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Asymptotic Analysis on the Early Exercise Boundary of American Options*

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1 Introduction

European-style options, which can only be exercised at its maturity, have closed-form formulas for their values in the standard model pioneered by Black and Scholes [9] and Merton [33]. Although a vast majority of traded options are of American-style optimally exercised before maturity, there are no closed-form formulas for their values even in the standard model called *vanilla*. The original statements of the American options problem are dating back to the work of Samuelson [37] and McKean [32]; see Barone-Adesi [3] for a concise review of the American options problem. The principal difficulty in analyzing American options may be the absence of an explicit expression for the *early exercise boundary* (EEB), which is an optimal level of critical asset value where early exercise occurs.

Kim [24] provided an integral representation for the American put value as a function of the EEB; see Jacka [20] and Carr et al. [13] for related studies. The integral representation suggests the idea of computing the option value via numerical integration. To implement this idea in practice, we need to obtain an accurate EEB approximation possibly in closed form. Various but non-closed form approximations have been developed for the EEBs: An early work toward approximating EEBs was Geske and Johnson [16], in which the option values are represented by a series of compound options with multivariate normal terms, and the EEB is evaluated only at a very limited number of points of time. The approximation developed by MacMillan [30] and Barone-Adesi and Whaley [5] is usually referred to as the quadratic approximation, and that is known to be consistent with the exact result for the perpetual case. The quadratic approximation for the EEB is given by a solution of a nonlinear equation, and hence we need some root-finding algorithm such as the Newton-Raphson algorithm. The Barone-Adesi and Whaley original quadratic approximation scheme by MacMillan [30] and [5] generates large pricing errors in some cases, and hence some refined approximations have been proposed, e.g., by Barone-Adesi and Elliot [4], Ju and Zhong [22] and Andrikopoulos [1]. Bunch and Johnson [10] derived a nonlinear equation for the EEB, based on the tangent approximation for the first passage probability of time to early exercise. This nonlinear equation also needs to be solved iteratively; see Zhu and He [41] for a refinement. We should mention that Zhu [39, 40] has been trying to develop closed-form approximations for the EEBs, but there still remain complicated expressions in his formulas, e.g., they are given in an infinite-series form and/or in an integral form.

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No doubt, the simplest approximation is a flat boundary. Barone-Adesi and Whaley [5] proposed a flat approximation as an initial guess of their iterative procedure to find the optimal EEB. Bjerksund and Stensland [7] have slightly modified this approximation to value the American option as a barrier option with knockout feature. Bjerksund and Stensland [8] further proposed an extended model by dividing the trading period into two parts according to the *golden rule*, each with a flat boundary. A strategy following from a flat EEB is feasible, but *not* optimal, which means that the option value with this strategy represents a lower bound to the true option value. Toward the optimal strategy, Huang et al. [18] assumed the EEB as a piecewise-constant function of time to maturity, and provided a recursive algorithm for obtaining a suboptimal exercise levels; see also Ingersoll [19] for the constant case and Sbuelz [38] for the two-step case. Instead of the step-function approximations, Omberg [36] and Ju [21] assumed an exponential function and a piecewise-exponential function for the EEB, respectively. In both approximations, however, there are no closed-form solutions for the bases and the exponents of those exponential functions, which must be computed numerically in their approaches; see Ingersoll [19] and Nunes [35] for numerical comparison of their pricing errors.

The multipiece EEB approximations in Huang et al. [18] and Ju [21] naturally have discontinuous points in the boundary, but the EEB should be smooth intrinsically [34]. Clearly, the discontinuity in the multipiece EEB approximations become an serious obstacle for accurate decision making of the option holders. If we regard the EEB approximation as a tool for quick decision-making in optimal-stopping situations as well as a tool for pricing, it should be a continuous and explicit function of time. A class of exponential functions would be an appropriate choice for the EEB approximation. Our goal in this paper is to develop approximations for the EEB in the form of a constant plus a single exponential function with an explicit exponent, satisfying two obvious consistency conditions at time to close to expiry and at infinite time to expiry; see Kim [25] for a regression approach to this class of approximations for EEBs.

2 Black-Scholes-Merton Formulation

Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $(S_t)_{t\geq 0}$ be the asset price governed by the risk-neutralized diffusion process

$$\frac{\mathrm{d}S_t}{S_t} = (r-\delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0, \tag{2.1}$$

where $r > 0$ is the risk-free interest rate, $\delta \geq 0$ is a continuous dividend rate, $\sigma > 0$ is a volatility of the asset returns. In (2.1), $(W_t)_{t\geq 0}$ is a standard Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to W and the probability measure $\mathbb{P}$ is chosen risk-neutrally so that the asset has mean rate of return r. We consider an American *put* option written on the asset price process $(S_t)_{t\geq 0}$, which has maturity $T > 0$ and strike price $K > 0$. Let

$$P \equiv P(t, S_t) = P(t, S_t; K, r, \delta), \quad 0 \leq t \leq T,$$

denote the value of the American put option at time t. Similarly, let $C \equiv C(t, S_t) = C(t, S_t; K, r, \delta)$ ($0 \le t \le T$) denote the value of the associated American call option with the same parameters as those in the put option.

