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Structural model based analysis on the period of past default memories

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Abstract

Ongoing financial crisis has revealed a serious issue on contagion effects for both credit risk management and evaluating portfolio credit derivatives. Default contagion is a phenomenon where a default by one firm has direct impact on the health of other surviving firms. Several credit models such as reduced-form model and incomplete information structural model have recently incorporated default contagion. In this study, we present a multi-name incomplete information structural model, which possess the contagion mechanism. Here, we suppose that investors can observe firm values and defaults but are not informed of the threshold level at which a firm is deemed to default. Also, to analyze the contagion effects under general settings, we consider the dependence structure of firm value dynamics and joint distribution of default thresholds. Additionally, in order to model the possibility of crisis normalization, we introduce the concept of *memory period* after default. During the memory period after a default, public investors remember when the previous default occurred and directly reflect that information for updating their belief. When the memory period after a default finish, investors forget about that default and shift their interest to recent defaults if exist. Simple Monte Carlo algorithm is proposed to make default distribution.

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1 Introduction

Interaction of financial default events plays a central role for both credit risk management and credit derivatives valuation. Recent financial crisis has revealed a necessity of quantitative methodology to analyze default contagion effects which are observed in several financial markets. Default contagion is a phenomenon where a default by one firm has direct impact on the health of other surviving firms. Existing dynamic credit risk models which deal with default contagion include, among others, [1], [2], [5], [6], [7], [8], [9], [13], [14] and comprehensive surveys can be found in Chapter 9 of [11]. Generally, credit risk modeling methodologies are categorized to either reduced form approach or structural approach. In the reduced form approach, by introducing interacting intensities, default contagion can be captured by the jump up of the default intensity immediately after the default as in [1], [2], [5], [9] and [14]. However it is not easy to incorporate this mechanism of the crisis mode which will cool down after some period. Information based default contagion described in Chapter 9 of [11] and [6] might be promising methods that allow to represent normalization of crisis via belief updating, however, explicit formulation of normalization and its effects to future defaults are not thoroughly studied. On the other hand, [7] and [8] have studied multi-name structural model under incomplete information and proposed a simulation method for sequential defaults without covering the explicit formulation of normalization. In this paper, we present a multi-name incomplete information structural model which possess a default contagion mechanism naturally in the sense that the sudden change of default probabilities arise from the investors revising their perspectives towards unobserved factors which characterize the joint density of default thresholds. Here, in our model, default thresholds are assumed to be unobservable from public investors and firms are deemed to default when firm values touch this level of threshold for the first time. This formulation is a slight generalization of [8]. Additionally, in order to model the possibility of crisis normalization, we introduce the concept of *memory period* after default. The model is designed so as to confine investor's attention to the recent defaults. During the memory period after a default, public investors remember when the previous default occurred and directly reflect that information for updating their belief. When the memory period after a default finish, investors forget about that default and shift their interest to recent defaults if exist. When all the existing memory periods terminate, we can consider the situation as a complete return to the normal economic condition. In order to evaluate the credit risk under the presence of the default contagion and possibilities of normalization, Monte Carlo simulation is the most reasonable method because of their non Markovian environment.

The rest of this paper is organized as follows. Section 2 introduce our model and deduces an expression for the conditional joint distribution of the default thresholds. Section 3 develops standard Monte Carlo simulation algorithm. Section 4 provides numerical examples and Section 5 concludes.

2 Incomplete information credit risk model

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ that describes the information flow over time. We impose two additional technical conditions, often called the usual conditions. The first is that $\mathcal{F}_t$ is right-continuous and the second is that $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets, meaning that one can always identify a sure event. Without mentioning it again, these conditions will be imposed on every filtration that we introduce in the sequel. The probability measure $\mathbb{P}$ serve as the statistical real world measure in risk management applications, while in derivatives pricing applications, $\mathbb{P}$ is a risk-neutral pricing measure. On the financial market, investors can trade credit risky securities such as bonds and loans issued by several firms indexed by i $(i = 1, 2, \cdots, n)$. In the following, we extend naturally the CreditGrades model in the sense that we consider more than two firms in the portfolio and their asset correlation as well as the dependence structure of the default barriers. Furthermore, we give a slight modification of the CreditGrades model reflecting the fact that the surviving firm's default barrier is lower than its historical path of asset value.

2.1 Model setting

Let $V_i(t)$ $(i = 1, 2, \cdots, n)$ represent the time t asset value of the firm i on a per share basis which solves the next stochastic differential equation

$$\frac{dV_i(t)}{V_i(t)} = \delta_i dW_i(t), \quad i = 1, 2, \cdots, n, \tag{1}$$

$$V_i(0) = v_i, \tag{2}$$

where $\delta_i \in \mathbb{R}$ is the asset volatility and v_i is the firm value at time 0 at which we stand. We assume that the asset value processes have correlations, i.e., $d\langle W_i(\cdot), W_j(\cdot)\rangle_t = \rho_{ij}dt$, where $\langle W_i(\cdot), W_j(\cdot)\rangle$ is the quadratic covariation. Filtrations generated by observed asset values are denoted by $\mathcal{G}_t^i = \sigma(V_i(s) : 0 \leq s \leq t)$. There is a random default threshold L_iD_i such that firm i default as soon as the asset value falls to the level L_iD_i, where L_i denotes the recovery rate at default and D_i is a positive constant representing debt per share, which may given by accounting reports. Then the default time of the firm i is a random variable $\tau_i \in (0, \infty]$ given by

$$\tau_i = \inf\{t > 0 : V_i(t) \leq L_iD_i\}. \tag{3}$$

Here random variables L_i $(i = 1, 2, \cdots, n)$ are mutually independent of the $V_i(t)$ $(i = 1, 2, \cdots, n)$. After a default, bond holders have high expectations to fully recover debt D_i, however, in many cases, recovery of original principal ends in failure and only the amount of its recovery rate L_i is returned. More complicated stochastic processes for $V_i(t)$ such as stochastic volatility may be possible, however, we shed lights to the multi-name setting and model the so-called default contagion. Let $H_i(t) = \mathbf{1}_{\{\tau_i \leq t\}}$ be a right-continuous process which indicate the default status of the firm i at time t and we denote by $\mathcal{H}_t^i = \sigma(H_i(s) : 0 \leq s \leq t)$ the associated filtration.

