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A brief survey on a fast Monte Carlo scheme for risk analyses using a probability measure transformation technique

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1 Introduction

Estimation of small probability is one of the most important theme in many application fields such as reliability engineering for structural systems or risk analysis. The well-known and most widely used Monte Carlo simulation method is crucial for such purposes because of its very slow convergence property. Transforming probability measure works quite well for reducing the variance inherent in the Monte Carlo simulation procedure, which enables us to estimate very small probability with good accuracy.

The author has been proposed a method for accelerating the Monte Carlo simulation by applying the probability measure transformation based upon the well-known Maruyama-Girsanov theorem[1] and its variations. In Refs.[2], [3] and [4], the method has been applied to stochastic systems driven by Wiener processes, in which a systematic procedure has been constructed for selecting the optimal probability measure under which an importance sampling simulation is executed by the use of a concept of *design point* playing a quite important role in the structural reliability engineering[5][6]. In Refs.[7] and [8], the method has been extended for treating a stochastic system driven by compound Poisson processes.

Recently, more generalized stochastic models have been applied for modeling various phenomena, especially, Lévy processes have been widely used for modeling dynamics of securities or wealth[9] For instance, a variance gamma process has been widely used for modeling wealth processes in credit risk analysis[10]. Thus, we need to refine the probability measure transformation technique so that it can be applied to stochastic systems driven by Lévy processes.

In this paper, we give a brief survey of an application of probability measure transformation technique to reduce the variance inherent in Monte Carlo simulations. Further, two examples of its application are shown.

2 Basic Formulation

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_t; 0 \leq t \leq T\}$ be a filtration on $(\Omega, \mathscr{F}, P)$. We consider a system such as

- An input noise disturbing the system behavior is described by a real-valued and temporally homogeneous Lévy process denoted by $\mathbb{Z}(\omega) = \{Z_t(\omega); 0 \leq t \leq T\}$.
- An output is described by a real-valued stochastic process denoted by $\mathbb{X}(\omega) = \{X_t(\omega); 0 \leq t \leq T\}$, which is supposed to be adapted to the filtration.

- The output $\mathbb{X}$ is related to the input noise $\mathbb{Z}$ as

$$\mathbb{X}(\omega) = \mathscr{H}[\mathbb{Z}(\omega)] \quad \text{(a.s.)}, \tag{2.1}$$

where $\mathscr{H}$ is a functional representing the system.

Next we introduce an indicator functional f for identifying a *risk event* which is our main subject of analysis, i.e.,

$$f[\mathbb{X}] = \begin{cases} 1 & \text{(risk event occurs in } [0,T]) \\ 0 & \text{(otherwise)} \end{cases} . \tag{2.2}$$

The main target of our analysis is to estimate its expectation, denoted by $\psi(T)$, as

$$\psi(T) = \int_\Omega f[\mathbb{X}(\omega)]P(d\omega) = \int_\Omega f\left[\mathscr{H}\left[\mathbb{Z}(\omega)\right]\right]P(d\omega) = \mathrm{E}_P\left\{f\left[\mathscr{H}\left[\mathbb{Z}\right]\right]\right\}, \tag{2.3}$$

where E_P denotes an operator to take expectation under the original probability measure P. If the risk event represents a failure of a system, $\psi(T)$ represents probability of system failure up to time T, which is a main target quantity in the reliability engineering. On the other hand, if X_t represents a wealth of a company at time t and the risk event represents an occurrence of ruin of the company, $\psi(T)$ represents probability of ruin (or frequently called probability of default) with in time T, which is a main subject in the field of collective risk theory[11].

In many fields of application including reliability analysis as well as risk analysis, it is frequently required to estimate $\psi(T)$ when it takes on a very small value. As the well-known Monte Carlo method does not work well for estimating such small probability, we have to execute simulation procedure based upon another probability measure.

Suppose that Q is such a probability measure defined on the same measurable space $(\Omega, \mathscr{F})$, which is equivalent to the original probability measure P. Using the measure Q, we can rewrite Eq.(2.3) as

$$\psi(T) = \int_\Omega f\left[\mathscr{H}\left[\mathbb{Z}(\omega)\right]\right]\frac{dP}{dQ}(\omega)Q(d\omega) = \mathrm{E}_Q\left\{f\left[\mathscr{H}\left[\mathbb{Z}\right]\right]\frac{dP}{dQ}\right\}, \tag{2.4}$$

where dP/dQ expresses the Radon-Nikodym derivative and E_Q denotes an operator to take expectation under Q. The Monte Carlo estimator under Q based upon Eq.(2.4) is then given as follows;

$$\hat{\psi}(T;N) = \frac{1}{N}\sum_{k=1}^{N} f\left[\mathscr{H}\left[\mathbb{Z}_Q^{(k)}\right]\right]\left(\frac{dP}{dQ}\right)_Q^{(k)}, \tag{2.5}$$

where $\mathbb{Z}_Q^{(k)}$ and $(dP/dQ)_Q^{(k)} (k = 1, \ldots, N)$ are independent samples of $\mathbb{Z}$ and dP/dQ respectively generated under Q.