From the theory of arbitrage pricing, the fair value of the American put option at time t is given by solving an *optimal stopping problem*

$$P(t, S_t) = \operatorname*{ess\,sup}_{\mathcal{T}_t \in [t,T]} \mathbb{E}\left[e^{-r(\mathcal{T}_t - t)}(K - S_{\mathcal{T}_t})^+ \,\middle|\, \mathcal{F}_t \right], \quad 0 \le t \le T, \tag{2.2}$$

where $\mathcal{T}_t$ is a stopping time of the filtration $(\mathcal{F}_t)_{t\ge 0}$ and the conditional expectation is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\mathcal{T}_t^* \in [t, T]$ is called an *optimal stopping time* if it gives the supremum value of the right-hand side of (2.2). The relationship between the early exercise feature of American options and optimal stopping problems was first analyzed by McKean [32] who studied the problem (2.2) under an actual probability measure rather than $\mathbb{P}$. Mathematically rigorous treatment of the problem (2.2) was first established by Bensoussan [6] and Karatzas [23]. Solving the optimal stopping problem (2.2) is equivalent to find the points (t, S_t) for which early exercise is optimal. Let $\mathcal{S}$ and $\mathcal{C}$ denote the *stopping region* and *continuation region*, respectively. The stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \left\{(t, S) \in [0,T] \times \mathbb{R}_+ \mid P(t, S) = (K - S)^+\right\}.$$

Of course, the continuation region $\mathcal{C}$ is the complement of $\mathcal{S}$ in $[0,T] \times \mathbb{R}_+$. The boundary that separates $\mathcal{S}$ from $\mathcal{C}$ is the EEB, which is defined by

$$B_p(t) = \sup\left\{S \in \mathbb{R}_+ \mid P(t, S) = (K - S)^+\right\}, \quad 0 \le t \le T.$$

McDonald and Schroder [31] proved that a *symmetric* relation holds between the American put and call values, i.e.,

$$C(t, S_t; K, r, \delta) = P(t, K; S_t, \delta, r). \tag{2.3}$$

See Carr and Chesney [12] for another symmetric relation in more general settings. If we define the EEB for the American call option by

$$B_c(t) = \inf\left\{S \in \mathbb{R}_+ \mid C(t, S) = (S - K)^+\right\}, \quad 0 \le t \le T,$$

then we also have a simple symmetric relation between the two boundaries $B_p(t) \equiv B_p(t; r, \delta)$ and $B_c(t) \equiv B_c(t; r, \delta)$ [12] such that

$$B_c(t; r, \delta) B_p(t; \delta, r) = K^2, \quad 0 \le t \le T. \tag{2.4}$$

McKean [32] showed that the American put value and the EEB can be obtained by jointly solving a *free boundary problem*, which is specified by the Black-Scholes-Merton partial differential equation (PDE)

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \delta) S \frac{\partial P}{\partial S} - rP = 0, \quad S > B_p(t), \tag{2.5}$$

together with the boundary conditions

$$\left|\begin{array}{l}\lim_{S\uparrow\infty} P(t,S) = 0 \\ \lim_{S\downarrow B_p(t)} P(t,S) = K - B_p(t) \\ \lim_{S\downarrow B_p(t)} \dfrac{\partial P}{\partial S} = -1,\end{array}\right. \tag{2.6}$$

and the *terminal condition*

$$P(T,S) = (K-S)^+. \tag{2.7}$$

The second condition in (2.6) is often called the *value-matching condition*, while the third condition is called the *smooth-pasting* or *high-contact condition.*

It is sometimes convenient to work with the equations where the current time t is replaced by the time to expiry $\tau \equiv T-t$. For the sake of notational convenience, we write $\widetilde{S}_\tau \equiv S_{T-\tau} = S_t$ and $\widetilde{B}_p(\tau) \equiv B_p(T-\tau) = B_p(t)$, and we refer to $(\widetilde{S}_\tau)_{\tau\leq T}$ as the *backward running process* of $(S_t)_{t\geq 0}$. From (2.5)–(2.7), the put price for the backward running process $\widetilde{P}(\tau,\widetilde{S}_\tau) \equiv P(T-\tau, S_{T-\tau}) = P(t,S_t)$ satisfies the PDE

$$-\frac{\partial \widetilde{P}}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{P}}{\partial S^2} + (r-\delta)S\frac{\partial \widetilde{P}}{\partial S} - r\widetilde{P} = 0, \quad S > \widetilde{B}_p(\tau), \tag{2.8}$$

with the boundary conditions

$$\left|\begin{array}{l}\lim_{S\uparrow\infty} \widetilde{P}(\tau,S) = 0 \\ \lim_{S\downarrow \widetilde{B}_p(\tau)} \widetilde{P}(\tau,S) = K - \widetilde{B}_p(\tau) \\ \lim_{S\downarrow \widetilde{B}_p(\tau)} \dfrac{\partial \widetilde{P}}{\partial S} = -1,\end{array}\right. \tag{2.9}$$

and the *initial condition*

$$\widetilde{P}(0,S) = (K-S)^+. \tag{2.10}$$

3 Valuation in the Laplace Domain

3.1 Laplace-Carson Transforms

In order to value American vanilla options, Carr [11] developed a fast and accurate method, which is called the *randomization* approach. The name "randomization" originates in its initial step of randomizing the maturity date T by an exponentially distributed random variable with mean $\lambda^{-1} = T$; see Chapter II of Feller [15] for a more general framework of randomization.