2.2 Incomplete information

With the view to analyzing how the period of past default memories affect succeeding defaults, we consider the incomplete information framework which is known to represent contagion. In order to depict the incomplete information structure more concretely, in addition to the assumption of the randomness of the default threshold, we postulate the following assumptions.

Assumption 2.1. *Public investors can observe firm values and default events although they can not directly observe the firm's default thresholds L_iD_i $(i = 1, 2, \cdots, n)$ except for the default time τ_i.*

Define the set of survived firms $\mathcal{S}_t = \{i \in \{1, 2, \cdots, n\}; \tau_i > t\}$ and the set of defaulted firms $\mathcal{D}_t = \{i \in \{1, 2, \cdots, n\}; \tau_i \leq t\}$ at the time t. We write $r_t = \#\mathcal{S}_t$, the number of elements in the set $\mathcal{S}_t$.

Assumption 2.2. *At time $t = 0$, we assume every firm in the portfolio are surviving, i.e., $r_0 = n$ and then the inequality $v_i > L_iD_i$ holds for all $i \in \{1, 2, \cdots, n\}$ under the condition $\mathcal{H}_0 = \mathcal{H}_0^1 \vee \cdots \vee \mathcal{H}_0^n$.*

Let $(\log L_1^*, \cdots, \log L_n^*)^\top$ be normally distributed random variable with mean vector $\boldsymbol{\mu} = \left(\log \bar{L}_1 - \frac{\gamma_{11}}{2}, \cdots, \log \bar{L}_n - \frac{\gamma_{nn}}{2}\right)^\top$ and variance-covariance matrix $\boldsymbol{\Gamma} = (\gamma_{ij})_{\leq i,j \leq n}$. Here, $\bar{L}_i, i = 1, 2, \cdots, n$, are some constants. And we assume that $(\log L_1, \cdots, \log L_n)^\top$ be the truncation of $(\log L_1^*, \cdots, \log L_n^*)$ above $\boldsymbol{c} = (\log(v_1/D_1), \cdots, \log(v_n/D_n))^\top$. We denote $\ell_i \overset{def}{=} \log L_i$.

Remark 2.3. *The definition of the mean vector* $\boldsymbol{\mu} = \left(\log \bar{L}_1 - \frac{\gamma_{11}}{2}, \cdots, \log \bar{L}_n - \frac{\gamma_{nn}}{2}\right)^\top$ *is given along the line of original CreditGrades model. [4] proposed that the random recovery rate* L_i *is modeled as* $L_i = \bar{L}_i e^{\gamma_{ii} Z - \gamma_{ii}^2/2}$ *with* $Z \sim N(0,1)$.

Assumption 2.4. *There is a consensus on the prior joint distribution of firm's default thresholds among the public investors. More concretely, investor's uncertainty about the default thresholds* $L_i^* D_i$ *is expressed by*

$$(\log L_1^*, \cdots, \log L_n^*)^\top \quad \sim \quad N_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}) \tag{4}$$

where N_n *is a n-dimensional Normal distribution.*

Assumption 2.5. *For each default time* τ_i, *public investors update their belief on the joint distribution function of surviving firm's default thresholds based on the assumption (4) and newly arrived information, i.e., the realized recovery rate* $V_i(\tau_i)/D_i$.

Remark 2.6. *Since public investors observe all the history of the firm value, they know that the unobservable threshold should be located below the running minimum of the firm value. Despite these knowledge, we assume that public investors treat the logarithm of the recovery rate* $\ell_i^* = \log L_i^*$ *as normally distributed random variable.*

Assumption 2.1, Assumption 2.4 and Assumption 2.5 provide the default contagion mechanism; The default of a firm reveals information about the default threshold and then public investors update their beliefs on surviving firm's joint distribution of thresholds. From public investors' perspective, this naturally causes the sudden change of default probabilities of survived firms, which is just what we wanted to model. The situation of contagious defaults can be translated to the recession, however, it will not continue forever. In our model, we further assume that public investors view the crisis will return to normal condition after some finite time interval.

Assumption 2.7. *The covariance parameter jumps from* γ_{ij} *to* 0 *at time* $\min\{\tau_i + s_i, \tau_j + s_j\}$ *for some constants* $s_i \in (0, \infty)$ *and* $s_j \in (0, \infty)$. *This can be captured by introducing time-depending covariance parameters* $\gamma_{ij;t}$ *defined as*

$$\gamma_{ij;t} \quad = \quad \gamma_{ij} \mathbf{1}_{\{t < \tau_i + s_i\}} \mathbf{1}_{\{t < \tau_j + s_j\}}, \qquad 1 \le i, j \le n, \tag{5}$$

and then assume that the elements of the variance-covariance matrix $\boldsymbol{\Gamma}$ *are given by (5). We call* s_i *the memory period of* i *after* τ_i.

Thus the mean vector $\boldsymbol{\mu}_t$ and the variance-covariance matrix $\boldsymbol{\Gamma}_t$ at time t can be defined as

$$\boldsymbol{\mu}_t \quad = \quad \left(\log \bar{L}_1 - \frac{\gamma_{11;t}}{2}, \cdots, \log \bar{L}_n - \frac{\gamma_{nn;t}}{2}\right)^\top \tag{6}$$

$$\boldsymbol{\Gamma}_t \quad = \quad \left(\gamma_{ij;t}\right)_{1 \le i,j \le n} \tag{7}$$

Assumption 2.8. $V_i(t) = V_i(\tau_i)$ *for* $t \le \tau_i + s_i$.