If we select a suitable measure Q so that we can generate many samples which contribute to the estimation of $\psi(T)$, the variance of $\hat{\psi}(T;N)$ inherent in the Monte Carlo procedure can be effectively reduced, which is one of variance reduction techniques known as *importance sampling.* Thus, we call the measure Q *importance sampling measure* in what follows. To execute a Monte Carlo simulation based upon the importance sampling measure, we need to select Q under which we can easily generate independent samples of both $\mathbb{Z}$ and dP/dQ with giving an effective reduction of the variance.

3 Probability measure transformation based upon the Lévy-Itô decomposition

In this section, we give a basic framework of the probability measure transformation from P to Q available for Lévy processes by the use of the well-known Lévy-Itô decomposition[12][13] based upon Ref.[14].

3.1 The Lévy-Itô decomposition

The Lévy-Itô decomposition for temporally homogeneous Lévy processes are expressed as follows;

$$Z_t = \sigma_{\mathrm{B}} B_t + q_{\mathrm{B}} t + \int_{|u|>0} \left\{ u\mu_t^Z(du) - t \cdot h(u) m_Z(du) \right\}, \tag{3.1}$$

in which B_t is a Wiener process, σ_{B} and q_{B} are constants, $\mu_t^Z(A)$represents the number of discontinuous jumps of Z_t appearing in $[0, t]$ with jump variation belonging to a set A. The measure μ_t^Z is the so-called Poisson random measure such that $\mu_t^Z(A)$ obeys a Poisson distribution with mean

$$\mathrm{E}_P \left\{ \mu_t^Z(A) \right\} = m_Z(A), \tag{3.2}$$

which determines a measure called *Lévy measure.* The function $h(u)$ appearing in the integral in the third term is required so that the accumulation of small jumps does not diverge for some subclass of Lévy processes. For example, the following function is frequently used;

$$h(u) = \begin{cases} -1 & (u < -1) \\ u & (-1 \leq u \leq 1) \\ 1 & (1 < u) \end{cases}. \tag{3.3}$$

Provided that $h(u)$ is given, σ_{B}, q_{B} and the Lévy measure m_Z determine a Lévy process Z_t under the weak uniqueness. Thus, the triplet $(\sigma_{\mathrm{B}}, q_{\mathrm{B}}, m_Z)$ is called *characteristic quantities* of Lévy processes.

For example, if $q_{\mathrm{B}} = 0$, $\sigma_{\mathrm{B}} = 1$ and $m_Z \equiv 0$, Z_t gives a Wiener process B_t. On the other hand, if $q_{\mathrm{B}} = \sigma_{\mathrm{B}} = 0$ and the Lévy measure is given as, with a positive constant λ,

$$m_Z(A) = \begin{cases} \lambda & (1 \in A) \\ 0 & (\text{otherwise}) \end{cases}, \tag{3.4}$$

Z_t is reduced to a Poisson process with an intensity λ.

3.2 Probability measure transformation based upon the Lévy-Itô decomposition

Next, we give a probability measure transformation procedure[15] based upon the Lévy-Itô decomposition given by Eq.(3.1).

The target measure Q, which is a probability measure defined on a measurable space $(\Omega, \mathscr{F})$, is assumed to be equivalent with the original probability measure P. Then, the Radon-Nikodym derivative dQ/dP, as well as dP/dQ, exists, which generates a P-martingale M_t given as

$$M_t = \mathrm{E}_P\left\{ \frac{dQ}{dP}\bigg| \mathscr{F}_t \right\}. \tag{3.5}$$

