数理解析研究所講究録
第 1886 巻 2014 年 60-81

Identification Problem in the Joint Estimation of Default Intensity and Recovery Rate using only Credit Default Swap Data*

Dara Sim
Graduate School of Economics
Osaka University

Masamitsu Ohnishi
Graduate School of Economics & Center for the Study of Finance and Insurance
Osaka University

1 Introduction

In recent years, a number of studies has been devoted to looking for ways to estimate market-implied recover rates. This quantity is important for risk management as well as for credit derivative pricing. Unlike historical recovery rates which are backward-looking, market-implied recovery rates subsume information about market's expectation of future economic condition. Although this quantity varies significantly over time and is negatively correlated with default rates (See e.g. Altman et al., 2005, Acharya et al., 2007), most studies assume constant, and fix at 40% for senior unsecured bonds. The reason is because in most credit risk approach, default probability and recovery rate, that enter into the formula of credit derivatives, are multiplicatively linked. This fact makes separate identification difficult. This identification problem is known since Duffie & Singleton (1999).

Many techniques has been considered in order to be able to separately estimate implied default probabilities and recovery rates from credit spreads (See Schläfer, 2011). One technique is to specify a link between implied default and recovery rates. For example, Bakshi et al. (2006) specify recovery rates as a function of default intensity. One other technique is to use credit default swap term structure information. Pan & Singleton (2008) use term structure information on sovereign CDS to estimate constant implied recovery rates of sovereign bonds. They show that default risk and recovery rate can be identified under the assumption of recovery of face value. Christensen (2007) use CDS data of Ford Moto Corp., and concludes that default and recovery risk can be jointly identified from CDS term structure data. He assumes both constant and stochastic recovery. Schneider et al. (2011) assume constant implied recovery rates and use CDS on senior unsecured bonds of 278 U.S. corporates. Doshi (2011) use senior and subordinate CDS of 46 firms to estimate jointly default intensity and stochastic recovery rates.

In this paper, we attempt to use CDS term structure information to decompose market-implied default intensity and recovery rate. We want to investigate if this is really possible. In order to do so, we first specify a joint model of interest rates, default intensity, and recovery rates, which incorporates negative correlation between default intensity and recovery rates while trying to preserve tractability. We consider both constant and stochastic recovery models. For interest rate term structure model, we consider the modified arbitrage-free Nelson-Siegel model proposed by Sim & Ohnishi (2012). In the stochastic recovery case, logistic model is assumed for loss given default (LGD) (so that the recovery rate value is ensured between 0 and 1), and negative correlation between default intensity and recovery rate is captured via an interest rate factor. Using generalized transforms of affine processes, CDS pricing formula can be explicitly obtained even under the logistic LGD assumption. Then, we investigate whether it is possible to jointly estimate default intensity and recovery rate through an empirical estimation and a simulation

*This work was supported by JSPS KAKENHI Gra nt Number 25-6058.

study. For interest rate data, we use JPY Libor and swap rates. For CDS data, we consider two sets of data: Sony's senior CDS and Mitsubishi UFJ's subordinate and senior CDS data. With two sets of data, and two types of recovery model, we have four cases: constant recovery model with one seniority CDS, constant recovery model with two seniorities CDS, stochastic recovery model with one seniority CDS, and stochastic recovery model with two seniorities CDS. First, interest rate model is estimated using the extended Kalman filter. Then, given the estimates of interest rate parameters and state variables, default intensity and recovery rate are jointly estimated using the square-root unscented Kalman filter.

From the results of the empirical estimations, we find that in all four cases when jointly estimating default intensity and loss given default together, we tend to largely overestimate implied default probabilities, and largely underestimate (or overestimate) implied loss given default (recovery rates), the result which does not seem plausible at all. However, in the constant recovery case, when fixing the loss given default, we are able to obtain much more plausible result. We conclude that separate identification of default intensity and recovery rates is not possible. This is verified by a simple simulation study on the four cases. From the simulation results, we find that for the constant recovery model, when jointly estimated, the estimates of the default intensity's parameters distribute far away from the true values to the right, and the estimates of the loss given default's parameters distribute far away from the true values to the left. However, when one of the unknown components to be estimated is given, the parameter estimates of the other components distribute very close to the true. This suggests that the default intensity factor and the loss given default parameter cannot be separately identified. In addition, in the stochastic recovery model with one seniority included, even if we fix the loss given default's parameters, the estimates of default intensity's parameters are far from the true values. This is an obvious evidence that the paths of default risk factor and that of the recovery risk factor are not jointly identified by the CDS data.

The organization of the paper is as follow. Joint model specification is described in section 2. Specifically, the model for term structure of interest rates, the model for default intensity, as well as the model for loss given default, are presented. Section 3 describes data, estimation process, and estimation results. The simulation study is conducted in section 4. Finally, section 5 draws a conclusion.

2 Joint Model Specifications

2.1 Model Setup

In this paper, we follow the reduced-form approach. In reduced-form approach, default is not based on the economic notion of bankruptcy. Instead it is treated as an event governed by an exogenously specified counting process. Lando (1998) and Duffie & Singleton (1999) use a cox process (also called doubly stochastic Poisson process), i.e., Poisson process with stochastic intensity, in the modelling of default event. Duffie & Singleton (1999) considers "Recovery of Market Value" just before default time. Though this assumption results in mathematically elegant models, it does not allow DI and LGD to be separately identified (See, Duffie & Singleton (1999) and Houweling & Vorst (2005)). We assume "Recovery of Par Value" paid at the time of default, as is the case in CDS markets.

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and a risk-neutral probability measure $\mathbb{Q}$. The information filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed satisfying the usual condition. Let $\mathbf{X} \in \mathbb{R}^d$ be a càdlàg $(\mathcal{F}_t)$-adapted process, $\mathcal{G}_t \equiv \sigma(\mathbf{X}_s : 0 \leq s \leq t)$ be σ-algebra generated by the process $\mathbf{X}$, $h = h(\mathbf{X}) : \mathbb{R}^d \to \mathbb{R}$ be non-negative and integrable on any finite time interval. Also, let N be a cox process with stochastic intensity h, that is, conditional on the realization of the path of h, N becomes an inhomogenous Poisson process,

$$\mathbb{Q}(N(T) - N(t) = n|\mathcal{G}_t) = \frac{1}{n!}\left(\int_t^T h_s \mathrm{d}s\right)^n e^{-\int_t^T h_s \mathrm{d}s}. \tag{1}$$

The default occurs the first time the process N is no longer zero. We denote the default time by T_d, i.e., $T_d = \inf\{t \in \mathbb{R}_0^+ | N(t) > 0\}$, and the σ-algebra generate by the default indicator $1_{\{T_d \leq s\}}$ by $\mathcal{H}_t$, i.e., $\mathcal{H}_t = \sigma(1_{\{T_d \leq s\}} : 0 \leq s \leq t)$, which contains the information whether default occurs by time t or not. Also, $\mathcal{F}_t$ can be thought of as the smallest σ-algebra containing both $\mathcal{H}_t$ and $\mathcal{G}_t$, i.e., $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. Thus, T_d is a stopping time w.r.t $\mathcal{H}_t$ and $\mathcal{F}_t$, but not $\mathcal{G}_t$. This means that knowing the sample path of $\mathbf{X}$ does not tell when the default happens.

For $t \leq u$, we have

$$\mathbb{Q}(t < T_d \leq u|\mathcal{G}_\infty \vee \mathcal{H}_t) \quad = 1_{\{T_d > t\}}(1 - e^{-\int_t^u h_s \mathrm{d}s}),$$

which is the $\mathcal{G}_\infty \vee \mathcal{H}_t$-conditional distribution of T_d given $\{T_d > t\}$ (See Appendix A.1). Differentiate w.r.t u yields its density function

$$1_{\{T_d > t\}} h_u e^{-\int_t^u h_s \mathrm{d}s}. \tag{2}$$

Finally, denote by $\varphi \equiv \varphi(\boldsymbol{X}) : \mathbb{R}^d \longrightarrow \mathbb{R}$ the loss given default (LGD) assumed to be a function of the latent state variable factors $\boldsymbol{X}$.

2.2 Interest Rates Model

We use the Modified Arbitrage-Free Nelson-Siegel (MAFNS) of Sim & Ohnishi (2012) because it has good empirical fit and its factors have Nelson-Siegel interpretations, namely level, slope, and curvature. This model is a modified version of the Arbitrage-Free Nelson-Siegel (AFNS) proposed by Christensen et al. (2011). Sim & Ohnishi (2012) show that MAFNS has better fits to the yield curves than AFNS. MAFNS belongs to a 3-factor affine class of dynamic term structure models of Duffie & Kan (1996) with the following specification.