Define the set $\tilde{\mathcal{D}}_t = \mathcal{D}_t \cap \{i \in \{1, 2, \cdots, n\}; \tau_i \le t < \tau_i + s_i\}$ at time t and let $\tilde{r}_t = \#\tilde{\mathcal{D}}_t$ be the number of elements in the set $\tilde{\mathcal{D}}_t$. Rearrange the order of firm identity numbers in such a way that the elements of $\tilde{\mathcal{D}}_t$ come after the elements of $\mathcal{S}_t$ and the elements of $\mathcal{D}_t \backslash \tilde{\mathcal{D}}_t$ are located the last. Let $\bar{\mathbf{\Gamma}}_t$ be a $(r_t+\tilde{r}_t)\times(r_t+\tilde{r}_t)$ submatrix formed by selecting the rows and columns from the subset $\mathcal{S}_t\cup\tilde{\mathcal{D}}_t$ and let $\bar{\boldsymbol{\ell}}_t$ and $\bar{\boldsymbol{\mu}}_t$ be corresponding $(r_t+\tilde{r}_t)$-dimensional vectors respectively.

$$\bar{\boldsymbol{\ell}}_t = \Big(\log L_1,\cdots,\log L_{r_t},\log L^*_{r_t+1},\cdots,\log L^*_{r_t+\tilde{r}_t}\Big)^\top \tag{8}$$

$$\bar{\boldsymbol{\mu}}_t = \Big(\log\bar{L}_1-\frac{\gamma_{11;t}}{2},\cdots,\log\bar{L}_{r_t}-\frac{\gamma_{r_t,r_t;t}}{2},\cdots,\log\bar{L}_{r_t+\tilde{r}_t}-\frac{\gamma_{r_t+\tilde{r}_t,r_t+\tilde{r}_t;t}}{2}\Big)^\top \tag{9}$$

Assumption 2.8 implies that during the memory period, public investors remember the firm values at which the defaults occurred. We note that

$$\{\tau_i\le t\}=\Big\{\min_{0\le s\le t}V_i(s)\le L_iD_i\Big\},\quad \forall i\in\mathcal{D}_t. \tag{10}$$

By virtue of Assumption 2.4, we can deduce the conditional joint distribution of the default thresholds as follows. Here we don't eliminate the possibility of simultaneous defaults, i.e., we don't need to assume $\mathbb{P}(\tau_i=\tau_j)=0$.

Proposition 2.9. *Let $\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}$ be a $\tilde{r}_t$-dimensional vector consists of the logarithm of the realized recovery rate at time t. Partition the vector $\bar{\boldsymbol{\ell}}_t$, $\bar{\boldsymbol{\mu}}_t$ and the matrix $\bar{\mathbf{\Gamma}}_t$ into*

$$\bar{\boldsymbol{\ell}}_t=\begin{pmatrix}\boldsymbol{\ell}_{\mathcal{S}_t}\\ \boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}\end{pmatrix},\quad \bar{\boldsymbol{\mu}}_t=\begin{pmatrix}\boldsymbol{\mu}_{\mathcal{S}_t}\\ \boldsymbol{\mu}_{\tilde{\mathcal{D}}_t}\end{pmatrix},\quad \bar{\mathbf{\Gamma}}_t=\begin{pmatrix}\mathbf{\Gamma}_{\mathcal{S}_t\mathcal{S}_t} & \mathbf{\Gamma}_{\mathcal{S}_t\tilde{\mathcal{D}}_t}\\ \mathbf{\Gamma}_{\tilde{\mathcal{D}}_t\mathcal{S}_t} & \mathbf{\Gamma}_{\tilde{\mathcal{D}}_t\tilde{\mathcal{D}}_t}\end{pmatrix},$$

where $\boldsymbol{\ell}_{\mathcal{S}_t}$ and $\boldsymbol{\mu}_{\mathcal{S}_t}$ are r_t dimensional vectors, $\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}$ and $\boldsymbol{\mu}_{\tilde{\mathcal{D}}_t}$ are $\tilde{r}_t$ dimensional vectors, $\mathbf{\Gamma}_{\mathcal{S}_t\mathcal{S}_t}$ is a $r_t\times r_t$ matrix, and $\mathbf{\Gamma}_{\tilde{\mathcal{D}}_t\tilde{\mathcal{D}}_t}$ is a $\tilde{r}_t\times\tilde{r}_t$ matrix. Then $\mathcal{F}_t$-conditional joint density of $\boldsymbol{\ell}_{\mathcal{S}_t}$ is given by

$$\frac{f(\boldsymbol{\ell}_{\mathcal{S}_t}|\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t})}{\int_{-\infty}^{\mathbf{c}_t}f(\boldsymbol{\ell}_{\mathcal{S}_t}|\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t})d\boldsymbol{\ell}_{\mathcal{S}_t}}\mathbf{1}_{\{\boldsymbol{\ell}_{\mathcal{S}_t}\le\mathbf{c}_t\}} \tag{11}$$

where

$$f(\boldsymbol{\ell}_{\mathcal{S}_t}|\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}) = \frac{1}{(\sqrt{2\pi})^{r_t}\sqrt{\det\mathbf{\Gamma}_{11.2,t}}}\exp\Big(-\frac{1}{2}(\boldsymbol{\ell}_{\mathcal{S}_t}-\boldsymbol{\mu}_{1,t})'\mathbf{\Gamma}^{-1}_{11.2,t}(\boldsymbol{\ell}_{\mathcal{S}_t}-\boldsymbol{\mu}_{1,t})\Big), \tag{12}$$

$$\boldsymbol{\mu}_{1,t} = \boldsymbol{\mu}_{\mathcal{S}_t}+\mathbf{\Gamma}_{\mathcal{S}_t\tilde{\mathcal{D}}_t}\mathbf{\Gamma}^{-1}_{\tilde{\mathcal{D}}_t\tilde{\mathcal{D}}_t}(\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}-\boldsymbol{\mu}_{\tilde{\mathcal{D}}_t}), \tag{13}$$

$$\mathbf{\Gamma}_{11.2,t} = \mathbf{\Gamma}_{\mathcal{S}_t\mathcal{S}_t}-\mathbf{\Gamma}_{\mathcal{S}_t\tilde{\mathcal{D}}_t}\mathbf{\Gamma}^{-1}_{\tilde{\mathcal{D}}_t\tilde{\mathcal{D}}_t}\mathbf{\Gamma}_{\tilde{\mathcal{D}}_t\mathcal{S}_t}, \tag{14}$$

$$\mathbf{c}_t = \left(\log\Big(\min_{0\le s\le t}V_1(s)\Big/D_1\Big),\cdots,\log\Big(\min_{0\le s\le t}V_{r_t}(s)\Big/D_{r_t}\Big)\right)^\top. \tag{15}$$