Since $\mathrm{E}_P\{M_t\}$ clearly equals to unity, M_t can be expressed, by the use of a suitable P-martingale N_t, as

$$M_t = \mathscr{E}(N)_t, \tag{3.6}$$

where $\mathscr{E}(N)_t$ is the well-known Doléans-Dade exponential defined as follows;

$$\mathscr{E}(N)_t = \exp\left\{ N_t - \frac{1}{2}[N,N]_t^c \right\} \prod_{s\leq t} (1+\Delta N_s)\,\mathrm{e}^{-\Delta N_s}, \tag{3.7}$$

in which $[N,N]_t$ represents a quadratic variation of N_t, $[N,N]_t^c$ represents its continuous part and $\Delta N_s = N_s - N_{s-}$ represents a discontinuous jump of N_t at time s. In this study, we confine ourselves to the case in which the martingale N_t is a Lévy process with mean zero under P, whose Lévy -Itô decomposition is supposed to be given as

$$N_t = \eta\sigma B_t + \int_{|u|>0} \{g(u)-1\}\left\{\mu_t^Z(du) - t\cdot m_Z(du)\right\}, \tag{3.8}$$

where η is a constant, $g(u)$ is a certain deterministic, as well as integrable, function. Substituting Eq.(3.8) into Eq.(3.7), we can obtain

$$M_t = \mathrm{e}^{\hat{N}_t}, \tag{3.9}$$

$$\hat{N}_t = \eta\sigma B_t - \frac{1}{2}\eta^2\sigma^2 t + \int_{|u|>0} \left\{\log g(u)\mu_t^Z(du) - t(g(u)-1)m_Z(du)\right\}. \tag{3.10}$$

The probability measure Q is finally constructed by substituting Eq.(3.10) into Eq.(3.5). In what follows, we call $(\eta, g(u))$ *characteristics* of the probability measure transformation from P to Q.

If the probability measure can be fully determined by the information up to $t = T$, we can obtain, from Eq.(3.6) and Eq.(3.10), as

$$\frac{dQ}{dP} = \mathrm{e}^{\hat{N}_T}, \tag{3.11}$$

since dQ/dP is $\mathscr{F}_T$-measurable. Thus, we can give an analytical formula for the Radon-Nikodym derivative between two measures.

Since the Wiener process and the accumulation of discontinuous jumps characterized by the Lévy measure, appearing in the Lévy -Itô decomposition, are statistically independent, the above measure transformation is reduced to a combination of the following two independent measure transformation as

(a) The process B_t^Q defined as

$$B_t^Q = B_t - \eta\sigma_{\mathrm{B}} t \tag{3.12}$$

is a Wiener process under Q.

(b) The Lévy measure under Q, denoted by m_Z^Q, is given as follows;

$$m_Z^Q(A) = \int_A g(u)m_Z(du). \tag{3.13}$$

The transformation given by Eq.(3.12) is the well-known Maruyama-Girsanov transformation[1].

4 Optimal selection of Q

In this section, we briefly review a method[14] for selecting optimal measure Q so that the variance of Monte Carlo simulation under Q can be most effectively reduced.

To make discussion clear, we suppose that the risk event occurs when the system process arrives at a certain risk set A, i.e., the indicator functional f is given as

$$f[\mathbb{X}] = \begin{cases} 1 & (0 \le \exists t \le T \quad \text{s.t} \quad X_t \in A) \\ 0 & (\text{otherwise}) \end{cases} . \tag{4.1}$$

In Ref.[14], the author has clarified that the convergence is effectively accelerated when the mean behavior of $\mathbb{X}$ under Q arrives at the risk set A at time t_d, which is called *design time* in Ref.[14]. That is,

$$\tilde{\mathrm{E}}_Q\{X_{t_d}\} = x_c, \tag{4.2}$$

in which $\tilde{\mathrm{E}}_Q\{X_{t_d}\}$ represents an approximated mean value of X_{t_d} under Q and x_c represents a point on the boundary of A nearest to the initial state. If Q is selected so that Eq.(4.2) is satisfied, risk event occurs for about 50 % of generated samples under Q, which has been used for realizing the most effective reduction of simulation time in structural system reliability analysis[6].

Next, a variance of the estimator under Q is given as

$$\mathrm{Var}_Q\left\{\hat{\psi}(T;N)\right\} = \frac{1}{N}\left(\mathrm{E}_P\left\{f\left[\mathscr{X}[\mathbb{Z}]\right]^2 \frac{dP}{dQ}\right\} - \psi(T)^2\right). \tag{4.3}$$

It should be noted that, although we can perfectly reduce the variance given by Eq.(4.3) by selecting Q as

$$dQ = \frac{1}{\psi(T)} f\left[\mathscr{X}[\mathbb{Z}]\right] dP, \tag{4.4}$$

which is clearly an impossible selection for estimating $\psi(T)$[16]. That is, we can not reduce the variance itself in selecting the measure Q.