The short rate r_t is given by

$$r_t = X_{1,t} + X_{2,t}, \tag{3}$$

where the dynamics of the three interest rate factors $\boldsymbol{X}_r = (X_1, X_2, X_3)'$ under risk-neutral measure $\mathbb{Q}$ is

$$\mathrm{d}\begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} = \begin{pmatrix} k_1^Q & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] \mathrm{d}t + \begin{pmatrix} \sigma_1\sqrt{X_{1,t}} & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \mathrm{d}\begin{pmatrix} W_{1,t}^Q \\ W_{2,t}^Q \\ W_{3,t}^Q \end{pmatrix}. \tag{4}$$

and the dynamics of $\boldsymbol{X}_r$ under the physical probability measure $\mathbb{P}$ can be specified as follow,

$$\mathrm{d}\begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} = \begin{pmatrix} k_1^P & 0 & 0 \\ 0 & k_2^P & 0 \\ 0 & 0 & k_3^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \end{pmatrix} - \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] \mathrm{d}t + \begin{pmatrix} \sigma_1\sqrt{X_{1,t}} & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \mathrm{d}\begin{pmatrix} W_{1,t}^P \\ W_{2,t}^P \\ W_{3,t}^P \end{pmatrix}, \tag{5}$$

where W_1, W_2, W_3 are independent Brownian motions.

Under this affine specification, the price at time t of a zero-coupon bond with maturity τ is explicitly obtained as follow.

$$P(\boldsymbol{X}_{r,t}, \tau) = \exp(-y(\boldsymbol{X}_{r,t}, \tau)\tau), \tag{6}$$

where $y(\boldsymbol{X}_{r,t},\tau)$ is the zero-coupon yield at time t of maturity τ, the explicit formula of which is given in the Appendix A.2.

Also, since in this paper Japanese Yen (JPY) Libor and swap rates are used for risk-free interest rates, expressions for Libor and swap rates are needed. Having obtained the formula for zero-coupon bond prices, we can straightforwardly do the calculation as these can be expressed in terms of prices of zero-coupon bonds with different maturities. Let m denote the number of payments per year for swap rates, then LIBOR and swap rates at time t with maturity τ can be expressed respectively as follow:

$$
\begin{aligned}
LIBOR(\boldsymbol{X}_{r,t},\tau) &= \frac{100}{\tau}\left(P(\boldsymbol{X}_{r,t},\tau)^{-1}-1\right), && (7)\\
SWAP(\boldsymbol{X}_{r,t},\tau) &= 100m\frac{1-P(\boldsymbol{X}_{r,t},\tau)}{\sum_{i=1}^{m\tau}P(\boldsymbol{X}_{r,t},\frac{i}{m})}. && (8)
\end{aligned}
$$

2.3 Credit Risk Model

To evaluate credit default swap or corporate bond in the reduced-form framework, we need to specify models for DI and REC (or LGD). In this section we specify these models for the case of constant recovery and the case of stochastic recovery. Many studies assume constant recovery (and fix it at 40% for senior unsecured bond); however, many empirical evidences find that REC is time-varying and negatively correlated with DI. As already mentioned, the purpose of our study is to verify if it is possible to separately identify DI and REC's parameters (and risk factors) from CDS data of different maturities. Even in the case that REC or LGD's parameter and DI cannot be separately identified, it could be that it is possible under the stochastic recovery case when correlation between the two is modelled, and even in the case the parameters can be jointly estimated, it could be that the factors cannot be separately identified. Therefore, it is paramount to consider both cases.

Constant Recovery Model

In this case, we assume that DI is governed by only one risk factor which follows a CIR process, i.e., $h(\boldsymbol{X}) = X_4$, where the dynamics of the factor X_4 under both risk-neutral and physical probability measures are assumed as follows:

$$
\begin{aligned}
\mathrm{d}X_{4,t} &= k_4^{\mathbb{Q}}(\theta_4^{\mathbb{Q}}-X_{4,t})\mathrm{d}t+\sigma_4\mathrm{d}W_{4,t}^{\mathbb{Q}},\\
\mathrm{d}X_{4,t} &= k_4^{\mathbb{P}}(\theta_4^{\mathbb{P}}-X_{4,t})\mathrm{d}t+\sigma_4\mathrm{d}W_{4,t}^{\mathbb{P}}.
\end{aligned}
$$

We assume essentially risk premium specification, i.e., $\mathrm{d}W_4^{\mathbb{Q}} = \mathrm{d}W_4^{\mathbb{P}}+\gamma\mathrm{d}t$, in which case we have $k_4^{\mathbb{Q}}\theta_4^{\mathbb{Q}} = k_4^{\mathbb{P}}\theta_4^{\mathbb{P}}$. W_4 is a Brownian motion independent of W_1, W_2, W_3.

Needless to say, the loss given default LGD is assumed constant and takes on different values for different seniorities.

Therefore, in this case we have 4 factors $\boldsymbol{X} = (X_1, X_2, X_3, X_4)'$ directly entered into our credit default swap model, three of which are those of the interest rates $\boldsymbol{X}_r = (X_1, X_2, X_3)'$, and one is the default risk factor X_4.

Stochastic Recovery Model

In this case, we assume that DI is governed by two factors: the interest rate factor corresponding to the level factor, and another factor which can be considered as a default-risk-specific factor. That is, DI has the following form:

$$h(\boldsymbol{X}) = \rho X_1 + X_4, \tag{9}$$

where the dynamics of factor X_4 under both risk-neutral and physical probability measures are assumed as follows:

$$
\begin{aligned}
\mathrm{d}X_{4,t} &= k_4^{\mathbb{Q}}(\theta_4^{\mathbb{Q}} - X_{4,t})\mathrm{d}t + \sigma_4 \mathrm{d}W_{4,t}^{\mathbb{Q}}, \\
\mathrm{d}X_{4,t} &= k_4^{\mathbb{P}}(\theta_4^{\mathbb{P}} - X_{4,t})\mathrm{d}t + \sigma_4 \mathrm{d}W_{4,t}^{\mathbb{P}}.
\end{aligned}
$$

Again, we assume essentially risk premium specification, i.e., $\mathrm{d}W_4^{\mathbb{Q}} = \mathrm{d}W_4^{\mathbb{P}} + \gamma \mathrm{d}t$, in which case we have $k_4^{\mathbb{Q}}\theta_4^{\mathbb{Q}} = k_4^{\mathbb{P}}\theta_4^{\mathbb{P}}$. W_4 is a Brownian motion independent of W_1, W_2, W_3. ρ is a constant and is assumed to take value in the interval $[-1, 1]$. It should be noted that it is well-known that DI is negatively correlated with instantaneous short rate, but we do not assume that ρ takes on only negative value. In practice, when we estimate our model, we consider two cases. First, we restrict that $\rho \in [-1, 0]$, and then $\rho \in [0, 1]$. We choose the case that has better fit to the data. Also note that when ρ takes on negative value the DI process will no longer be guaranteed non-negative, which will violate the mathematical requirement that it is a non-negative process. In that case we neglect this requirement. However, when ρ takes on non-negative value, the DI process is guaranteed non-negative because it is a linear function of two CIR processes. This is as well another reason we prefer MAFNS to AFNS.

For LGD model, we assume Logistic function which can be considered one of the most appropriate assumption and which ensures that the value of REC rate is guaranteed between 0 and 1. That is φ take the following form.

$$
\varphi(\boldsymbol{X}) = \frac{1}{1 + \exp(-b_0 - b_1 X_{1,t} - b_2 X_{2,t} - b_3 X_{3,t} - b_5 X_{5,t})}, \tag{10}
$$

where X_5 can be thought of as the recovery-risk-specific factor. Its dynamics under both risk-neutral and physical probability measures are assumed as follows:

$$
\begin{aligned}
\mathrm{d}X_{4,t} &= \sigma_5 \mathrm{d}W_{5,t}^{\mathbb{Q}}, \\
\mathrm{d}X_{4,t} &= k_5^{\mathbb{P}}(\theta_5^{\mathbb{P}} - X_{5,t})\mathrm{d}t + \sigma_5 \mathrm{d}W_{5,t}^{\mathbb{P}},
\end{aligned}
$$

where W_5 is a Brownian motion assumed independent of W_1, W_2, W_3, and W_4. Note that by specifying its dynamics under both measures, we implicitly specify the risk premium. The parameters b_0, b_1, b_2, b_3, b_5 are assumed different for different seniorities. As will be mentioned bellow, in this paper, two cases are considered: the case when only one seniority are included, and the case when two seniorities are included. Without losing generality, we restrict b_5 to 1 when only one seniority CDS data are included, and restrict one of the two b_5's to 1 when two seniorities CDS data are used in the estimation. In addition, to capture the negative correlation between DI and REC, we impose restriction on the parameter b_1 to have the same sign as ρ. Moreover, we further impose restriction on the parameter b_2 to be positive. This is equivalent to assume that X_2 and LGD are positively correlated. Recall that X_2 is corresponding to the negative slope of the yield curve, i.e., the difference between short- and long-term yields. When the short-term rate exceeds the long-term rate (inverted yield curve), we may expect future economic down-turn, in which case REC rate may be lower than that in the situation of normal yield curves if the default event occurs.

To sum up, in the stochastic recovery model we have 5 factors $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)'$ directly entered into our credit default swap model – three of which are those of the interest rates $\boldsymbol{X}_r = (X_1, X_2, X_3)'$, one is the default risk factor X_4, and the other one is the recovery-risk-specific factor X_5.

2.4 Credit Default Swap Pricing Formula

A CDS is a single-name credit derivative contract that involves 2 parties: the protection buyer and the protection seller. It provides protection against default of a 3rd party called the reference entity. A CDS contract consists of a time-to-maturity T, premium payment dates $t_1, t_2, ..., t_n = T$, and the CDS premium $S(t,T)$ that need to be determined at time $t = t_0$ as the contract is made. Thus, $S(t,T)$ is $\mathcal{F}_t$-measurable.