Mathematically, the randomization approach is closely related to the Laplace-Carson transform (LCT): Let $f(\tau)$ be a function of exponential order, i.e., there exist some constants M and $\lambda_0 \geq 0$, for which $|f(\tau)| \leq M\mathrm{e}^{\lambda_0\tau}$ for all $\tau \geq 0$. Then, the LCT $f^*(\lambda)$ of a function $f(\tau)$ is defined by

$$f^*(\lambda) \equiv \mathcal{LC}[f(\tau)] = \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau} f(\tau)\mathrm{d}\tau,$$

where λ is a complex number with $\mathrm{Re}(\lambda) > \lambda_0$. There is *no* essential difference between LCT and Laplace transform. The principal reason why LCT is often preferred to Laplace transform in the context of option pricing would be that LCT generates relatively simpler formulas for option pricing problems because constant values are invariant after transformation [26, 27, 28]. Since the time-reversed quantities $\widetilde{P}(\tau, S)$ and $\widetilde{B}_p(\tau)$ are bounded functions of $\tau \in \mathbb{R}_+$, we can define the LCTs of these functions for $\mathrm{Re}(\lambda) > 0$. The randomization approach can be interpreted to mean that the LCT $P^*(\lambda, S) = \mathcal{LC}[\widetilde{P}(\tau, S)]$ is an exponentially weighted sum (integral) of the time-reversed value $\widetilde{P}(\tau, S)$ for (infinitely many) different values of the maturity $T = \lambda^{-1} \in \mathbb{R}_+$, which makes $\widetilde{P}(\tau, S)$ and $P^*(\lambda, S)$ well defined for $\tau \geq 0$ and $\lambda > 0$, respectively.

From (2.8)–(2.10), the LCT $P^*(\lambda, S)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 P^*}{\mathrm{d}S^2} + (r-\delta)S\frac{\mathrm{d}P^*}{\mathrm{d}S} - (\lambda + r)P^* + \lambda(K-S)^+ = 0, \quad S > B_p^*, \tag{3.1}$$

together with the boundary conditions

$$\left|\begin{array}{l} \lim\limits_{S\uparrow\infty} P^*(\lambda, S) = 0 \\ \lim\limits_{S\downarrow B_p^*} P^*(\lambda, S) = K - B_p^* \\ \lim\limits_{S\downarrow B_p^*} \dfrac{\mathrm{d}P^*}{\mathrm{d}S} = -1, \end{array}\right. \tag{3.2}$$

where $B_p^* \equiv B_p^*(\lambda) = \mathcal{LC}[\widetilde{B}_p(\tau)]$ is a constant in the Laplace world due to the memoryless property of the exponential distribution. Solving this boundary-value problem, Kimura [26, Theorems 3.1 and 3.3] proved that

$$P^*(\lambda, S) = \begin{cases} K - S, & S \leq B_p^* \\ p^*(\lambda, S) + e_p^*(\lambda, S), & S > B_p^*, \end{cases} \tag{3.3}$$

where $p^*(\lambda, S)$ is the LCT of $\widetilde{p}(\tau, S)$, the time-reverse value of the European put option associated with the American put option on target, which is given by

$$p^*(\lambda, S) = \begin{cases} \xi(S) + \dfrac{\lambda K}{\lambda + r} - \dfrac{\lambda S}{\lambda + \delta}, & S < K \\ \eta(S), & S \geq K, \end{cases} \tag{3.4}$$

with

$$\begin{cases} \xi(S) = \dfrac{K}{\theta_1 - \theta_2}\dfrac{\lambda}{\lambda+\delta}\left(1 - \dfrac{r-\delta}{\lambda+r}\theta_2\right)\left(\dfrac{S}{K}\right)^{\theta_1}, & S < K \\ \eta(S) = \dfrac{K}{\theta_1 - \theta_2}\dfrac{\lambda}{\lambda+\delta}\left(1 - \dfrac{r-\delta}{\lambda+r}\theta_1\right)\left(\dfrac{S}{K}\right)^{\theta_2}, & S \geq K, \end{cases} \tag{3.5}$$

and the parameters $\theta_i \equiv \theta_i(\lambda)$ ($i = 1, 2$, $\theta_1 > 1$, $\theta_2 < 0$) are two roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - (\lambda + r) = 0, \tag{3.6}$$

i.e.,

$$\theta_i = \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) - (-1)^i\sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)}\right\}, \quad i = 1, 2.$$

In (3.3), the function $e_p^*(\lambda, S)$ can be regarded as the LCT of the time-reverse early exercise premium of the American put option, which is given by

$$e_p^*(\lambda, S) = -\frac{1}{\theta_2}\left\{\theta_1\xi(B_p^*) + \frac{\delta}{\lambda+\delta}B_p^*\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2}, \quad S > B_p^*,$$

and B_p^* ($\leq K$) is a unique positive solution of the functional equation

$$\lambda\left(\frac{B_p^*}{K}\right)^{\theta_1} + \delta\theta_1\frac{B_p^*}{K} + r(1-\theta_1) = 0. \tag{3.7}$$

3.2 Put-Call Symmetry

For the backward running process $(\widetilde{S}_\tau)_{\tau\leq T}$, let $\widetilde{C}(\tau, \widetilde{S}_\tau) \equiv C(T-\tau, S_{T-\tau}) = C(t, S_t)$ and $\widetilde{B}_c(\tau) \equiv B_c(T-\tau) = B_c(t)$. Also, for $\lambda > 0$, let $C^*(\lambda, S) = \mathcal{LC}[\widetilde{C}(\tau, \widetilde{S}_\tau)]$ and $B_c^*(\lambda) = \mathcal{LC}[\widetilde{B}_c(\tau)]$. Then, for American put and call options in the Laplace domain, we have symmetric relations similar to (2.3) and (2.4):

Theorem 1 *Between the option values $P^*(\lambda, S) \equiv P^*(\lambda, S; K, r, \delta)$ and $C^*(\lambda, S) \equiv C^*(\lambda, S; K, r, \delta)$, there exists a symmetric relation such that*

$$C^*(\lambda, S; K, r, \delta) = P^*(\lambda, K; S, \delta, r), \quad \lambda > 0. \tag{3.8}$$