Proof. From the continuity of the asset process $V^i(t)$ and equation (10), public investors know that $L_iD_i=\min_{0<s\le\tau_i}V_i(s)$ for all defaulted firms $i\in\tilde{\mathcal{D}}_t$ and $L_iD_i<\min_{0<s\le t}V_i(s)$ for all survived firms $i\in\mathcal{S}_t$. Here, whenever defaults occur, let the order of the firms be rearranged in such a way that the elements of $\tilde{\mathcal{D}}_t$ come after the elements of $\mathcal{S}_t$. Define the set

$$R(\tilde{\mathcal{D}}_t)\stackrel{def}{=}\Big(0,\frac{\min_{0<s\le t}V_s^1}{D_1}\Big)\times\cdots\times\Big(0,\frac{\min_{0<s\le t}V_s^{r_t}}{D_{r_t}}\Big)\times\Big\{\frac{\min_{0<s\le t}V_s^{r_t+1}}{D_{r_t+1}}\Big\}\times\cdots\times\Big\{\frac{\min_{0<s\le t}V_s^{r_t+\tilde{r}_t}}{D_{r_t+\tilde{r}_t}}\Big\},$$

with the special case

$$R(\emptyset)=\Big(0,\frac{\min_{0<s\le t}V_s^1}{D_1}\Big)\times\cdots\times\Big(0,\frac{\min_{0<s\le t}V_s^n}{D_n}\Big)$$

to be the possible range of the recovery rate vector $L = (L_1, L_2, \cdots, L_n)$ under the condition of $\mathcal{F}_t$. In particular, for $i \in \tilde{\mathcal{D}}_t$, L_i takes value $\frac{\min_{0<s\le t} V_s^i}{D_i}$. Let $\mathcal{G}_t = \mathcal{G}_t^1 \vee \mathcal{G}_t^2 \vee \cdots \vee \mathcal{G}_t^n$. As in the proof of the lemma 4.1 of [7], from Bayes' Theorem,

$$\begin{aligned}
\mathbb{P}(L \in A|\mathcal{F}_t) &= \mathbb{P}(L \in A|\tilde{\mathcal{D}}_t, \mathcal{G}_t) \\
&= \mathbb{P}(L \in A|L \in R(\tilde{\mathcal{D}}_t), \mathcal{G}_t) \\
&= \frac{\mathbb{P}(L \in A \cap R(\tilde{\mathcal{D}}_t)|\mathcal{G}_t)}{\mathbb{P}(L \in R(\tilde{\mathcal{D}}_t)|\mathcal{G}_t)} \\
&= \frac{\mathbb{P}(L \in A \cap R(\tilde{\mathcal{D}}_t))}{\mathbb{P}(L \in R(\tilde{\mathcal{D}}_t))}.
\end{aligned}$$

The last equality holds because L is independent of $\mathcal{G}_t$. Hence the joint distribution of the surviving firm's logarithm of recovery rates are given by the conditional distribution of $\boldsymbol{\ell}_{\mathcal{S}_t}$ given $\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t}$ at which the realization $L_i D_i = \min_{0<s\le\tau_i} V_i(s)$ hold for all $i \in \tilde{\mathcal{D}}_t$. Conditional distributions of the multivariate normal distribution are well known. See for example [3] for details. However, from Assumption 2.1, public investors have already know the following inequalities hold.

$$L_i D_i < \min_{0\le s\le t} V_i(s), \quad i \in \mathcal{S}_t. \tag{16}$$

Therefore the conditional distribution $f(\boldsymbol{\ell}_{\mathcal{S}_t}|\boldsymbol{\ell}_{\tilde{\mathcal{D}}_t})$ should be truncated above $\boldsymbol{c}_t$ given by (15). □

In the case $\rho_{ij} = 0$ for all $i \neq j$, the problem become quite easy because first-passage time of 1-dimensional Geometric Brownian motion is well known. In fact, in such a case, [8] showed that the counting process $\sum_{i=1}^n H_i(t)$ has intensity process and they proposed the simulation method based on the total hazard rate even if $V_i, i = 1, 2, \cdots, n$ are not observable.

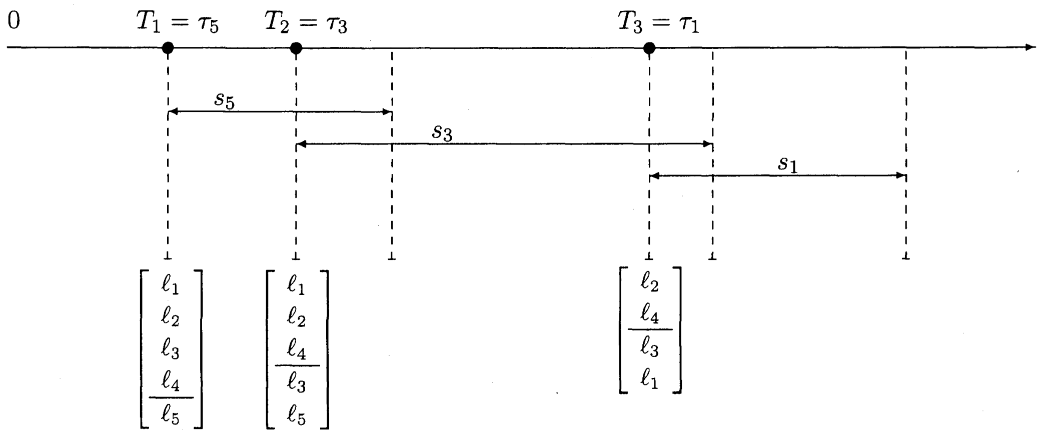

Figure 1: Sequence of defaults and the corresponding memory periods

Let $T_1, T_2, \cdots, T_n$ be an ordered default times of $\{\tau_1, \tau_2, \cdots, \tau_n\}$. Figure 1 illustrate an example of sequence of defaults with $n = 5$ and the corresponding memory periods. At time