Let $\delta_i = t_i - t_{i-1}, i = 1, 2, ..., n$ be the time between premium payments, then the protection buyer needs to pay to the protection seller a premium amount equal to $\delta_i S(t,T)$ at time $t_i, i = 1, 2, ..., n$ if default does not yet occur and the accrued premium $(T_d - t_i)S(t,T)$ if default occurs between the interval $[t_{i-1}, t_i]$. On the other hand, the protection seller is obligated to pay to the protection buyer at time of default the amount equal to the LGD at default time φ_{T_d}. Thus, under the arbitrage-free assumption, the CDS spread at time t with maturity τ is given by the following formula (See Appendix A.3).

$$S(t,t+\tau) = \frac{\int_t^{t+\tau} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s (r_u+h_u)\mathrm{d}u}\varphi_s h_s \middle| \mathcal{G}_t\right] \mathrm{d}s}{\sum_{i=1}^n \delta_i \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_i}(r_u+h_u)\mathrm{d}u}\middle|\mathcal{G}_t\right] + \sum_{i=1}^n \int_{t_{i-1}}^{t_i}(s-t_{i-1})\mathbb{E}^{\mathbb{Q}}\left[h_s e^{-\int_t^s (r_u+h_u)du}\middle|\mathcal{G}_t\right] \mathrm{d}s}. \quad (11)$$

In the constant recovery case, we only need to evaluate

$$\mathbb{E}^{\mathbb{Q}}\left[h_s \exp\left\{-\int_0^s (r_u+h_u)\mathrm{d}u\right\}\right], \text{and } \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_0^s (r_u+h_u)\mathrm{d}u\right\}\right].$$

These can be evaluated directly or as transforms of affine processes applying the results of Duffie et al. (2000).

In the stochastic recovery case, in addition to the above expectations, we also need to obtain the expression for

$$\mathbb{E}^{\mathbb{Q}}\left[h_s\varphi_s \exp\left\{-\int_0^s (r_u+h_u)\mathrm{d}u\right\}\right]. \quad (12)$$

Following the demonstration of Chen & Joslin (2012), first note that φ can be written (See Appendix A.5)

$$\varphi_t = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{(iv-1)Y_t}}{Cosh\left(\frac{\pi v}{2}\right)}\mathrm{d}v,$$

where $Y_t = -\frac{1}{2}\{b_0 + b_1X_{1,t} + b_2X_{2,t} + b_3X_{3,t} + b_5X_{5,t}\}$. Then substituting this into (12), we have

$$\mathbb{E}^{\mathbb{Q}}\left[h_s\varphi_s \exp\{(r_u+h_u)\mathrm{d}u\}\right] = Re\left\{\frac{1}{2}\int_0^{\infty} \frac{1}{Cosh\left(\frac{\pi v}{2}\right)}\mathbb{E}^{\mathbb{Q}}\left[h_s \exp\left\{-\int_0^s (r_u+h_u)\mathrm{d}u + (iv-1)Y_s\right\}\right]\right\},$$

where $Re\{.\}$ denotes the real part of a complex number.
Once again, the explicit expression for $\mathbb{E}^{\mathbb{Q}}\left[h_s \exp\left\{-\int_0^s (r_u+h_u)\mathrm{d}u + (iv-1)Y_s\right\}\right]$ can be obtained as a transform of affine processes applying the results of Duffie et al. (2000). Thus, even in the case of stochastic recovery model we still obtain semi-analytical expression for CDS spreads.

3 Empirical Estimation

In this section, we estimate our joint models using CDS data of different maturities. We consider four different cases: a constant recovery model with only senior CDS data included, a constant recovery model with two seniorities of CDS data included, a stochastic recovery model with only senior CDS data included, and finally a stochastic recovery model with two seniorities of CDS data included. First, we describe the data used in the estimation, and then the estimation procedure and methods, as well as the estimation results. The purpose of this empirical estimation is to see if DI and REC can be separately identified. To do so, we compute implied default probabilities and implied recovery rates, and examine if these values are plausible by comparing our result with that of Schläfer (2011). Schläfer (2011) introduces a new approach for estimating market-implied recovery rates. The approach uses information of CDS on different-seniority debt instruments (Senior unsecured bonds, subordinated bonds, and secured loans) as well as firm's capital structure.

3.1 Data

For risk-free interest rates data, we use JPY Libor rates and JPY swap rates. Included in the estimation of our term structure model are the Libor rates with maturity of 3 months, 6 months, and 12 months, as well as the swap rates with maturity of 2, 3, 4, 5, 7, and 10 years. For CDS data, we consider data of two Japanese firms: Sony Corp. and Mitsubishi UFJ Financial Group. For Sony, we have senior CDS data with maturity of 1,3,5,7, and 10 years. For MUFG, we have both senior and subordinate CDS data with maturity of 1,3,5,7, and 10 years. The sample used consists of 100 weekly observations for the period from Jan. 2006 to Nov. 2007. This period is just before the period of financial market turbulence, which started from the end of 2007 as consequence of sub-prime loan crisis. For our purpose, we restrict to study only the period of relatively stable market. Small sample of nearly 2 years of weekly data is considered for ease of estimation and in order to avoid impact of structural change. The data is acquired from Thomson Reuters Datastream. It should be noted for the sample period considered, the Mood's rating of Sony was A2, and of MUFG was A1 and was later upgraded to Aa2.

3.2 Estimation Procedure and Methods

Estimation procedure is divided into 2 steps. In the first step, we estimate the MAFNS model using JPY Libor rates and swap rates. The maximum likelihood estimation method based on the Extended Kalman Filter (EKF) is used. We will need expressions for first derivatives of Libor and Swap rates w.r.t. interest rate factors. These are given in Appendix A.6. In step2, given the parameter estimates and estimated paths of interest rate factors, we jointly estimate DI and REC rates using CDS data. The maximum likelihood estimation method based on the Square-Root Unscented Kalman Filter (SR-UKF) is used. The Unscented Kalman Filter (UKF) proposed by Julier & Uhlmann (1997) is derivative-free and outperforms EKF in terms of prediction and estimation errors. The SR-UKF proposed by Van Der Merwe & Wan (2001) is better than UKF in terms of numerical stability, and guarantees positive semi-definiteness of the state covariance. Note that since models to be estimated can be formulated as non-linear dynamic systems with additive noises, state variables are not augmented.

3.3 Estimation Results

3.3.1 Results for the Term Structure Model

First, note that heteroskedasticity is assumed for the measurement errors, and the estimates of their standard deviations are reported in Table 2.

Table 1 reports parameter estimates and standard errors for the term structure model. In order that each factor be identified as meaningful factor, the parameter λ, which determines the speed of decay of loadings on factor 2 and 3, is fixed at 0.57. Observing the speeds of mean reversion rates, we can see that all the three factors are highly persistent, and the most persistent is the first factor followed by the second and the third. As expected, these factors have Nelson-Siegel interpretations. The first factor is closely related to long term yields (level), whereas the second and third factors are respectively closely related to negative slope, and curvature of the yield curve. This is confirmed in Table 3, which reports correlations between the first factor and 10-year yields (0.86 for their levels, and 0.91 for their first differences), between the second factor and the difference between 3-month and 10-year yields (0.99 for their levels and 0.96 for their first differences), between the third factor and empirical curvature (two times of 2-year yields less 3-month yields and less 10-year yield)(0.98 for both levels and first differences).

To see the fitting performance of the model, we refer to Table 2 which reports Root Mean Square Errors (RMSE), Mean Absolute Percentage Errors (MAPE), as well as Measurement Errors Standard Deviation (ME-SD). RMSE and ME-SD are reported in basis points, whereas MAPE is reported in percents. Except for 3-month Libor rates for which our model is slightly difficult to fit (with MPAE of 12.7%), our model is able to achieve almost perfect fit for most Libor and swap rates of different maturities, with MAPE for 6-month Libor rate of 0.00%, and MPAE for 3, 4, 5, 7, and 10-year swap rates of 0.38%, 0.02%, 0.14%, 0.12%, and 0.66% respectively. Having difficulty fitting short-term rates is not restricted to our model; most term structure models do. However, the overall performance is very good. Not surprisingly, this is one of the most important feature of the Nelson-Siegel type models.

3.3.2 Results for Credit Default Swap model

Using the parameter estimates and estimated paths of risk factors, we are able to calculate the j-year-horizon implied default probabilities as follow:

$$\mathbb{Q}(t < T_d \leq t + j|\mathcal{F}_t) = 1 - \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_t^{t+j} h_s \mathrm{d}s\right\}\Big|\mathcal{F}_t\right].$$

Similarly, for the stochastic recovery, we calculate the j-year-horizon implied recovery rates as expected recovery rates if the firm were to default j years later from now. That is,

$$REC(t,j) = 1 - \mathbb{E}^{\mathbb{Q}}\left[LGD_{t+j}|\mathcal{F}_t\right].$$

Note that the measurement errors are assumed homoskedastic for CDS data of the same seniority through out the paper, so we have one measurement errors' standard deviation parameter when one seniority data is included, and two standard deviation parameters when two seniorities are included, to be estimated. This assumption is especially important for the below simulation study which is computationally intensive.