In addition, between the early exercise boundaries $B_p^*(\lambda) \equiv B_p^*(\lambda; r, \delta)$ and $B_c^*(\lambda) \equiv B_c^*(\lambda; r, \delta)$, there exists a symmetric relation such that

$$B_c^*(\lambda; r, \delta)B_p^*(\lambda; \delta, r) = K^2, \quad \lambda > 0. \tag{3.9}$$

Proof. Let $V_p \equiv V_p(x)$ and G be the solution of the following boundary value problem

$$\frac{1}{2}\sigma^2x^2\frac{\mathrm{d}^2V_p}{\mathrm{d}x^2} + (\delta - r)x\frac{\mathrm{d}V_p}{\mathrm{d}x} - (\lambda+\delta)V_p + \lambda(K-x)^+ = 0, \quad x > G, \tag{3.10}$$

with the boundary conditions

$$\left|\begin{array}{l} \lim_{x\uparrow\infty} V_p(x) = 0 \\ \lim_{x\downarrow G} V_p(x) = K - G \\ \lim_{x\downarrow G} \dfrac{\mathrm{d}V_p(x)}{\mathrm{d}x} = -1. \end{array}\right. \tag{3.11}$$

Comparing (3.10) and (3.11) with (3.1) and (3.2), we see that $V_p(x) = P^*(\lambda, x; K, \delta, r)$ and $G = B_p^*(\lambda; \delta, r)$; note that the parameters r and δ are exchanged. With the changes of variables $y := K^2/x$ and $H := K^2/G$, define a transformed function

$$V_c(y) = \frac{K}{x}V_p(x)\bigg|_{x=K^2/y} = \frac{y}{K}V_p\left(\frac{K^2}{y}\right), \quad 0 < y < H.$$

Then, in (3.11), the first boundary condition is rewritten for $V_c(y)$ as

$$\lim_{y\downarrow 0} V_c(y) = 0 \tag{3.12}$$

and the value-matching condition and the smooth-pasting condition respectively become

$$\lim_{y\uparrow H} V_c(y) = \frac{K}{G}(K-G)\bigg|_{G=K^2/H} = \frac{H}{K}\left(K-\frac{K^2}{H}\right) = H-K \tag{3.13}$$

and

$$\begin{aligned}\lim_{y\uparrow H}\frac{\mathrm{d}V_c(y)}{\mathrm{d}y} &= \lim_{y\uparrow H}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{K}{x}V_p(x)\right)\frac{\mathrm{d}x}{\mathrm{d}y} = \lim_{y\uparrow H}\left(-\frac{K}{x^2}V_p+\frac{K}{x}\frac{\mathrm{d}V_p}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}y}\\ &= \left\{-\frac{K}{G^2}(K-G)-\frac{K}{G}\right\}\lim_{y\uparrow H}\left(-\frac{K^2}{y^2}\right) = \left(-\frac{K^2}{G^2}\right)\left(-\frac{K^2}{H^2}\right) = 1. \end{aligned}\tag{3.14}$$

Next we will derive the ODE for $V_c(y)$ $(0<y<H)$. By straightforward calculation, we have

$$x\frac{\mathrm{d}V_p}{\mathrm{d}x} = \frac{K}{y}\left(V_c - y\frac{\mathrm{d}V_c}{\mathrm{d}y}\right) \quad\text{and}\quad x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}V_p}{\mathrm{d}x}\right) = \frac{K}{y}\left\{y\frac{\mathrm{d}}{\mathrm{d}y}\left(y\frac{\mathrm{d}V_c}{\mathrm{d}y}\right) - 2y\frac{\mathrm{d}V_c}{\mathrm{d}y} + V_c\right\},$$

from which the ODE (3.10) for $V_p(x)$ can be rewritten as

$$\begin{aligned}0 &= \frac{1}{2}\sigma^2 x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}V_p}{\mathrm{d}x}\right) + (\delta - r - \tfrac{1}{2}\sigma^2)x\frac{\mathrm{d}V_p}{\mathrm{d}x} - (\lambda+\delta)V_p + \lambda(K-x)^+\\ &= \frac{K}{y}\left[\frac{1}{2}\sigma^2\left\{y\frac{\mathrm{d}}{\mathrm{d}y}\left(y\frac{\mathrm{d}V_c}{\mathrm{d}y}\right) - 2y\frac{\mathrm{d}V_c}{\mathrm{d}y}+V_c\right\} + (\delta-r-\tfrac{1}{2}\sigma^2)\left(V_c - y\frac{\mathrm{d}V_c}{\mathrm{d}y}\right) - (\lambda+\delta)V_c + \lambda(y-K)^+\right]\\ &= \frac{K}{y}\left[\frac{1}{2}\sigma^2 y^2\frac{\mathrm{d}^2V_c}{\mathrm{d}y^2} + (r-\delta)y\frac{\mathrm{d}V_c}{\mathrm{d}y} - (\lambda+r)V_c + \lambda(y-K)^+\right].\end{aligned}$$

Hence, we obtain the ODE

$$\frac{1}{2}\sigma^2 y^2\frac{\mathrm{d}^2V_c}{\mathrm{d}y^2} + (r-\delta)y\frac{\mathrm{d}V_c}{\mathrm{d}y} - (\lambda+r)V_c + \lambda(y-K)^+ = 0, \quad 0<y<H. \tag{3.15}$$