0, since all the firms are active and then the unconditional joint density of $(\ell_1, \cdots, \ell_5)^\top = (\log L_1, \cdots, \log L_5)^\top$ is given by

$$\frac{f(\boldsymbol{\ell}_{\tilde{S}_0})}{\int_{-\infty}^{\boldsymbol{c}_0} f(\boldsymbol{\ell}_{\tilde{S}_0}) d\boldsymbol{\ell}_{\tilde{S}_0}} \mathbf{1}_{\{\boldsymbol{\ell}_{\tilde{S}_0} \leq \boldsymbol{c}_0\}}. \tag{17}$$

At the first default time $T_1 = \tau_5$, updated default threshold is sampled under the condition $\ell_5 = \log(V_5(\tau_5)/D_5)$ and this condition remains effective until $\tau_5 + s_5$. This is shown by a square bracket $[\ \ell_1,\ \ell_2,\ \ell_3,\ \ell_4 \mid \ell_5\ ]^\top$ which indicate that the random vector $(\ \ell_1,\ \ell_2,\ \ell_3,\ \ell_4\ )^\top$ should be sampled under the condition $\ell_5 = \log(V_5(\tau_5)/D_5)$ at time T_1. If the second default occurred at time $T_2 = \tau_3 < \tau_5 + s_5$, then the updated default threshold at $T_2 = \tau_3$ should be sampled under the condition $(\ell_3, \ell_5) = (\log(V_3(\tau_3)/D_3), \log(V_5(\tau_5)/D_5))$. However, at the third default time $T_3 = \tau_1$, investor's interest have changed from the first default to the second default completely, i.e., the memory period of 5 after τ_5 have finished. Therefore updated default threshold should be sampled under the condition $(\ell_3, \ell_1) = (\log(V_3(\tau_3)/D_3), \log(V_1(\tau_1)/D_1))$. Notice also that $\mathcal{S}_{T_1} = \{1,2,3,4\}$, $\tilde{\mathcal{D}}_{T_1} = \{5\}, \mathcal{S}_{T_2} = \{1,2,4\}$, $\tilde{\mathcal{D}}_{T_2} = \{5\}$.

2.3 Default Contagion

In this subsection we see how the conditional distribution of the default threshold change at the default time. Suppose that the first default occurred at time $\tau_j > 0$. Let $g_i(x)$ denote the unconditional density of $L_i D_i$ and let $g_i(x|T_1 = \tau_j)$ denote the conditional density of $L_i D_i$ given $L_j = V_j(\tau_j)/D_j$. The distributions of $L_i D_i$, $i \in \{1, 2, \cdots, n\}\backslash j$ change at $T_1 = \tau_j$ from $g_i(x)$ to $g_i(x|T_1 = \tau_j)$ then the default probabilities $\mathbb{P}(\tau_i < t | t < T_2)$, which is restricted before T_2, change from

$$\int_{-\infty}^{c_i(t)} \mathbb{P}(V_i(t) < x) g_i(x) dx \tag{18}$$

to

$$\int_{-\infty}^{c_i(t)} \mathbb{P}(V_i(t) < x) g_i(x | T_1 = \tau_j) dx. \tag{19}$$

Figure 2 and 3 shows the conditional distribution of $L_i D_i$ at τ_j- and τ_j with $\bar{L}_i = 0.4, D_i = 0.85, c_i(\tau_j-) = 0.95$. Distributions are truncated above the running minimum of the firm value 0.95. We see that the γ_{ij} control the contagion impact effectively.

3 Monte Carlo method

This section develops a numerical method to compute the distribution of the number of defaults via Monte Carlo simulation. Complicating matters is the fact that new information of defaults changes the mean and covariance of the joint distribution of the thresholds. At each moment, covariance matrix should be calculated relying upon whether the memory period have terminated or not. Therefore, the simulation depends on the path, i.e. the order of occurrence of sequential defaults.

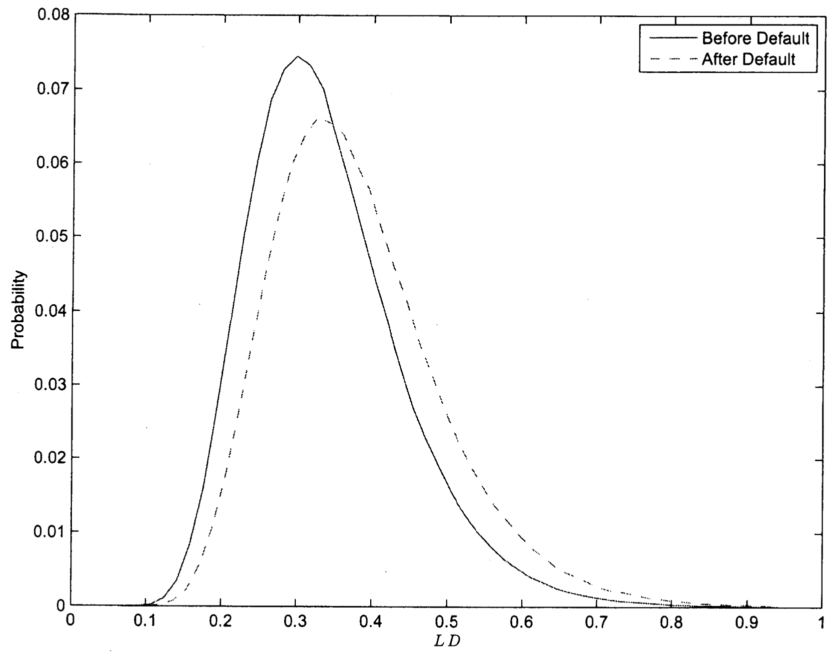

Figure 2: $g_i(x)$ and $g_i(x|T_1 = \tau_j)$ whith $\gamma_{ij} = 0.01$.