As already mentioned above, we have the estimation results for 4 different cases. The results for Case1–Constant recovery & Sony's senior CDS data– are reported in Table 4, 5, and 6. First of all, the fitting performance of the model can be seen from Table 5, which reports RMSE and MPAE for the CDS spreads of the five maturities. Although the model has difficulty fitting 1-year CDS spreads, it fits quite well for the 3-year, 5-year, 7-year, and 10-year CDS spreads with MPAE of 7.32%, 5.79%, 9.79%, and 10.78% respectively. Table 4 reports the parameter estimates and standard errors. The estimated parameter of LGD is 0.0318, which is then converted to an implied recovery rate of over 96%. This value is so much larger than the usually-assumed recovery rate for senior CDS, which is 40%, and is of course much larger than the implied recovery rates estimated by Schläfer (2011) which range from 10.00% to 42.90% with an average of 24.00% for senior unsecured bonds. However, with this result alone, it is too early to draw any conclusion that it is incorrect. Next, we examine the implied probabilities of default which is reported in Table 6. The values reported are average values across times. The average values of 1 to 5-year-horizon implied PD are respectively 1.93%, 5.21%, 10.36%, 17.81%, and 27.62%. These values are too high to be plausible. According to Moody'sInvestorService (2012), the average one and five-year historical PD for a Moody's-rated 'A' firm are 0.061% and 0.598% respectively. Even with the risk-premium incorporated, we may not expect the implied PD to be so much higher than the historical PD as this result shows. This result suggests that we may have largely overestimated implied PD and underestimated (or overestimated) implied LGD (or recovery rates). This may be the consequence of inability of the CDS data to separately identify DI and LGD (which is in fact the case as will be seen from the simulation results below). To see what happens if we do not estimate LGD along with DI, we fix it at 0.6 (i.e., assume 40% constant recovery rate) and re-estimate the model. From Table 16, we see that in such a case the average values of implied PD of 1 to 5-year horizons are quite plausible.

The results for Case2–Constant recovery & MUFG's senior and subordinate CDS data– are reported in Table 7, 8, and 9. Table 8 reports RMSE in basis points and MPAE in percents for the 2 seniorities and 5 maturities CDS spreads. The constant recovery model seems to have poor fitting performance for the subordinate CDS spreads with an average MPAE of 22.93%. As we can also see in Table 7 which shows parameter estimates and standard errors, the measurement errors' standard deviations are estimated to be 7.09 basis points for subordinate CDS spreads, and 1.13 basis points for senior CDS spreads. The estimate of LGD is 0.0510 for the senior, and 0.0955 for the subordinate CDS. These can be then converted to implied recovery rates of 94.90% for the senior, and 90.45% for the subordinate CDS. These values are far from plausible since according to Moody'sInvestorService (2012), the average historical recovery rates are reported to be from around 40% to around 60% for senior bonds, and around 20% for subordinate bonds. This result implies a large negative recovery risk premium. In other words, the result suggests that investors prefer to take on a high recovery risk, which is implausible. Schläfer (2011)'s estimates of implied recovery rates for subordinated bonds range from 1.40% to 11.20%, which suggests that investors demand positive recovery risk premium. Table 9 shows average values across time of implied PD for 1 to 5-year horizons, which are respectively 0.72%, 2.04%, 4.21%, 7.48%, and 11.97%. Once again, according to Moody'sInvestorService (2012), the average one and five-year historical PD for a Moody's-rated 'Aa' firm are respectively 0.02%, and 0.22%. Thus, the values of the implied PD does not seem reasonable because since the implied PD are probabilities under risk-neutral measure, this will imply a too high default risk premium demanded by investors. Therefore, it appears that in the case when two seniorities are included, the joint estimation of the three unknown components (DI, senior LGD, and subordinate LGD) may result in an overestimation of PD and recovery rates. Similar to Case1 above, we consider what happens when one of the LGD parameter is fixed. The implied PD and implied recovery rates of senior CDS after fixing subordinate LGD parameter at 1 (ie., assume zero recovery rate for the subordinate contracts) are given in Table 17. The implied recovery rate for senior CDS is 47%, and the implied PD is 0.07% for the 1-year horizon, and 1.20% for the 5-year horizon. This result looks much more reasonable than that of the joint estimation case, suggesting that DI and senior LGD may have been probably separately identified. In fact, this is the case as we will see in the simulation result below.

The results for Case3–Stochastic recovery & Sony's senior CDS data– are reported in Table 10, 11, and 12. From Table 11, we can see that with the stochastic recovery model, there is quite an improvement in the fitting performance with average MAPE of 7.95% versus 11.53% of the constant recovery model. Nevertheless, even the stochastic recovery model has difficulty fitting the 1-year CDS spreads. Just as in case1, the average values across time of the implied PD and recovery rates, reported in Table 12, are too high to be plausible. Even in this case, when jointly estimate DI and LGD together, we tend to largely overestimate implied PD and recovery rates.

Finally, the results for Case4– Stochastic recovery & MUFG's senior and subordinate CDS data– are reported in Table 13, 14, and 15. There is significant improvement in fitting performance especially for subordinate CDS spreads. However, just as in Case2, the average values of the implied PD and implied recovery rates for both seniorities are far from plausible.

4 Simulation Study

From the estimation results above, we see that when jointly estimate DI and REC, the implied probabilities of default tend to be largely overestimated, whereas the implied LGD (or REC) tend to be largely underestimated (or overestimated). Such failure to estimate plausible values of implied PD and LGD (or REC) may be attributed to the fact that the information contained within CDS data of different maturities alone is not sufficient for separately identifying DI's factor and LGD's parameter in the constant recovery case, and DI's parameters and factor and LGD's parameters and factor in the stochastic recovery case. In this section, simulation studies are conducted for the four different cases in order to verify that this is really the case. We first describe the methodology and then examine the simulation results.

4.1 Methodology

The simulation procedures for the four different cases are almost identical. They slightly differ only in how to determine true parameters to be chosen. Below we describe how we determine true parameters for the four different cases considered so far.

- *Case1: Constant recovery & only one seniority*
  Fixing LGD parameter at 0.6, we estimate the constant recovery model using Sony's senior CDS data. We consider the acquired parameter estimates and the fixed LGD as true parameters, and consider the fitted values of CDS spreads as theoretical CDS spreads.
- *Case2: Constant recovery & two seniorities*
  Fixing the subordinate LGD parameter at 1, we estimate the constant recovery model using MUFG's senior and subordinate CDS data. We consider the acquired parameter estimates and the fixed subordinate LGD as true parameters, and consider the fitted values of CDS spreads as theoretical CDS spreads.
- *Case3: Stochastic recovery & only one seniority*
  First, fixing LGD parameter at 0.6, we estimate a constant recovery model (but with DI's model as that in the stochastic recovery model) using Sony's senior CDS data. After acquiring the DI's parameter estimates and its estimated path, we then fix these parameters and path and estimate the stochastic recovery model for the LGD's parameters and for the path of recovery risk factor, using once again the Sony's senior CDS data. We consider these DI and LGD's parameter estimates as true parameters, and the fitted values of CDS spreads from the stochastic recovery model as theoretical CDS spreads.
- *Case4: Stochastic recovery & two seniorities*
  First, fixing subordinate LGD parameter at 0.8, we estimate a constant recovery model (but with DI's model as that in the stochastic recovery model) using MUFG's senior and subordinate CDS data. After acquiring the DI's parameter estimates and its estimated path, we then fix these parameters and path and estimate the stochastic recovery model for the senior and subordinate LGD's parameters and for the path of recovery risk factor, using once again the MUFG's senior CDS data. We consider these DI and LGD's parameters

estimates as true parameters, and the fitted values of CDS spreads from the stochastic recovery model as theoretical CDS spreads.

To generate CDS data for simulation from the models, we simply take the generate data to be the sum of theoretical CDS spreads and measurement errors, where measurement errors are generated from independently identically normal distribution with zero mean and a homoskedastic variance. This is done n times to obtain n sets of CDS data, which will be used to estimate the models' parameters and factors n times so as to finally obtain their distributions. Note that for simplicity, when generating artificial CDS data, we generate the random source only from the measurement errors. This means that we do not generate paths of default risk factor and paths of recovery risk factor. For our purpose, we think that it is not necessary for doing so.

4.2 Simulation Results

The simulation results are reported in Table 18, 19, 20, and 21. The tables report the true values along with mean and standard deviation of the parameter estimates. Note that * indicates that the true lies within 1 standard deviation from the mean, and ** indicates that the true lies within 2 standard deviations from the mean.