In much the same way as in (3.1) and (3.2) for the put case, the ODE (3.15) together with the boundary conditions (3.12)–(3.14) is no more than the boundary-value problem for the call case, which means that $V_c(y) = C^*(\lambda, y; K, r, \delta)$ and $H = B_c^*(\lambda; r, \delta)$. By the definition of V_c and a change of *numéraire*, we obtain

$$C^*(\lambda, S; K, r, \delta) = V_c(S) = \frac{S}{K}V_p\left(\frac{K^2}{S}\right) = \frac{S}{K}P^*\left(\lambda, \frac{K^2}{S}; K, \delta, r\right) = P^*(\lambda, K; S, \delta, r),$$

which proves (3.8). From the relation $GH = K^2$, we immediately have (3.9). □

Let $\nu_1 \equiv \nu_1(\lambda) > 1$ and $\nu_2 \equiv \nu_2(\lambda) < 0$ be two real roots of the quadratic equation

$$\frac{1}{2}\sigma^2\nu^2 + (\delta - r - \tfrac{1}{2}\sigma^2)\nu - (\lambda+\delta) = 0, \tag{3.16}$$

i.e.,

$$\nu_i = \frac{1}{\sigma^2}\left\{-(\delta - r - \tfrac{1}{2}\sigma^2) - (-1)^i\sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+\delta)}\right\}, \quad i = 1,2.$$

Clearly, $\nu_i(\lambda) \equiv \nu_i(\lambda; r, \delta)$ and $\theta_i(\lambda) \equiv \theta_i(\lambda; r, \delta)$ $(i=1,2)$ are symmetric with respect to r and δ, namely, $\theta_i(\lambda; \delta, r) = \nu_i(\lambda; r, \delta)$. In addition, there is an important relation among these roots:

Lemma 1 *For $\lambda > 0$, we have*

$$\begin{cases} \theta_1(\lambda) + \nu_2(\lambda) = 1 \\ \theta_2(\lambda) + \nu_1(\lambda) = 1. \end{cases}$$

Proof. We only prove the first equation $\theta_1 + \nu_2 = 1$. The second one follows similarly.

$$\begin{aligned} \nu_2 &= \frac{1}{\sigma^2}\left\{-(\delta - r - \tfrac{1}{2}\sigma^2) - \sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + \delta)}\right\} \\ &= 1 - \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(\delta - r - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r) + 2\sigma^2(\delta - r)}\right\} \\ &= 1 - \frac{1}{\sigma^2}\left\{-(r - \delta - \tfrac{1}{2}\sigma^2) + \sqrt{(r - \delta - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda + r)}\right\} = 1 - \theta_1, \end{aligned}$$

and hence $\theta_1(\lambda) + \nu_2(\lambda) = 1$ for $\lambda > 0$. □

From Lemma 1, we can calculate $C^*(\lambda, S)$ from the results (3.3)–(3.7) for $P^*(\lambda, S)$ without directly solving a boundary-value problem associated with (3.1) and (3.2).

Theorem 2 *The LCT $C^*(\lambda, S)$ for the American call value is given by*

$$C^*(\lambda, S) = \begin{cases} S - K, & S \geq B_c^* \\ c^*(\lambda, S) + e_c^*(\lambda, S), & S < B_c^*, \end{cases}$$

where $c^*(\lambda, S)$ is the LCT of $\widetilde{c}(\tau, S)$, the time-reverse value of the European call option associated with the American call option on target, and $e_c^*(\lambda, S)$ is the LCT of the time-reverse early exercise premium, which are

$$c^*(\lambda, S) = \begin{cases} \xi(S), & S < K \\ \eta(S) + \dfrac{\lambda S}{\lambda + \delta} - \dfrac{\lambda K}{\lambda + r}, & S \geq K, \end{cases}$$

$$e_c^*(\lambda, S) = \frac{1}{\theta_1}\left\{\frac{\delta}{\lambda + \delta}B_c^* - \theta_2\eta(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1}, \quad S < B_c^*.$$

The functions $\xi(\cdot)$ and $\eta(\cdot)$ are defined in (3.5), and the LCT $B_c^* \equiv B_c^*(\lambda)$ $(\geq K)$ is a unique positive solution of the functional equation

$$\lambda\left(\frac{B_c^*}{K}\right)^{\theta_2} + \delta\theta_2\frac{B_c^*}{K} + r(1 - \theta_2) = 0. \tag{3.17}$$

Proof. We prove only the functional equation (3.17) for the LCT B_c^*, because this equation plays a key role in this paper. The LCT $C^*(\lambda, S)$ for the American call value can be proved in a similar and straightforward manner: If we exchange the two parameters r and δ in the functional equation (3.7) for B_p^*, θ_1 should be replaced by ν_1 and B_p^*/K by K/B_c^*, due to (3.9) and (3.16). Hence, using Lemma 2, we have

$$\begin{aligned} 0 &= \lambda\left(\frac{K}{B_c^*}\right)^{\nu_1} + r\nu_1\frac{K}{B_c^*} + \delta(1 - \nu_1) \\ &= \lambda\left(\frac{K}{B_c^*}\right)^{1-\theta_2} + r(1 - \theta_2)\frac{K}{B_c^*} + \delta\theta_2 \\ &= \frac{K}{B_c^*}\left\{\lambda\left(\frac{B_c^*}{K}\right)^{\theta_2} + \delta\theta_2\frac{B_c^*}{K} + r(1 - \theta_2)\right\}, \end{aligned}$$

from which (3.17) holds for the LCT B_c^*. □

4 Asymptotic Approximations

4.1 Asymptotic Properties

Prior to approximating the EEB of American options, we briefly review some known asymptotic properties of the time-reverse EEB as $\tau \to 0$ and $\tau \to \infty$: From the initial-value theorem in the theory of Laplace transforms, we obtain

$$\overline{B}_p \equiv B_p(T) = \lim_{\tau \to 0} \widetilde{B}_p(\tau) = \lim_{\lambda \to \infty} B_p^*(\lambda) = \min\left(\frac{r}{\delta}, 1\right) K. \tag{4.1}$$