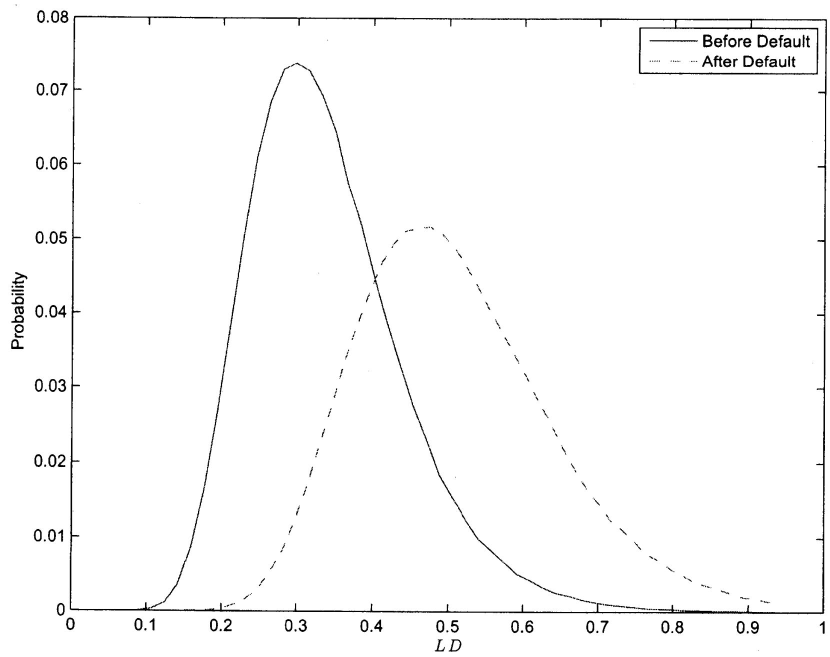

Figure 3: $g_i(x)$ and $g_i(x|T_1 = \tau_j)$ whith $\gamma_{ij} = 0.04$.

The time interval $[0,T]$ is partitioned into sub-intervals of equal length Δ and firm value processes evolve along the discretized time step $k\Delta$, $k=1,2,\cdots n$, where $n\Delta = T$. With the discretization of the time variable, we redefine the default time as

$$\tau_i = \inf\{k\Delta > 0 : V_i(k\Delta) \le L_i D_i\} \tag{20}$$

in analogy with its continuous time version (3).

Algorithm 3.1. *To generate a one sample path of the total default* $\{\sum_i H_t^i\}_{t\le T}$, *perform the following:*

Step 0. *Initialize* V_0, H_0. *Set* $\mathcal{S}_0 = \{1,2,\cdots,n\}$, $r_0 = n$, $\mathcal{D}_0 =$ *and* $k=1$. *Draw the random barrier* L_iD_i *for all firms in portfolio and fix them until the first default occurred.*

Step 1. *Generate the* $r_{(k-1)\Delta}$*-dimensional path* $V_{k\Delta} = \left(V_{k\Delta}^1,\cdots,V_{k\Delta}^{r_{(k-1)\Delta}}\right)$ *and calculate the running minimum* $\mathcal{M}_{k\Delta}^i \stackrel{def}{=} \min_{0\le s\le k\Delta} V_{k\Delta}^i$ *for each* $i \in \mathcal{S}_{(k-1)\Delta}$.

Step 2. *Determine whether default occurred or not at time* $k\Delta$ *and renew the set* $\mathcal{S}_{k\Delta}$ *as follows.*
If $\mathcal{M}_{k\Delta}^i \le L_iD_i$, *then the firm* i *gets default at time* $k\Delta$, *and then set* $H_{k\Delta}^i = 1$.
Else, set $H_{k\Delta}^i = 0$.
Let $\{i_1, i_2, \cdots, i_m\}$ *be a set consists of defaulted firms at time* $k\Delta$ *and go to Step 3.*
If $\mathcal{M}_{k\Delta}^i > L_iD_i$ *hold for all* $i \in \mathcal{S}_{k\Delta}$, *set* $H_{k\Delta}^i = 0$ *for all* i *and go to Step 1.*

Step 3. *Determine* $\{s_{i_1}, s_{i_2}, \cdots, s_{i_m}\}$ *for all defaulted firms and calculate the realized barrier for the defaulted firms and store* $\{\ell_{i_1}, \ell_{i_2}, \cdots, \ell_{i_m}\}$.

Step 4. *Renew the matrix* $\mathbf{\Gamma}_{k\Delta} = (\gamma_{ij,k\Delta})_{1\le i,j\le n}$ *and the set* $\tilde{\mathcal{D}}_{k\Delta}$. *Draw the random barrier* L_iD_i *for all survived firms* $i \in \mathcal{S}_{k\Delta}$ *and fix them until next default occurred. Sampling is based on the distribution truncated above* $\left(\mathcal{M}_{k\Delta}^1,\ \mathcal{M}_{k\Delta}^2,\ \cdots,\mathcal{M}_{k\Delta}^{r_{k\Delta}}\right)$.