The result for Case1 for 1000 trials is reported in Table 18. In this case, we further consider 3 subcases. First, we fix LGD's parameter and then estimate the model for DI's parameters together with its path as well as the measurement error standard deviation. The result is reported in Panel(A). For the DI's parameters, the true value lies within 2 standard deviations from the mean, and the standard deviation is so small compared to the true value. This suggests that the estimates distribute very close to the true values. Next, we consider fixing the DI's parameters at true values, and estimate the model for the LGD's and ME-SD's parameter as well as the path of DI. From Panel(B), we find that the mean of both parameter estimates are not significantly different (the absolute difference between the mean and the true value is less than two standard errors, where the standard error is $SD/\sqrt{1000}$) from the true values. Finally, we consider the joint estimation, the result of which is shown in Panel(C). All the DI's and LGD's parameter estimates distribute far away from the true values. It is interesting to note that the only parameter with a mean not significantly different from the true is the ME-SD. This suggests that there are many combinations of different values (with different levels) of DI's parameters and LGD's parameter that give almost equivalent fitting performances. Also notice that while the estimates of LGD distribute far to the left, the estimates of DI's parameters distribute far to the right. This is the reason why we tend to overestimate PD and underestimate LGD. To sum things up, in this case, attempting to jointly estimate the two components (DI and LGD) together, we will end up with far incorrect parameter estimates. When information about one of the two component is given, we are able to obtain quite accurate estimates of the parameters. This is clear evidence of identification problem.

The result for Case2 for 1000 trials is shown in Table 19. Similar to Case1, we consider four subcases. First, we fix subordinate LGD at the true, and estimate the model for DI's parameters, its path, senior LGD, and both ME-SDs. The result is in Panel(A), for which we find that for all of the parameters, the true lies within one or two standard deviations from the mean. Also, the value of the deviations is relatively very small. This means that there is high probability of obtaining estimates that are very close to the true. Second, we fix senior LGD at the true, and estimate the rests of the parameter along with DI's path. As shown in Panel(B), the result is similar to the first subcase. Third, we fix DI's parameters, and estimate its path along with the two LGDs and the two ME-SDs, the result of which is given in Panel(C). From similar observation, we can draw the same conclusion as in the first and second subcases. Finally, the joint estimation result is given in Panel(D). Except for the two ME-SDs, the other parameter estimates distribute far away from the true value. Similar conclusion can be drawn just as in Case1. When the three components (DI, senior LGD, and subordinate LGD) are jointly estimated, due to identification problem we will not be able to obtain accurate results. However, when one of the three unknown components is given, the identification problem is clear.

Next, the result for Case3 for 100 trials is reported in Table 20. We have two state variables to estimate: the default risk factor, and the recovery risk factor. First, we fix the LGD's parameters, and estimate DI's parameters, its path, and the path of the recovery risk factor. The result is shown in Panel(A), for which we find that except for the risk-neutral parameter $\kappa_4^{\mathbb{Q}}$, for the other parameters, the true lies within one or two standard deviations. However, it should be notice that the mean and standard deviation of the long-term mean parameter $\theta_4^{\mathbb{P}}$ are many times larger than the true value. This means that there is small probability of obtaining an estimate that is close to its true value. That is we will greatly overestimate the default intensity. What happens is that even though we fixed the parameters of the recovery risk factor, its path can still fluctuate in a range allowed by its parameters. This, together with the fact that we can obtain accurate estimate of ME-SD , suggests that many combinations of different values with different levels of DI's parameters and its path, and the path of recovery risk factor could produce almost equivalent fitting performances. This is a clear evidence of factor identification problem. Next, we consider fixing DI's parameters without fixing its path. Examining result shown in Panel(B), we find that except for $\kappa_5^{\mathbb{P}}$, for all the other parameters, the true value lies within one or two standard deviation, but since the standard deviation is not relatively so small, it is likely that we will obtain estimates that are less accurate. However, compared to the first subcase when we fixed LGD's parameters, the identification problem in this subcase is not so serious. The reason is that in credit spreads the most prominent component is not the recovery risk factor, but the default risk factor. So, when the default risk factor's parameters are fixed, to achieve equivalent fitting performances, its path does not have enough freedom to take on different levels. Without having to consider the joint estimation, we can conclude in this case that not only parameters but also factors can not be separately identified.

Finally, the result for Case4 for 95 trials is reported in Table 21. As from the analysis of the results above, we are obvious about the existence of parameter and factor identification, in this case we consider the subcase of joint estimation only. Although for all parameters, except ρ, the true value lies within one or two standard deviations, the standard deviation for most parameters, especially the long-term means and mean-reversion rates, is so large compared to its true value. This means that the parameter estimates have disperse distributions. Once again, together with the fact that the estimates of both ME-SDs distribute very close to their true values, this suggests that many combinations of different values with different levels of DI's parameters, its path, LGD's parameters, and its path, could produce equivalent fitting performance. This is obviously an evidence of parameter and factor identification problem.

5 Conclusion

In this paper, we examined if it is possible to jointly estimate default intensity and recovery rates using only information embedded in the CDS term structure of different seniorities. We first specified a joint model of risk-free term structure of interest rates, default intensity, and loss given default. We considered a joint model with constant recovery assumption, and a joint model with stochastic recovery assumption. We used JPY Libor and swap rates as risk-free interest rates. Taking the parameter estimates and estimated path of the interest rate factors as given, we estimated credit default swap model using two sets of data: Sony's senior CDS, and MUFG's senior and subordinate CDS. We then divided our study into four cases: constant recovery

& Sony's senior CDS, constant recovery & MUFG's senior and subordinate CDS, stochastic recovery & Sony's senior CDS, and stochastic recovery & MUFG's senior and subordinate CDS.

From the results of the empirical estimations, we found that in all four cases when jointly estimating default intensity and loss given default together, we tend to greatly overestimate implied default probabilities, and greatly underestimate (or overestimate) implied loss given default (recovery rates). We were not able to obtain plausible estimates of the model parameters at all. However, in the constant recovery case, when we fixed the loss given default, we were able to obtain much more plausible result. We conclude that separate identification between default intensity and recovery rates is not possible. In order to verify this, we conducted a simple simulation study on the four cases. From the simulation results, we found that, for the constant recovery model, when jointly estimated, the estimates of the default intensity parameters distribute far away from the true values to the right, and the estimates of the loss given default parameters distribute far away from the true values to the left. However, when one of the unknown components to be estimated is given, the parameter estimates of the other components distribute very close to the true. This suggests that default intensity factor and loss given default parameter cannot be separately identified. In addition, in the stochastic recovery model with one seniority included, even if we fixed the loss given default's parameters, the estimates of default intensity's parameters are far different from the true values. This is because the path of default risk factor and that of the recovery risk factor could not be jointly identified by the CDS data.

In conclusion, joint estimation of default intensity and recovery rates using only CDS data is not possible. The information subsumed in the different-seniority CDS term structure may not be enough to allow for such separation. Attempting to jointly estimate default intensity and recovery rates using only term structure of CDS premia could result in overestimation of implied recovery rates and probabilities of default. For example, Pan & Singleton (2008) and Schneider et al. (2011) may have overestimated implied recovery rates. Pan & Singleton (2008) estimate implied recovery rates for Mexico, Turkey at 78.9, 76.4% respectively. The market convention for sovereign CDS is to fix implied recovery rates at 25%. Schneider et al. (2011) obtain estimates of implied recovery rates ranging from 68.7% to 89.9% with average of 79.0% much higher than the market convention which is 40%. Therefore, it might be important to consider a more reliable approach for separately estimating default probabilities and recovery rates.

References

Acharya, V.V., Bharath, S.T., & Srinivasan, A. (2007). Does industry-wide distress affect defaulted firms? Evidence from creditor recoveries. *Journal of Financial Economics*, 85(3), 787–821.

Altman, E.I., Brady, B., Resti, A., & Sironi, A. (2005). The Link between Default and Recovery Rates: Theory, Empirical Evidence, and Implications*. *The Journal of Business*, 78(6), 2203–2228.

Bakshi, G., Madan, D., & Zhang, F. (2006). Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates. Working paper, University of Maryland.

Chen, H., & Joslin, S. (2012). Generalized transform analysis of affine processes and applications in finance. *Review of Financial Studies*, 25(7), 2225–2256.

Christensen, J.H., Diebold, F.X., & Rudebusch, G.D. (2011). The affine arbitrage-free class of Nelson–Siegel term structure models. *Journal of Econometrics*, 164(1), 4–20.

Christensen, J.H.E. (2007). Joint default and recovery risk estimation: an application to cds data. Working paper, Federal Reserve Bank of San Francisco.

Doshi, H. (2011). The term structure of recovery rates. Working paper, McGill University.

Duffie, D., & Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4), 379–406.

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6), 1343–1376.

Duffie, D., & Singleton, K.J. (1999). Modeling term structures of defaultable bonds. *Review of Financial studies*, 12(4), 687–720.

Houweling, P., & Vorst, T. (2005). Pricing default swaps: Empirical evidence. *Journal of International Money and Finance*, 24(8), 1200–1225.

Julier, S.J., & Uhlmann, J.K. (1997). New extension of the Kalman filter to nonlinear systems. In *AeroSense'97*, (pp. 182–193), International Society for Optics and Photonics.

Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives research*, 2(2-3), 99–120.

Moody'sInvestorService (2012). Annual Default Study: Corporate Default and Recovery Rates, 1920-2011.

Pan, J., & Singleton, K.J. (2008). Default and recovery implicit in the term structure of sovereign CDS spreads. *The Journal of Finance*, 63(5), 2345–2384.

Schläfer, T. (2011). *Recovery risk in credit default swap premia*. Springer.