See Kimura [26, Theorem 3.4] for details, and also see Kim [24] and Kwok [29, pp. 257–258] for alternative proofs. For the call case, due to the put-call symmetry in (3.9), we have

$$\underline{B}_c \equiv B_c(T) = \max\left(\frac{r}{\delta}, 1\right) K. \tag{4.2}$$

To see asymptotic behavior of the time-reverse EEB as $\tau \to \infty$, we consider the case that λ is sufficiently small, which is due to the final-value theorem in the theory of Laplace transforms.

Lemma 2 *For sufficiently small $\lambda > 0$, we have two different pairs of asymptotic approximations for $B_p^*(\lambda)$ and $B_c^*(\lambda)$, which are*

$$B_p^*(\lambda) \approx \frac{r}{\delta}\frac{\theta_1 - 1}{\theta_1} K \quad \textit{and} \quad B_c^*(\lambda) \approx \frac{r}{\delta}\frac{\theta_2 - 1}{\theta_2} K, \tag{4.3}$$

and

$$B_p^*(\lambda) \approx \frac{\theta_2}{\theta_2 - 1} K \quad \textit{and} \quad B_c^*(\lambda) \approx \frac{\theta_1}{\theta_1 - 1} K. \tag{4.4}$$

Proof. From (3.7) and (3.17), we obtain (4.3) by removing the first terms of the functional equations (3.7) and (3.17). Applying the basic relations in quadratic equations to (3.7)

$$\begin{cases} \lambda + r = -\frac{1}{2}\sigma^2\theta_1\theta_2 \\ r - \delta = -\frac{1}{2}\sigma^2\left(\theta_1 + \theta_2 - 1\right), \end{cases} \tag{4.5}$$

we have another expression of the equation (3.7) for B_p^*, which is

$$\lambda\left(1 - \frac{r-\delta}{\lambda + r}\theta_2\right)\left(\frac{B_p^*}{K}\right)^{\theta_1} + \delta(1 - \theta_2)\frac{B_p^*}{K} + r\theta_2\frac{\lambda + \delta}{\lambda + r} = 0. \tag{4.6}$$

Deleting the first term in (4.6) and using the approximation $(\lambda+\delta)/(\lambda+r) \approx \delta/r$ for sufficiently small λ, we obtain the approximation for $B_p^*(\lambda)$ in (4.4). Similarly, from (3.17), we can calculate the approximation for $B_c^*(\lambda)$ in (4.4) by replacing θ_1 with θ_2. □

From Lemma 2, we immediately obtain the exact limiting values when $\tau \to \infty$ [29, pp. 258–260] as

$$\begin{cases} \underline{B}_p \equiv \lim_{\tau \to \infty} \widetilde{B}_p(\tau) = \frac{r}{\delta}\frac{\theta_1^\circ - 1}{\theta_1^\circ} K = \frac{\theta_2^\circ}{\theta_2^\circ - 1} K \\ \overline{B}_c \equiv \lim_{\tau \to \infty} \widetilde{B}_c(\tau) = \frac{r}{\delta}\frac{\theta_2^\circ - 1}{\theta_2^\circ} K = \frac{\theta_1^\circ}{\theta_1^\circ - 1} K, \end{cases} \tag{4.7}$$

where $\theta_i^\circ = \lim_{\lambda\to 0}\theta_i(\lambda)$, i.e.,

$$\theta_i^\circ = \frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2) - (-1)^i\sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2 + 2\sigma^2 r}\right\}, \quad i = 1, 2.$$

The boundary values $\underline{B}_p \equiv \underline{B}_p(r,\delta)$ and $\overline{B}_c \equiv \overline{B}_c(r,\delta)$ are of the perpetual American options with infinite maturity, i.e. $T = \infty$. Note that the put-call symmetry also holds for these limiting values, i.e., $\underline{B}_p(\delta, r)\overline{B}_c(r,\delta) = K^2$.

4.2 Exponential Approximations

Lemma 3 *For sufficiently small $\lambda > 0$, we have*

$$\begin{cases} \theta_1(\lambda) = \theta_1^\circ + \dfrac{2}{\sigma^2}\dfrac{\lambda}{\theta_1^\circ - \theta_2^\circ} + o(\lambda) \\[2ex] \theta_2(\lambda) = \theta_2^\circ + \dfrac{2}{\sigma^2}\dfrac{\lambda}{\theta_2^\circ - \theta_1^\circ} + o(\lambda). \end{cases}$$

Proof. For simplicity, denote $\omega \equiv r - \delta - \frac{1}{2}\sigma^2$. Then, for $i = 1, 2$ and sufficiently small $\lambda > 0$, we have

$$\begin{aligned} \theta_i(\lambda) &= \frac{1}{\sigma^2}\left\{-\omega - (-1)^i\sqrt{\omega^2 + 2\sigma^2(\lambda + r)}\right\} \\ &= \frac{1}{\sigma^2}\left\{-\omega - (-1)^i\sqrt{\omega^2 + 2\sigma^2 r}\sqrt{1 + \frac{2\sigma^2\lambda}{\omega^2 + 2\sigma^2 r}}\right\} \\ &= \frac{1}{\sigma^2}\left\{-\omega - (-1)^i\sqrt{\omega^2 + 2\sigma^2 r}\left(1 + \frac{\sigma^2\lambda}{\omega^2 + 2\sigma^2 r}\right)\right\} + o(\lambda) \\ &= \theta_i^\circ - (-1)^i\frac{\lambda}{\sqrt{\omega^2 + 2\sigma^2 r}} + o(\lambda) = \theta_i^\circ - (-1)^i\frac{2}{\sigma^2}\frac{\lambda}{\theta_1^\circ - \theta_2^\circ} + o(\lambda), \end{aligned}$$

where we have used the relation $\theta_1^\circ - \theta_2^\circ = \frac{2}{\sigma^2}\sqrt{\omega^2 + 2\sigma^2 r}$. □