Step 5. *Set* $k = k+1$ *and go to step1.*

4 Numerical examples

This section demonstrates the effects of the memory period through numerical examples with a sample portfolio consists of 25 firms. The basic set of the model parameter values are summarized as follows;

- $V_0^i = 1$, $\delta^i = 0.2$, $D_i = 0.95$, $\overline{L}_i = 0.7$ for all $i = 1,2,\cdots,25$,
- $\rho_{ij} = 0.7$ for $i \ne j$ and $\rho_{ii} = 1$, $\gamma_{ii} = 0.09$,
- $s_i = s$ for all i, $T = 5$,

which would be employed throughout this section. In addition, we investigate four cases such as

- Case 1: $\gamma_{ij} = 0.02$ for all $i \ne j$,
- Case 2: $\gamma_{ij} = 0.03$ for all $i \ne j$,
- Case 3: $\gamma_{ij} = 0.04$ for all $i \ne j$,
- Case 4: $\gamma_{ij} = 0.05$ for all $i \ne j$.

In the following numerical examples, we performed 500000 trials that will be expected to achieve reasonably accurate values. Next Figures 4, 5, 6, 7 show how the default distributions change in response to the memory period $s_i = s$ taking values in $s \in \{5, 1, 0.25, 0.01\}$. One sees that the first default occurs with the same probability for each γ_{ij} but the second default occurs with different probability in response to the memory period s. The larger the memory periods get, the more tail gets fat.

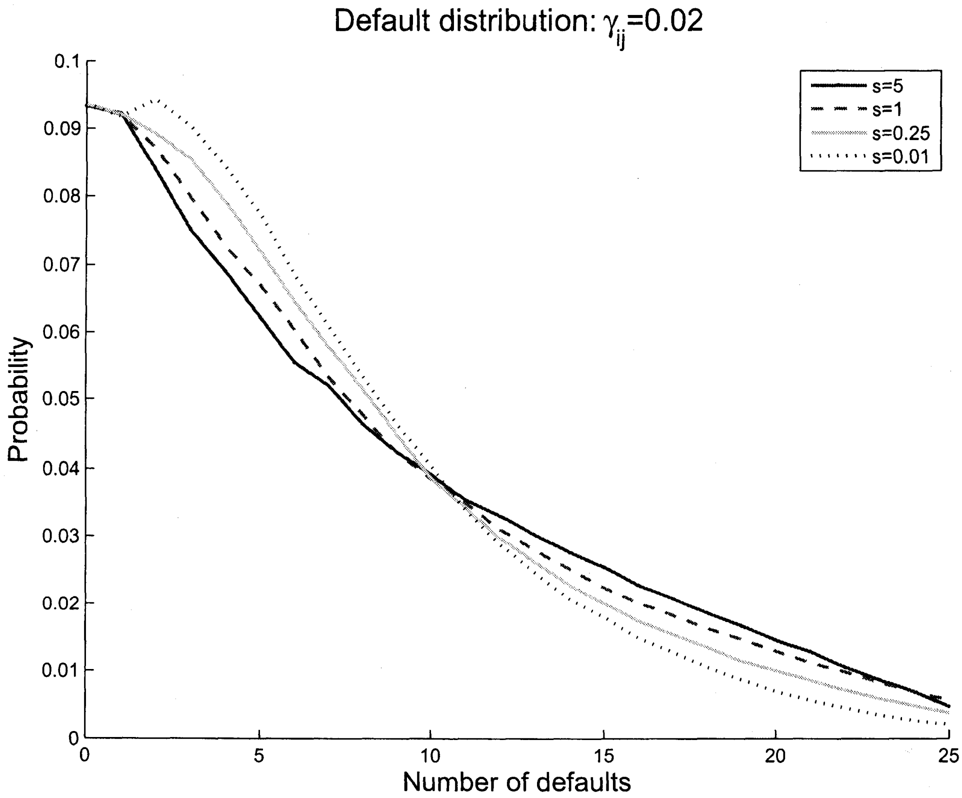

Figure 4: Case 1

5 Conclusion

This paper proposed incomplete information multi-name structural model. We extend naturally the CreditGrades model in the sense that we consider more than two firms in the portfolio and their asset correlation as well as the dependence structure of the default thresholds. Introduced the notion of the memory periods which control the loss distribution as if the default correlation changes.

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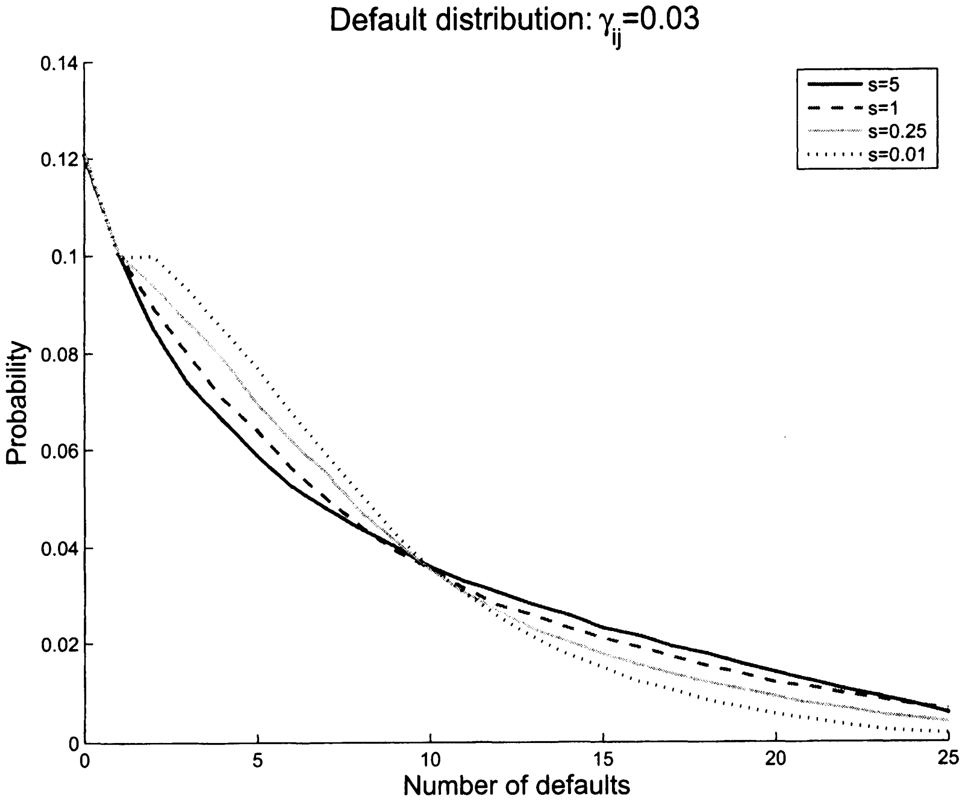