Equation(4.3) can be rewritten as the following inequality;

$$\mathrm{Var}_Q\left\{\hat{\psi}(T;N)\right\} \le \frac{1}{N}\mathrm{E}_P\left\{f\left[\mathscr{X}[\mathbb{Z}]\right]^2 \frac{dP}{dQ}\right\}. \tag{4.5}$$

Further applying the Schwarz inequality, we can obtain

$$\mathrm{Var}_Q\left\{\hat{\psi}(T;N)\right\} \le \frac{1}{N}\left(\mathrm{E}_P\left\{f\left[\mathscr{X}[\mathbb{Z}]\right]^4\right\}\right)^{1/2}\left(\mathrm{E}_P\left\{\left(\frac{dP}{dQ}\right)^2\right\}\right)^{1/2}, \tag{4.6}$$

which gives one of upper bounds of the variance. Since $\mathrm{E}_P\left\{f\left[\mathscr{X}[\mathbb{Z}]\right]^4\right\}$ does not depend on the measure Q, we can minimize the upper bound by minimizing $\mathrm{E}_P\{(dP/dQ)^2\}$.

Consequently, we can determine the optimal measure Q by solving the following conditional minimizing problem for the characteristics $(\eta, g(u))$;

$$\underset{(\eta, g(u))}{\textbf{minimize}} \quad \mathrm{E}_P\left\{\left(\frac{dP}{dQ}\right)^2\right\} \tag{4.7}$$

$$\textbf{subject to} \quad \tilde{\mathrm{E}}_Q\{X_{t_d}\} - x_c = 0 \tag{4.8}$$

5 Application to risk analysis for infrastructures

Fist, we apply the importance sampling simulation scheme constructed in this paper to a random damage growth model recently developed for tunnel concrete linings[17] as an important example of maintenance for infrastructures.

Let X_t be a quantified damage degree at time t for tunnel concrete linings, which is here supposed to obey the following stochastic differential equation;

$$dX_t = \mu X_t + X_{t-} dC_t, \quad X_0 = x_0 \text{ (a.s.)}, \tag{5.1}$$

where μ is a positive constant representing a damage growth resistance and $C = \{C_t; t \geq 0\}$ is a compound Poisson process drives random damage growth. A compound Poisson process C is a Lévy process whose Lévy measure, here denoted by m_C, is uniformly integrable, i.e.,

$$\int_{|u|>0} m_C(du) \equiv \lambda < +\infty, \tag{5.2}$$

which indicates that a measure ν_C defined as

$$\nu_C(A) = \frac{1}{\lambda} m_C(A), \tag{5.3}$$

is a probability measure. Therefore, according to the basic property of Lévy processes, C can be expressed as

$$C_t = \sum_{k=1}^{N_t^{\mathrm{HPP}}} Y_k, \tag{5.4}$$

in which N_t^{HPP} is a temporally homogeneous Poisson process with an intensity λ give by Eq.(5.2) and $\{Y_k\}$ is a set of i.i.d. random variables obeying the probability measure defined by Eq.(5.3). Since C drives the damage growth, $Y_k > 0$ (a.s.) for $\forall\ k$.

The risk event is here supposed to be a failure of tunnel concrete linings, which is here supposed to occur when the damage degree exceeds a certain critical level x_c ($> x_0$). Hence, the indicator functional is given by Eq.(4.1) with $A = \{X; X > x_c\}$.

As a Lévy measure of any compound Poisson process is uniformly integrable, C is again a compound Poisson process under Q provided that the function g satisfies

$$\int_{u>0} g(u) m_C(du) \equiv \frac{\lambda^Q}{\lambda} < +\infty, \tag{5.5}$$

where λ^Q gives an intensity of C under Q and jumps $\{Y_k\}$ obeys a probability measure defined by

$$\nu_C^Q(A) = \frac{1}{\lambda^Q} m_C^Q(A) = \frac{1}{\lambda^Q} \int_A g(u) m_C(du). \tag{5.6}$$

Applying the result obtained in Section 3, we can obatain the Radon-Nikodym derivative as

$$\frac{dP}{dQ} = \exp\left\{ (\lambda^Q - \lambda)T - \hat{C}_T \right\} \tag{5.7}$$

$$\hat{C}_t = \int_{|u|>0} \log g(u) \mu_t^C(du) = \sum_{k=1}^{N_t^{\mathrm{HPP}}} \log g(Y_k) \tag{5.8}$$

It should be noted that the characteristic η is not needed in this measure transformation.