Schneider, P., Sögner, L., & Veža, T. (2011). The economic role of jumps and recovery rates in the market for corporate default risk. *Journal of Financial and Quantitative Analysis*, 45(06), 1517–1547.

Sim, D., & Ohnishi, M. (2012). A Modified Arbitrage-Free Nelson Siegel Model: An Alternative Affine Term Structure Model of Interest Rates. Working paper, Research Institute for Mathematical Sciences, Kyoto University.

Van Der Merwe, R., & Wan, E.A. (2001). The square-root unscented Kalman filter for state and parameter-estimation. In *Acoustics, Speech, and Signal Processing, 2001. Proceedings.(ICASSP'01). 2001 IEEE International Conference on*, volume 6, (pp. 3461–3464), IEEE.

Graduate School of Economics
Osaka University, Toyonaka 560-0043, Japan
E-mail address: simdara@gmail.com

大阪大学経済学研究科 Dara Sim

A Appendices

A.1 Show that $\mathbb{Q}(t < T_d \leq u|\mathcal{G}_\infty \vee \mathcal{H}_t) = 1_{\{T_d>t\}}(1 - e^{-\int_t^u h_s ds})$

From (1), we obtain the following conditional probabilities.

$$\begin{aligned}
\mathbb{Q}(T_d > t|\mathcal{G}_\infty) &= \mathbb{Q}(T_d > t|\mathcal{G}_t) \\
&= \mathbb{Q}(N(t) - N(0) = 0|\mathcal{G}_t) \\
&= e^{-\int_0^t h_s ds}, t \geq 0, \\
\mathbb{Q}(T_d > t|\mathcal{G}_\infty \vee \mathcal{H}_t) &= 1_{\{T_d>t\}}\mathbb{Q}(T_d > T|\mathcal{G}_\infty \vee \mathcal{H}_t) + 1_{\{T_d\leq t\}}\mathbb{Q}(T_d > T|\mathcal{G}_\infty \vee \mathcal{H}_t) \\
&= 1_{\{T_d>t\}}\mathbb{Q}(T_d > T|\mathcal{G}_\infty \vee \mathcal{H}_t) \\
&= 1_{\{T_d>t\}}\frac{\mathbb{Q}(\{T_d > T\} \cap \{T_d > t\})}{Q(T_d > t|\mathcal{G}_\infty)} \\
&= 1_{\{T_d>t\}}e^{-\int_t^{T_d} h_s ds}.
\end{aligned}$$

A.2 Zero-coupon yield curves

$$y(\boldsymbol{X}_{r,t}, \tau) = \frac{2(1 - e^(- \eta\tau))}{\tau(\eta + k_1^Q + (\eta - k_1^Q)e^{-\eta\tau})}X_{1,t} + \frac{1 - e^{-\lambda\tau}}{\lambda\tau}X_{2,t} + \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right]X_{3,t} + \frac{A(\tau)}{\tau}, \quad (1)$$

where

$$\begin{aligned}
\eta &= \sqrt{(k_1^Q)^2 + 2\sigma_1^2} \\
\frac{A(\tau)}{\tau} &= \sigma_2^2\left[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3}\frac{1 - e^{-\lambda\tau}}{\tau} + \frac{1}{4\lambda^3}\frac{1 - e^{-2\lambda\tau}}{\tau}\right] \\
&\quad + \sigma_3^2\left[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2} - \frac{1}{4\lambda}\tau e^{-2\lambda\tau} - \frac{3}{4\lambda^2}e^{-2\lambda\tau} - \frac{2}{\lambda^3}(1 - e^{-\lambda\tau}) + \frac{5}{8\lambda^2}\frac{1 - e^{-2\lambda\tau}}{\tau}\right] \\
&\quad - \kappa_1^{\mathbb{Q}}\theta_1^{\mathbb{Q}}\left[\frac{2}{\tau\sigma_1^2}\log\left(\frac{\kappa_1^{\mathbb{Q}} + \eta + (\eta - \kappa_1^{\mathbb{Q}})e^{-\eta\tau}}{2\eta}\right) + \frac{2}{\eta + \kappa_1^{\mathbb{Q}}}\right].
\end{aligned}$$

A.3 Derivation of the General CDS Spread Formula

From (2), the present values at time t of the premium leg and default leg can be respectively calculated as follow.

Default leg:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_d} r_u du}\varphi_{T_d}1_{\{t\leq T_d\leq T\}}\middle|\mathcal{F}_t\right] &= 1_{\{T_d\geq t\}}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_d} r_u du}\varphi_{T_d}1_{\{t\leq T_d\leq T\}}\middle|\mathcal{G}_T \vee \mathcal{H}_t\right]\middle|\mathcal{F}_t\right] \\
&= 1_{\{T_d\geq t\}}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r_u+h_u)du}\varphi_s h_s ds\middle|\mathcal{F}_t\right] \\
&= 1_{\{T_d\geq t\}}\int_t^T \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s (r_u+h_u)du}\varphi_s h_s\middle|\mathcal{G}_t\right] ds.
\end{aligned}$$

Premium leg:

$$\begin{aligned}
&S(t,T)\sum_{i=1}^{n}\mathbb{E}^{\mathbb{Q}}\left[\delta_i e^{-\int_t^{t_i} r_u du}1_{\{T_d>T\}} + (T_d - t_{i-1}e^{-\int_t^T r_u du}1_{\{t_{i-1}<T_d<t_i\}}\middle|\mathcal{F}_t\right] \\
&= 1_{\{T_d\geq t\}}S(t,T)\left(\sum_{i=1}^{n}\delta_i\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t} t_i(r_u+h_u)du}\middle|\mathcal{G}_t\right] + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(s - t_{i-1})\mathbb{E}^{\mathbb{Q}}\left[h_s e^{-\int_t^s (r_u+h_u)du}\middle|\mathcal{G}_t\right]\right).
\end{aligned}$$

Since the default leg and premium leg must be equal in order to ensure arbitrage-free, we must have

$$S(t,t+\tau)=\frac{\int_t^{t+\tau}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s(r_u+h_u)\mathrm{d}u}\varphi_s h_s\middle|\mathcal{G}_t\right]\mathrm{d}s}{\sum_{i=1}^n\delta_i\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_i}(r_u+h_u)\mathrm{d}u}\middle|\mathcal{G}_t\right]+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(s-t_{i-1})\mathbb{E}^{\mathbb{Q}}\left[h_s e^{-\int_t^s(r_u+h_u)\mathrm{d}u}\middle|\mathcal{G}_t\right]\mathrm{d}s}.$$

A.4 Explicit Formula for $\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_u+h_u)\mathrm{d}u}h_T e^{(iv-1)Y_T}\right]$

Applying the result of Duffie et al. (2000), we obtain

$$\begin{aligned}
\psi(i,v,T-t,\boldsymbol{X}_t) &:= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_u+h_u)\mathrm{d}u}e^{(iv-1)Y_T}\right]=\exp\left\{\alpha+\sum_{j=1}^5\beta_j X_{j,t}\right\}, &(2)\\
\hat{\psi}(i,v,T-t,\boldsymbol{X}_t) &:= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T(r_u+h_u)\mathrm{d}u}h_T e^{(iv-1)Y_T}\right]\\
&= (A+B_1X_{1,t}+B_4X_{4,t})\,\psi(i,v,T-t,\boldsymbol{X}_t), &(3)
\end{aligned}$$

where $\alpha,\beta_j,j=1,2,3,4,5$, and A,B_1,B_4 satisfy the following Riccati equations:

$$\begin{aligned}
\dot{\beta}_1(t) &= -(1+\rho)-k_1^{\mathbb{Q}}+\frac{\sigma_1^2}{2}\beta_1^2,\\
\dot{\beta}_2(t) &= -1-\lambda\beta_2(t),\\
\dot{\beta}_3(t) &= -\lambda\beta_3(t)+\lambda\beta_2(t),\\
\dot{\beta}_4(t) &= -1-k_4^{\mathbb{Q}}\beta_4(t)+\frac{\sigma_4^2}{2}\beta_4^2(t),\\
\dot{\beta}_5(t) &= 0,\\
\dot{\alpha}(t) &= k_1^{\mathbb{Q}}\theta_1\mathbb{Q}+\frac{\sigma_2^2}{2}\beta_2^2(t)+\frac{\sigma_3^2}{2}\beta_3^2(t)+k_4^{\mathbb{Q}}\theta_4^{\mathbb{Q}}\beta_4(t)+\frac{\sigma_5^2}{2}\beta_5^2(t),\\
\dot{B}_1(t) &= -k_1^{\mathbb{Q}}B_1(t)+\sigma_1^2B_1(t)\beta_1(t),\\
\dot{B}_4(t) &= -k_4^{\mathbb{Q}}B_4(t)+\sigma_4^2B_4(t)\beta_4(t),
\end{aligned}$$

with boundary conditions: $\beta_j(0)=\frac{1}{2}b_j(1-iv),j=1,2,3,5$, and $\beta_4(0)=0,\alpha(0)=\frac{1}{2}b_0(1-iv),B_1(0)=\rho,B_4(0)=1,A(0)=0$.

