bar{\phi} : \partial X \to \partial Y$?

(II) If $\phi : G \to H$ is an isomorphism then when does ϕ' induce an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$?

(III) If $X = Y$ and $Gx_0 = Hx_0$ then when are groups G and H virtually isomorphic (i.e. there exist finite-index subgroups G' and H' of G and H respectively such that G' and H' are isomorphic)?

(IV) If $X = Y$ and $Gx_0 = Hx_0$ then when do there exist finite-index subgroups G' and H' of G and H respectively such that G' and H' are conjugate in the isometry group $\mathrm{Isom}(X)$ of X?

(V) If there is an isomorphism $\phi : G \to H$ then when does there exist a homeomorphism (or homotopy equivalence) $\psi : X/G \to Y/H$?

Here it seems that (III)–(V) are relate to [1], [8], [9], [14], [18], [19], [20], [22] and [23].

References

[1] A. Bartels and W. Lück, *The Borel conjecture for hyperbolic and CAT(0)-groups*, Ann. of Math. (2) 175 (2012), 631–689

[2] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. 43 (1996), 123–139.

[3] P. Bowers and K. Ruane, *Boundaries of nonpositively curved groups of the form $G \times \mathbb{Z}^n$*, Glasgow Math. J. 38 (1996), 177–189.

[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[5] M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Math. 1539, Springer-Verlag, 1993.

[6] C. B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology 39 (2000), 549–556.

[7] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. 117 (1983), 293–324.

[8] A. Furman, *Orbit equivalence rigidity*, Ann. of Math. (2) 150 (1999), 1083–1108.

[9] A. Furman, *Mostow-Margulis rigidity with locally compact targets*, Geom. Funct. Anal. 11 (2001), 30–59.

[10] R. Geoghegan and P. Ontaneda, *Boundaries of cocompact proper CAT(0) spaces*, Topology 46 (2007), 129–137.

[11] E. Ghys and P. de la Harpe (ed), *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Progr. Math. vol. 83, Birkhäuser, Boston MA, 1990.

[12] M. Gromov, *Hyperbolic groups*, Essays in group theory (Edited by S. M. Gersten), pp. 75–263, M.S.R.I. Publ. 8, 1987.

[13] M. Gromov, *Asymptotic invariants for infinite groups*, Geometric Group Theory (G.A. Niblo and M.A. Roller, eds.), LMS Lecture Notes, vol. 182, Cambridge University Press, Cambridge, 1993, pp. 1–295.

[14] M. Gromov and P. Pansu, Rigidity of lattices: an introduction, Geometric topology: recent developments (Montecatini Terme, 1990) (Berlin), Lecture Notes in Math., vol. 1504, Springer, 1991, pp. 39–137.

[15] T. Hosaka, *Reflection groups of geodesic spaces and Coxeter groups*, Topology Appl. 153 (2006), 1860–1866.

[16] T. Hosaka, *Parabolic subgroups of Coxeter groups acting by reflections on CAT(0) spaces*, Rocky Mount. J. Math. 42 (2012), 1207–1214.

[17] T. Hosaka, *On equivariant homeomorphisms of boundaries of CAT(0) groups and Coxeter groups*, preprint.

[18] Y. Kida, *Measure equivalence rigidity of the mapping class group*, Ann. of Math. (2) 171 (2010), 1851–1901.

[19] N. Monod, *Superrigidity for irreducible lattices and geometric splitting*, J. Amer. Math. Soc. 19 (2006), 781–814.

[20] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. (2) 164 (2006), 825–878.

[21] C. Mooney, *Examples of non-rigid CAT(0) groups from the category of knot groups*, Algebr. Geom. Topology 8 (2008), 1667–1690.

[22] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Etudes Sci. Publ. Math. (1968), no. 34, 53–104.

[23] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973, Annals of Mathematics Studies, no. 78.

[24] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. thesis, Ohio State University, 1988.

[25] D. Radcliffe, *Unique presentation of Coxeter groups and related groups*, Ph.D. thesis, The University of Wisconsin-Milwaukee, 2001.

[26] J. M. Wilson, *A CAT(0) group with uncountably many distinct boundaries*, J. Group Theory 8 (2005), 229–238.

DEPARTMENT OF MATHEMATICS, SHIZUOKA UNIVERSITY, SURUGA-KU, SHIZUOKA 422-8529, JAPAN
E-mail address: `sthosak@ipc.shizuoka.ac.jp`