Equation(4.2) is here approximated as

$$\tilde{\mathrm{E}}_Q\{X_{t_d}\} = x_0 \exp\left\{(a + \lambda^Q q_1^Q)t_d\right\} = x_c, \tag{5.9}$$

$$q_1^Q = \mathrm{E}_Q\{Y_1\} = \int_0^\infty u\nu_C^Q(du) = \int_0^\infty yg(y)\nu_C(dy), \tag{5.10}$$

where $x_0 = X_0$. Further, we can calculate Eq.(4.7) as

$$\mathrm{E}_P\left\{\left(\frac{dP}{dQ}\right)^2\right\} = \exp\left[-3\lambda T + \lambda T \int_0^\infty \left\{2g(y) + g(y)^{-2}\right\} \nu_C(dy)\right] \tag{5.11}$$

Here we assume that $\{Y_k\}$ obeys an exponential distribution with mean q_1 under P, i.e.,

$$\nu_C(A) = \int_A \frac{1}{q_1} \exp\left(-\frac{y}{q_1}\right) dy. \tag{5.12}$$

Then, if we assume that $\{Y_k\}$ also obeys an exponential distribution with mean q_1^Q, i.e.,

$$\nu_C(A) = \int_A \frac{1}{q_1^Q} \exp\left(-\frac{y}{q_1^Q}\right) dy, \tag{5.13}$$

the function g is obtained as

$$g(y) = \frac{\lambda^Q}{\lambda} \frac{q_1}{q_1^Q} \exp\left\{\left(\frac{1}{q_1} - \frac{1}{q_1^Q}\right) y\right\}. \tag{5.14}$$

Substituting Eqs.(5.9), (5.11) and (5.14) into Eqs.(4.7) and (4.8), we can reduce the optimization procedure as

$$\underset{q_1^Q > 2q_1/3}{\textbf{minimize}} \quad \frac{2w}{q_1^Q} - \frac{\left(q_1^Q\right)^5}{w^2 q_1^2 (3q_1^Q - 2q_1)} \tag{5.15}$$

$$w = \frac{1}{\lambda}\left(\frac{1}{T}\log\frac{x_c}{x_0} - a\right) \tag{5.16}$$

The optimal q_1^Q is numerically obtained from Eq.(5.16), which determines the optimal intensity λ^Q as

$$\lambda^Q = \frac{\lambda w}{q_1^Q} = \frac{1}{q_1^Q T}\left(\log\frac{x_x}{x_0} - a\right) \tag{5.17}$$

Figure 1 shows estimated $\psi(T)$ under parameters

$$x_0 = 2.0,\ x_c = 15.0,\ a = 5.0 \times 10^{-3},\ q_1 = 0.06,\ \lambda = 0.5,$$

for T = 10, 20 and 30, where vertical axis is logarithmically plotted. In Fig.1, solid crosses represent estimated $\psi(T)$ obtained by our proposed scheme with 100 samples and error bars show range of estimated values for ten times independent simulations. On the other hand, gray triangle represent estimated $\psi(T)$ obtained by crude Monte Carlo simulation, i.e., Monte Carlo simulation executed under the original measure P with 5000 samples.

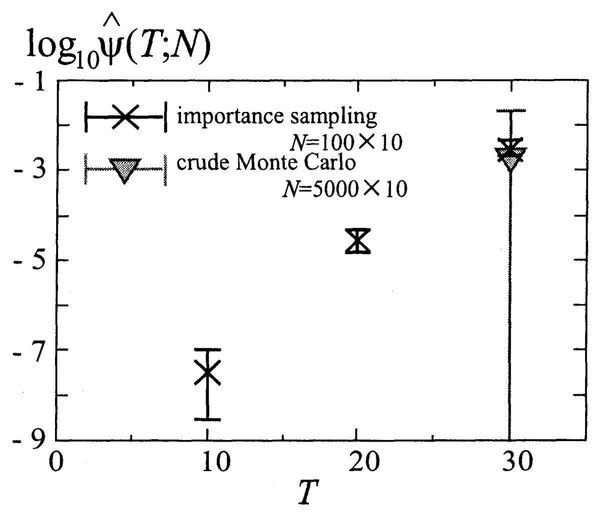

Fig.1 Estimated $\psi(T)$ by proposed scheme (solid crosses) and crude Monte Carlo scheme (gray triangle).