A.5 Show that φ_t can be written $\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{(iv-1)Y_t}}{Cosh(\frac{\pi v}{2})}\mathrm{d}v$

First, it can be written

$$\varphi_t=\frac{1}{1+e^{2Y_t}}=\frac{e^{-Y_t}}{2}\frac{1}{Cosh(Y_t)},$$

where $Cosh(Y)=\frac{1}{2}\left(e^y+e^{-y}\right)$, and $Y=-\frac{1}{2}\left(b_0+b_1X_1+b_2X_2+b_3X_3+b_5X_5\right)$.
The Fourier transform of $\frac{1}{Cosh(y)}$ is

$$\hat{g}(v)=\frac{\pi}{Cosh(\frac{\pi v}{2})}.$$

Then $\frac{1}{Cosh(y)}$ can be recovered by the Fourier inversion as follow:

$$\frac{1}{Cosh(y)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(v)e^{ivy}\mathrm{d}v=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ivy}}{Cosh(\frac{\pi v}{2})}\mathrm{d}v.$$

A.6 First derivative of Libor and Swap rates w.r.t. interest rate factors:

If $P(\boldsymbol{X}_{r,t},\tau) = \exp\{L(\tau) + m_1(\tau)X_{1,t} + m_2(\tau)X_{2,t} + m_3(\tau)X_{3,t}\}$, then

$$\frac{\partial LIBOR}{\partial X_{j,t}}(\boldsymbol{X}_{r,t},\tau) = -\frac{100}{\tau}P(\boldsymbol{X}_{r,t},\tau)^{-1}m_j(\tau)$$

$$\frac{\partial SWAP}{\partial X_{j,t}}(\boldsymbol{X}_{r,t},\tau) = -SWAP(\boldsymbol{X}_{r,t},\tau)\left[\frac{P(\boldsymbol{X}_{r,t},\tau)}{1-P(\boldsymbol{X}_{r,t},\tau)}m_j(\tau) - \sum_{l=1}^{h\tau} m_j\left(\frac{i}{h}\right)\frac{(\boldsymbol{X}_{r,t},\frac{l}{h})}{\sum_{l=1}^{h\tau}P(\boldsymbol{X}_{r,t},\frac{l}{h})}\right],$$

for $j = 1, 2, 3$.

Table 1: Risk-free term structure parameter estimates

	$\theta_1^{\mathbb{P}}$	$\theta_2^{\mathbb{P}}$	$\theta_3^{\mathbb{P}}$	$\kappa_1^{\mathbb{P}}$	$\kappa_2^{\mathbb{P}}$	$\kappa_3^{\mathbb{P}}$	σ_1	σ_2	σ_3	$\kappa_1^{\mathbb{Q}}$
Estimates	0.0042	-0.0042	-0.0038	0.6928	1.0199	4.8490	0.1248	0.0099	0.0131	0.0088
SE	0.0048	0.0000	0.0000	0.7986	0.7696	1.3048	0.0122	0.0000	0.0000	0.0072

Note: In order that each factor is identified, λ is fixed at 0.57.

Table 2: Fitting performance of the term structure model

	LIB3M	LIB6M	LIB12M	SW2Y	SW3Y	SW4Y	SW5Y	SW7Y	SW10Y
RMSE	4.70	0.00	4.03	1.21	0.52	0.04	0.25	0.24	1.61
MAPE	12.17	0.00	7.05	1.15	0.38	0.02	0.14	0.12	0.66
ME-SD	4.7	0.01	4.02	1.23	0.54	0.11	0.29	0.39	1.68

Note: LIB3M, LIB6M, and LIB12M denotes 3-month, 6-month, and 12-month Libor rates respectively, and SW2Y, SW3Y, SW4Y, SW5Y, SW7Y, and SW10Y denotes 2, 3, 4, 5, 7, and 10-year Swap rates respectively. RMSE is Root Mean Square Errors; MAPE is Mean Absolute Percentage Errors, and ME-SD is the standard deviation of the measurement errors. RMSE and ME-SD are reported in basis points, whereas MAPE are reported in percents.

Table 3: Correlations between interest rate factors and empirical level, slope, and curvature

	10y	[0.25y-10y]	[2*(2y)-0.25y-10y]		Δ10y	Δ[0.25y-10y]	Δ[2*(2y)-0.25y-10y]
X_1	0.8597			ΔX_1	0.9146		
X_2		0.9924		ΔX_2		0.9558	
X_3			0.9783	ΔX_3			0.9805

Note: '10y' denotes 10-year zero-coupon yields (or zero rates). '[0.25y-10y]' is the difference between 3-month and 10-year zero rates. '[2*(2y)-0.25y-10y]' means two times of 2-year zero rates less 3-month zero rates and less 10-year zero rates. Δ denotes changes (first differences).

Table 4: Parameter estimates and standard errors (Case1)

	$\theta_4^{\mathbb{P}}$	$\kappa_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	LGD	σ_e
Estimates	0.0106	0.2607	0.1037	-0.4688	0.0318	1.6972
SE	0.0039	0.0845	0.0032	0.0092	0.0024	0.1287

Note: Homoskedasticity is assumed for the measurement errors. σ_e is their standard deviation.

Table 5: Fitting performance (Case1)

	1	3	5	7	10	Mean	Median
RMSE	1.72	1.21	1.35	3.04	4.14	2.29	1.72
MAPE	23.98	7.32	5.79	9.79	10.78	11.53	9.79

Note: This table reports Root Mean Square Errors (RMSE) and Mean Absolute Percentage Errors (MAPE) for 1,3,5,7,10-year CDS spreads, and its mean and median. RMSE are in basis points, and MAPE are in percents.

Table 6: Implied Probabilities of Default (PD) (Case1)

	1-year	2-year	3-year	4-year	5-year
Implied PD	1.93	5.21	10.36	17.81	27.62

Note: All values are average values across times and are in percents.

Table 7: Parameter estimates and standard errors (Case2)

	$\theta_4^{\mathbb{P}}$	$k_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	LGD^{SEN}	LGD^{SUB}	σ_e^{SEN}	σ_e^{SUB}
Estimates	0.0014	0.9437	0.1310	-0.5027	0.0510	0.0955	1.1344	7.0903
SE	0.0002	0.0183	0.0023	0.0079	0.0063	0.0118	0.0217	0.0803

Note: Homoskedasticity is assumed for the measurement errors. $\sigma_e^{SEN}, \sigma_e^{SUB}$ are standard deviations of the measurement errors for senior CDS and subordinate CDS respectively.

Table 8: Fitting performance (Case2)

		1	3	5	7	10	Mean	Median
Senior	RMSE	0.89	1.10	1.51	2.11	3.19	1.76	1.51
	MAPE	19.42	8.89	4.42	6.41	5.87	8.99	6.41
Subordinate	RMSE	5.73	5.15	4.69	5.35	12.25	6.64	5.35
	MAPE	45.27	24.43	8.58	13.63	22.71	22.93	22.71

Note: This table reports Root Mean Square Errors (RMSE) and Mean Absolute Percentage Errors (MAPE) for 1,3,5,7,10-year senior and subordinate CDS spreads, and its mean and median. RMSE are in basis points, and MAPE are in percents.

Table 9: Implied Probabilities of Default (PD) (Case2)

	1-year	2-year	3-year	4-year	5-year
Implied-PD	0.72	2.04	4.21	7.48	11.97

Note: All values are average values across time and are in percents.

Table 10: Parameter estimates and standard errors (Case3)

	$\theta_4^{\mathbb{P}}$	$\kappa_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	ρ			
Estimates	0.0510	0.3316	0.3510	-0.2162	0.6582			
SE	0.0112	0.0587	0.0284	0.0273	0.0406			
	$\theta_5^{\mathbb{P}}$	$\kappa_5^{\mathbb{P}}$	σ_5	b_0	b_1	b_2	b_3	σ_e
Estimates	-2.2897	0.2276	0.9965	-0.9245	6.7406	4.6066	32.0076	1.1916
SE	0.2414	0.1001	0.0802	0.0569	1.2194	1.4381	2.8606	0.1127

Note: Homoskedasticity is assumed for the measurement errors. σ_e is their standard deviation.

Table 11: Fitting performance (Case3)

	1	3	5	7	10	Mean	Median
RMSE	1.61	1.04	1.48	1.35	1.13	1.32	1.35
MAPE	23.07	5.02	6.79	3.04	1.83	7.95	5.02

Note: This table reports Root Mean Square Errors (RMSE) and Mean Absolute Percentage Errors (MAPE) for 1,3,5,7,10-year senior CDS spreads, and its mean and median. RMSE are in basis points, and MAPE are in percents.

Table 12: Implied Probabilities of Default (PD) and Implied Recovery rates (REC) (Case3)

	1-year	2-year	3-year	4-year	5-year
Implied-PD	8.32	17.98	27.68	36.58	44.37
Implied-REC	99.20	98.67	97.99	97.21	96.36

Note: All values are average values across time and are in percents.