Figure 5: Case 2

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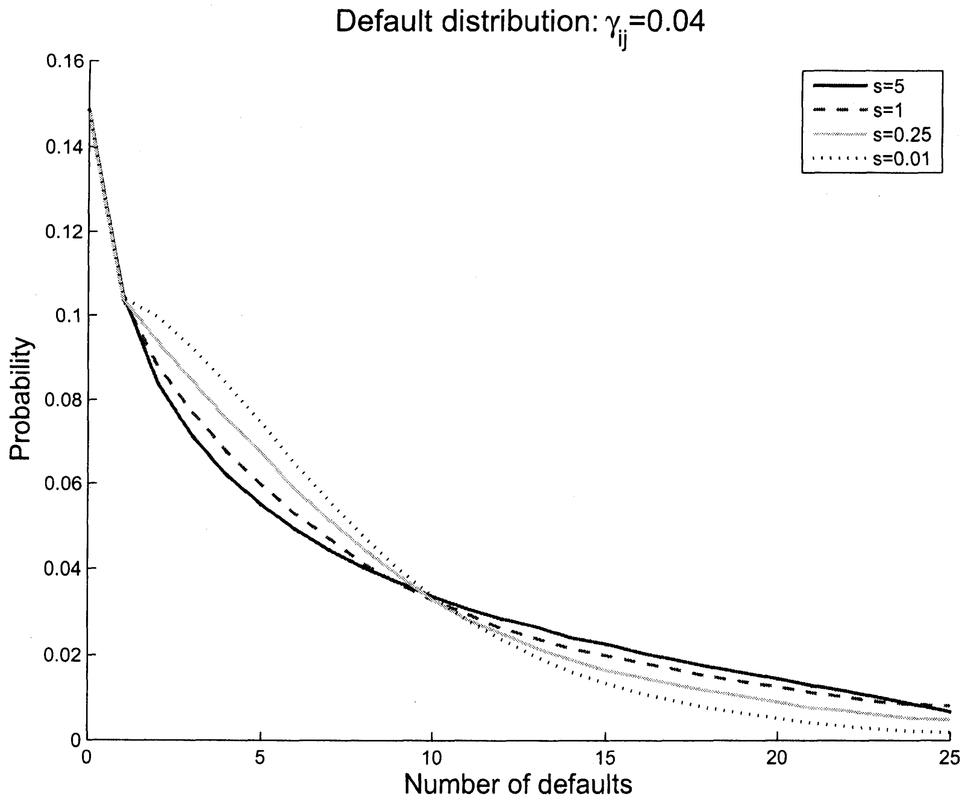

Figure 6: Case 3

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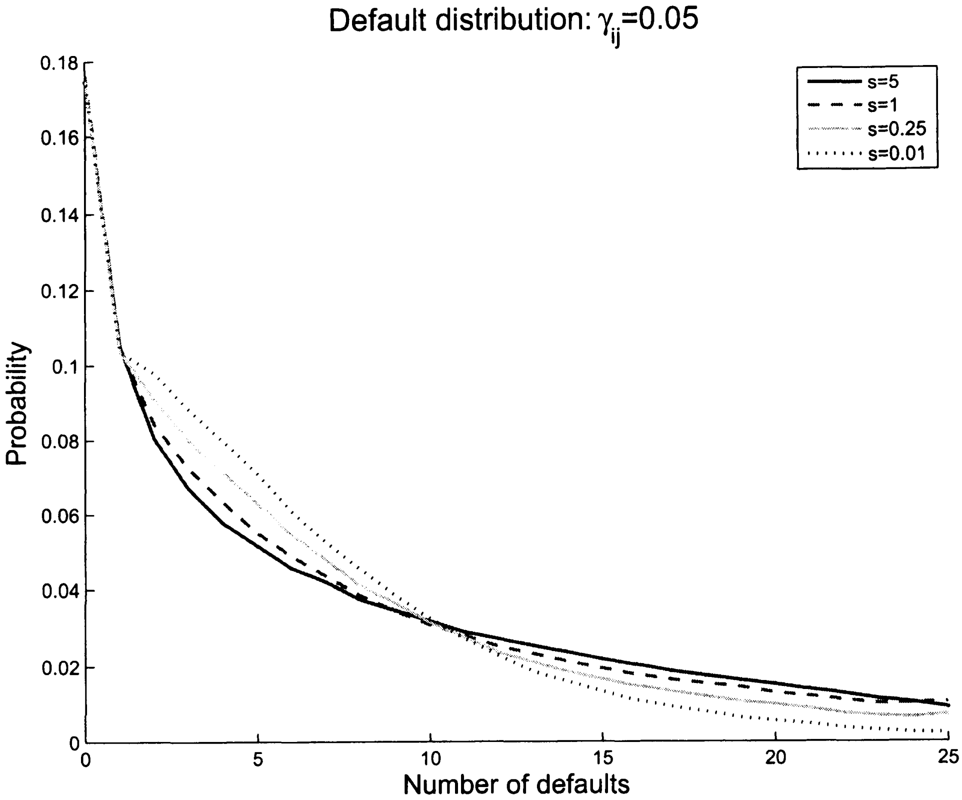

Figure 7: Case 4

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