From Lemmas 2 and 3, we shall derive asymptotic approximations for the time-reverse EEBs of the American put and call options. However, the asymptotic approximations (4.3) and (4.4) in Lemma 2 are subtly different for $\lambda > 0$, though they are exactly equivalent for the limit as $\lambda \to 0$ as shown in (4.7).

Theorem 3 *For sufficiently large τ, we have two different pairs of asymptotic approximations for the time-reverse early exercise boundaries $\widetilde{B}_p(\tau)$ and $\widetilde{B}_c(\tau)$, which are*

$$\begin{cases} \dfrac{\widetilde{B}_p(\tau)}{\underline{B}_p} \approx 1 + \dfrac{1}{\theta_1^\circ - 1}\exp\left\{-\tfrac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\right\} \\[2ex] \dfrac{\widetilde{B}_c(\tau)}{\overline{B}_c} \approx 1 + \dfrac{1}{\theta_2^\circ - 1}\exp\left\{-\tfrac{1}{2}\sigma^2\theta_2^\circ(\theta_2^\circ - \theta_1^\circ)\tau\right\}, \end{cases} \tag{4.8}$$

and

$$\begin{cases} \dfrac{\widetilde{B}_p(\tau)}{\underline{B}_p} \approx 1 - \dfrac{1}{\theta_2^\circ}\exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\right\} \\[2ex] \dfrac{\widetilde{B}_c(\tau)}{\overline{B}_c} \approx 1 - \dfrac{1}{\theta_1^\circ}\exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_1^\circ)(\theta_2^\circ - \theta_1^\circ)\tau\right\}. \end{cases} \tag{4.9}$$

Proof. First, let us start from $B_p^*(\lambda)$ in (4.3). Combining the asymptotic results for $B_p^*(\lambda)$ and $\theta_1(\lambda)$, for sufficiently small $\lambda > 0$, we have

$$\frac{B_p^*(\lambda)}{K} \approx \frac{r}{\delta}\left\{1 - \frac{\frac{1}{2}\sigma^2(\theta_1^\circ - \theta_2^\circ)}{\lambda + \frac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)}\right\},$$

which can be analytically inverted as

$$\begin{aligned}\frac{\widetilde{B}_p(\tau)}{K} &\approx \frac{r}{\delta}\left[1 - \tfrac{1}{2}\sigma^2(\theta_1^\circ - \theta_2^\circ)\int_0^\tau \exp\left\{-\tfrac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)t\right\}\mathrm{d}t\right]\\ &= \frac{r}{\delta}\left[\frac{\theta_1^\circ - 1}{\theta_1^\circ} + \frac{1}{\theta_1^\circ}\exp\left\{-\tfrac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\right\}\right]\\ &= \frac{\overline{B}_p}{K}\left[1 + \frac{1}{\theta_1^\circ - 1}\exp\left\{-\tfrac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\right\}\right].\end{aligned}$$

Hence, for sufficiently large $\tau > 0$, we obtain the put value in (4.8). Similarly, from $B_c^*(\lambda)$ in (4.3), we obtain the call value in (4.8). Secondly, from $B_p^*(\lambda)$ in (4.4), for sufficiently small $\lambda > 0$, we obtain

$$\frac{B_p^*(\lambda)}{K} \approx \frac{\lambda - \frac{1}{2}\sigma^2\theta_2^\circ(\theta_1^\circ - \theta_2^\circ)}{\lambda + \frac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)}.$$

Analytical inversion leads to

$$\begin{aligned}\frac{\widetilde{B}_p(\tau)}{K} &\approx \exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\right\} - \tfrac{1}{2}\sigma^2\theta_2^\circ(\theta_1^\circ - \theta_2^\circ)\int_0^\tau \exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)t\right\}\mathrm{d}t\\ &= \frac{\theta_2^\circ}{\theta_2^\circ - 1} - \frac{1}{\theta_2^\circ - 1}\exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\right\}\\ &= \frac{\underline{B}_p}{K}\left[1 - \frac{1}{\theta_2^\circ}\exp\left\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\right\}\right],\end{aligned}$$

and hence we obtain the put value in (4.9). Similarly, from $B_c^*(\lambda)$ in (4.4), we obtain the call value in (4.9). □