Table 13: Parameter estimates and standard errors (Case4)

	$\theta_4^{\mathbb{P}}$	$\kappa_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	ρ	$\theta_5^{\mathbb{P}}$	$\kappa_5^{\mathbb{P}}$	σ_5	σ_e^{SEN}	σ_e^{SUB}
Estimates	0.0006	0.4794	0.1463	-0.3777	0.1289	5.6879	4.9933	0.3747	1.2670	3.7832
SE	0.0005	0.3370	0.0211	0.0428	0.0171	1.3216	3.4145	0.0682	0.0807	0.1264
	b_0^{SEN}	b_1^{SEN}	b_2^{SEN}	b_3^{SEN}	b_5^{SEN}	b_0^{SUB}	b_1^{SUB}	b_2^{SUB}	b_3^{SUB}	
Estimates	3.1341	29.8532	30.0584	-4.6874	-1.0438	-7.7873	31.8520	0.0000	-58.9594	
SE	2.0746	0.9685	6.9665	4.4877	0.2060	1.2975	0.2460	1.0950	15.5034	

Note: b_5^{SUB} is fixed at 1. Homoskedasticity is assumed for the measurement errors. $\sigma_e^{SEN}, \sigma_e^{SUB}$ are standard deviations of the measurement errors for senior CDS and subordinate CDS respectively.

Table 14: Fitting performance (Case4)

		1	3	5	7	10	Mean	Median
Senior	RMSE	0.75	1.60	1.34	0.83	1.13	1.13	1.13
	MAPE	14.12	18.60	8.34	2.58	3.36	9.40	8.34
Subordinate	RMSE	3.70	2.38	3.29	1.76	5.53	3.33	3.29
	MAPE	25.89	9.99	5.54	3.47	5.26	10.03	5.54

Note: This table reports Root Mean Square Errors (RMSE) and Mean Absolute Percentage Errors (MAPE) for 1,3,5,7,10-year senior and subordinate CDS spreads, and its mean and median. RMSE are in basis points, and MAPE are in percents.

Table 15: Implied Probabilities of Default (PD) and Implied Recovery rates (REC) (Case4)

	1-year	2-year	3-year	4-year	5-year
Implied-PD	0.47	1.13	2.02	3.17	4.57
Implied-REC-SEN	91.84	89.71	87.63	85.71	83.96
Implied-REC-SUB	78.89	76.94	75.22	73.62	72.11

Note: All values are average values across time and are in percents. Case4: Stochastic recovery & MUFG's senior and subordinate CDS

Table 16: Implied Probabilities of Default (PD) (Fix LGD = 0.6) (Case1)

	1-year	2-year	3-year	4-year	5-year
Implied-PD	0.10	0.28	0.58	1.02	1.60

Notes: All values are reported in percents.

Table 17: Implied Probabilities of Default (PD) and Implied Recovery rates of senior CDS (REC) (Fix LGD^{SUB} = 1) (Case2)

	1-year	2-year	3-year	4-year	5-year
Implied-PD	0.07	0.20	0.42	0.74	1.20
Implied-REC-SEN	47.00	47.00	47.00	47.00	47.00

Notes: All values are reported in percents.

Table 18: Simulation results for case1 (n trials = 1000)

(A) Fix LGD=0.6						
	$\theta_4^{\mathbb{P}**}$	$\kappa_4^{\mathbb{P}**}$	σ_4^{**}	$\kappa_4^{\mathbb{Q}**}$	σ_e^*	
True	1.13E-04	3.3715	0.1686	-0.3873	2.1709	
Mean	1.40E-04	2.9894	0.1592	-0.3446	2.2017	
SD	2.64E-05	0.3933	0.0056	0.0221	0.0882	
(B) Fix DI's parameters at true						
	LGD*	σ_e^*				
True	0.6000	2.1709				
Mean	0.5995	2.1992				
SD	0.0140	0.0886				
(C) Joint estimation						
	$\theta_4^{\mathbb{P}}$	$\kappa_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	LGD	σ_e^*
True	1.13E-04	3.3715	0.1686	-0.3873	0.6000	2.1709
Mean	0.0055	0.6134	0.1408	-0.3950	0.0626	2.1777
SD	0.0009	0.2723	0.0054	0.0164	0.0393	0.0759

Note: * denotes the true lies within 1 standard deviation from the mean. ** denotes the true lies within 2 standard deviations from the mean.

Table 19: Simulation results for Case2 (n trials = 1000)

(A) Fix $\text{LGD}^{SUB} = 1$								
	$\theta_4^{\mathbb{P}*}$	$\kappa_4^{\mathbb{P}**}$	σ_4^{**}	$\kappa_4^{\mathbb{Q}**}$	LGD^{SEN*}	σ_e^{SEN*}	σ_e^{SUB*}	
True	2.44E-05	4.4910	0.1341	-0.4897	0.5300	1.3429	6.7101	
Mean	2.39E-05	5.4342	0.1368	-0.4760	0.5280	1.3446	6.7032	
SD	2.10E-06	0.6023	0.0022	0.0097	0.0050	0.0494	0.2206	
(B) Fix $\text{LGD}^{SEN} = 0.53$								
	$\theta_4^{\mathbb{P}*}$	$\kappa_4^{\mathbb{P}**}$	σ_4^{**}	$\kappa_4^{\mathbb{Q}**}$	LGD^{SUB*}	σ_e^{SEN*}	σ_e^{SUB*}	
True	2.44E-05	4.4910	0.1341	-0.4897	1.0000	1.3429	6.7101	
Mean	2.37E-05	5.4652	0.1368	-0.4760	0.9968	1.3446	6.7061	
SD	2.04E-06	0.6041	0.0022	0.0097	0.0058	0.0494	0.2208	
(C) Fix DI parameters								
	LGD^{SEN*}	LGD^{SUB*}	σ_e^{SEN*}	σ_e^{SUB*}				
True	0.5300	1.0000	1.3429	6.7101				
Mean	0.5269	0.9932	1.3379	6.7003				
SD	0.0082	0.0137	0.0482	0.2205				
(D) Joint estimation								
	$\theta_4^{\mathbb{P}}$	$\kappa_4^{\mathbb{P}}$	σ_4	$\kappa_4^{\mathbb{Q}}$	LGD^{SEN}	LGD^{SUB}	σ_e^{SEN*}	σ_e^{SUB*}
True	2.44E-05	4.4910	0.1341	-0.4897	0.5300	1.0000	1.3429	6.7101
Mean	0.0012	0.6947	0.1250	-0.4936	0.0658	0.1242	1.3283	6.6947
SD	0.0001	0.0614	0.0022	0.0094	0.0033	0.0062	0.0475	0.2204

Note: * denotes the true lies within 1 standard deviation from the mean. ** denotes the true lies within 2 standard deviations from the mean.

Table 20: Simulation results for Case3 (n trials=100)

(A) Fix LGD's parameters								
	$\theta_4^{\mathbb{P}*}$	$\kappa_4^{\mathbb{P}**}$	σ_4^{**}	$\kappa_4^{\mathbb{Q}}$	ρ^*	σ_e^*		
True	8.78E-05	3.4575	0.1651	-0.4363	0.0137	1.6806		
Mean	0.0054	1.7832	0.1813	-0.2312	0.0056	1.7443		
SD	0.0093	1.0442	0.0078	0.0665	0.0063	0.0704		
(B) Fix DI 's parameters								
	$\theta_5^{\mathbb{P}*}$	$\kappa_5^{\mathbb{P}}$	σ_5^{**}	b_0^*	b_1^*	b_2^*	b_3^*	σ_e^*
True	-0.8217	0.0138	0.5597	-0.1334	31.7809	1.5815	86.7421	1.6806
Mean	-0.5671	1.2246	0.7437	-0.0797	31.7261	-1.6696	90.335	1.6938
SD	0.3122	0.5464	0.1196	0.2164	0.0963	10.8787	23.2082	0.0688

Note: * denotes the true lies within 1 standard deviation from the mean. ** denotes the true lies within 2 standard deviations from the mean.

Table 21: Simulation results for Case4 (n trials=95)

Joint Estimation								
	$\theta_4^{\mathbb{P}*}$	$\kappa_4^{\mathbb{P}*}$	σ_4^{**}	$\kappa_4^{\mathbb{Q}*}$	ρ^{**}	$\theta_5^{\mathbb{P}*}$	$k_5^{\mathbb{P}**}$	σ_5^*
True	6.17E-05	2.5777	0.1405	-0.5307	0.0346	1.9360	2.5105	1.3159
Mean	1.43E-04	2.6675	0.1470	-0.5119	0.0755	2.0911	7.2707	1.4274
SD	1.62E-04	1.0918	0.0048	0.0300	0.0336	2.5517	4.6351	0.7552
	b_0^{SEN*}	b_1^{SEN*}	b_2^{SEN*}	b_3^{SEN*}	b_5^{SEN*}	σ_e^{SEN*}		
True	-0.6798	18.8779	12.4192	-5.4701	-0.2680	1.2436		
Mean	-1.2520	13.4349	13.7458	-9.0068	-0.5973	1.2381		
SD	1.4767	6.1003	7.9217	14.1821	0.6623	0.0464		
	b_0^{SUB*}	b_1^{SUB*}	b_2^{SUB*}	b_3^{SUB*}	b_5^{SUB}	σ_e^{SUB*}		
True	-2.1286	25.5554	0.0289	-406.8029	1.0000	4.2869		
Mean	-3.6328	22.7463	27.8052	-319.4820	1.0000	4.2149		
SD	2.6328	8.5403	29.2064	127.8082	0.0000	0.1733		

Note: * indicates the true lies within 1 standard deviation from the mean. ** indicates the true lies within 2 standard deviations from the mean.