These approximations are valid for sufficiently large τ, besides their values at maturity $\tau = 0$ partially coincide with the exact ones: For the first pair of approximations in (4.8), $\widetilde{B}_p(0) = \overline{B}_p = rK/\delta$ if $r < \delta$ and $\widetilde{B}_c(0) = \underline{B}_c = rK/\delta$ if $r > \delta$, whereas for the second pair in (4.9), $\widetilde{B}_p(0) = \overline{B}_p = K$ if $r \geq \delta$ and $\widetilde{B}_c(0) = \underline{B}_c = K$ if $r \leq \delta$. These observations suggest that a natural mixture of these approximations becomes consistent with the exact boundary behavior at maturity. That is, a candidate pair of approximations for the time-reverse EEBs is given by

$$\frac{\widetilde{B}_p(\tau)}{\underline{B}_p} \approx \beta_p(\tau) \equiv \begin{cases} 1 + \dfrac{1}{\theta_1^\circ - 1}\exp\Big\{-\tfrac{1}{2}\sigma^2\theta_1^\circ(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r < \delta \\ 1 - \dfrac{1}{\theta_2^\circ}\exp\Big\{-\tfrac{1}{2}\sigma^2(1 - \theta_2^\circ)(\theta_1^\circ - \theta_2^\circ)\tau\Big\}, & r \geq \delta. \end{cases} \tag{4.10}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\overline{B}_c} \approx \beta_c(\tau) \equiv \begin{cases} 1 + \dfrac{1}{\theta_2^\circ - 1}\exp\Big\{-\tfrac{1}{2}\sigma^2\theta_2^\circ(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r > \delta \\ 1 - \dfrac{1}{\theta_1^\circ}\exp\Big\{-\tfrac{1}{2}\sigma^2(1 - \theta_1^\circ)(\theta_2^\circ - \theta_1^\circ)\tau\Big\}, & r \leq \delta. \end{cases} \tag{4.11}$$

4.3 Heuristics near Expiry

Evans et al. [14] have derived explicit expressions valid near expiry for the EEBs of American put and call options, which are, as $\tau \to 0+$,

$$\frac{\widetilde{B}_p(\tau)}{\underline{B}_p} \sim \begin{cases} 1-\sigma\sqrt{\tau \ln\left(\dfrac{\sigma^2}{8\pi(r-\delta)^2\tau}\right)}, & r>\delta \\ 1-\sigma\sqrt{2\tau\ln\left(\dfrac{1}{4\sqrt{\pi}\delta\tau}\right)}, & r=\delta \\ 1-\kappa\sigma\sqrt{2\tau}, & r<\delta, \end{cases}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\overline{B}_c} \sim \begin{cases} 1+\sigma\sqrt{\tau \ln\left(\dfrac{\sigma^2}{8\pi(r-\delta)^2\tau}\right)}, & r<\delta \\ 1+\sigma\sqrt{2\tau\ln\left(\dfrac{1}{4\sqrt{\pi}\delta\tau}\right)}, & r=\delta \\ 1+\kappa\sigma\sqrt{2\tau}, & r>\delta, \end{cases}$$

where the constant $\kappa \approx 0.4517$ is the root of the transcendental equation

$$\int_\kappa^\infty \mathrm{e}^{-(x^2-\kappa^2)}\mathrm{d}x = \frac{2\kappa^2-1}{4\kappa^3}.$$

Clearly, the exponential approximations in Theorem 3 display different tangent behavior near expiry, e.g., for $r<\delta$,

$$\lim_{\tau\to 0+}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\widetilde{B}_p(\tau)}{\underline{B}_p}\right) \approx \beta_p'(0) = -\frac{\sigma^2}{2}\frac{\theta_1^\circ(\theta_1^\circ-\theta_2^\circ)}{\theta_1^\circ-1} < 0,$$

whereas the exact value is $-\infty$. This may implies that our approximations for put (call) tend to overestimate (underestimate) the true values for small $\tau > 0$. The asymptotic properties near expiry seems to be helpful for refining our approximations. However, the exact asymptotic expressions above cannot be directly applied to generating refined approximations for EEBs, because if $r \ge \delta$ ($r \le \delta$) for the put (call) case, (a) they cannot be defined for all $\tau > 0$; and (b) for the region of τ where they can be defined, they are not monotone functions of τ, being inconsistent with the exact results. In order to eliminate the defect (a), Barone-Adesi and Whaley [5, Equations (33) and (A10)] have provided a simple but rough approximation based on an asymptotic behavior near expiry; see Bjerksund and Stensland [7] for a minor modification. However, their approximations also have the same defect on the monotonicity, depending on the values of r and δ [5, p. 310].

To realize the tangent behavior near expiry, we further propose a pair of simple but *heuristic* approximations for the time-reverse EEBs as follows:

$$\frac{\widetilde{B}_p(\tau)}{\underline{B}_p} \approx \beta_p^\circ(\tau) \equiv \begin{cases} 1+\dfrac{1}{\theta_1^\circ-1}\dfrac{1}{1+\sigma\sqrt{\tau}}, & r<\delta \\ 1-\dfrac{1}{\theta_2^\circ}\dfrac{1}{1+\sigma\sqrt{\tau}}, & r\ge\delta. \end{cases} \tag{4.12}$$

and

$$\frac{\widetilde{B}_c(\tau)}{\overline{B}_c} \approx \beta_c^\circ(\tau) \equiv \begin{cases} 1 + \dfrac{1}{\theta_2^\circ - 1}\dfrac{1}{1+\sigma\sqrt{\tau}}, & r > \delta \\ 1 - \dfrac{1}{\theta_1^\circ}\dfrac{1}{1+\sigma\sqrt{\tau}}, & r \leq \delta. \end{cases} \tag{4.13}$$

It is easy to check that (a) the approximations above are defined for $\tau \geq 0$; (b) they are monotone functions of τ and they are consistent with the exact results at $\tau = 0$ as well as $\tau \to \infty$; and besides $\beta_p^{\circ\prime}(0) = -\infty$ and $\beta_c^{\circ\prime}(0) = +\infty$, being consistent with the exact tangent behavior. The approximations (4.12) and (4.13) are aimed basically at refining the tangent behavior near expiry, thus they are not used solely but are combined with the asymptotic approximations (4.10) and (4.11).

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