From the result, we can see that the proposed scheme works quite well for estimating very small $\psi(T)$ with only 100 samples for each simulation. However, the crude Monte Carlo simulation can not give estimation for $T \leq 20$ even though 5000 samples are generated in the simulation.

Table 1 shows optimally selected λ^Q and q_1^Q for each T. When T is small, the intensity λ^Q and mean q_1^Q are magnified so that we can strongly accelerate the growth of X. The magnification ratio gradually decreases as T increases.

	$T = 10$	$T = 20$	$T = 30$
λ	0.5		
q_1	0.06		
λ^Q	1.0102	0.7730	0.6618
q_1^Q	0.1945	0.1239	0.0939

Table 1 Comparison of parameters associated with the compound Poisson process C_t under the original measure P and the importance sampling measure Q.

6 Application to Credit Default Swap pricing

6.1 Firm asset dynamics

Next, we discuss an application of the proposed simulation scheme to pricing of Credit Default Swaps (CDS).

We suppose that dynamics of a firm asset, denoted by X_t, is given as a solution of the following stochastic differential equation[18];

$$dX_t = X_{t-} d\tilde{Z}_t, \tag{6.1}$$

where the driving noise $\tilde{Z}_t$ is given as

$$\tilde{Z}_t = Z_t + \frac{1}{2}[Z, Z]^c_t + \sum_{u\in(0,t]} \left\{ e^{\Delta Z_u} - 1 - \Delta Z_u \right\}, \tag{6.2}$$

where Z_t is supposed to be a temporally homogeneous Lévy process whose decomposition is given by Eq.(3.1). The explicit form of the solution X_t is given as

$$X_t = X_0 \exp(Z_t), \tag{6.3}$$

which gives an explicit expression of Eq.(2.1). It should be noted that the firm asset dynamics given by Eq.(6.3) is a natural extension of the well-known Black-Scholes model.

The risk event discussed here is a default of the firm. It is here assumed to occur when the firm asset X_t falls below a prespecified default boundary $x_d(< X_0)$, i.e., the indicator functional is given as

$$f[\mathbb{X}] = \begin{cases} 1 & (0 \leq^{\exists} t \leq T \text{ s.t. } X_t < x_d) \\ 0 & (\text{otherwise}) \end{cases} . \tag{6.4}$$

Then, the probability $\psi(T)$ represents probability of default up to time T. If we introduce a default time, denoted by τ_D, as

$$\tau_D = \inf\{t; X_t < x_d\}, \tag{6.5}$$

$\psi(T)$ can be expressed by the use of τ_D as

$$\psi(T) = P(\tau_D \leq T). \tag{6.6}$$

Thus, $\psi(T)$ as a function of T can be regarded as a probability distribution function of the default time τ_D.

6.2 Pricing of CDS

Let us consider a CDS with maturity T_0. A discounted income of the protection buyer, denoted by c_B, is given as

$$c_B = y e^{-r\tau_D} \mathbf{1}_{\{\tau_D \leq T_0\}}, \tag{6.7}$$

where y is the protection value, r is a risk-free interest rate and $\mathbf{1}_A$ is an indicator function of event A. On the other hand, a discounted income of the protection seller, denoted by c_S, is given as

$$c_S = \int_0^{T_0} q y e^{-rs} \mathbf{1}_{\{\tau_D \geq s\}} ds + R y e^{-r\tau_D} \mathbf{1}_{\{\tau_D \leq T_0\}}, \tag{6.8}$$

where q is a premium rate of the CDS and R is a recovery rate.

The CDS premium rate is determined under the condition that the expectation of c_B coincides with the expectation of c_S under the so-called equivalent martingale measure denoted by P^*, which can realize a kind of economical equilibrium. We denote $\psi(t)$ under P^* as $\psi^*(t)$, then the expectations of c_B and c_S under P^* are given as follows;

$$\mathrm{E}_{P^*}\{c_B\} = \int_0^{T_0} y e^{-rt} d\psi^*(t), \tag{6.9}$$

$$\mathrm{E}_{P^*}\{c_S\} = \int_0^{T_0} qy\mathrm{e}^{-rt}(1-\psi^*(t))dt + R\int_0^{T_0} y\mathrm{e}^{-rt}d\psi^*(t). \tag{6.10}$$

The equilibrium condition is given as

$$\mathrm{E}_{P^*}\{c_B\} = \mathrm{E}_{P^*}\{c_S\}. \tag{6.11}$$

Therefore we can express the CDS premium rate q as follows;

$$q = \frac{(1-R)\displaystyle\int_0^{T_0} \mathrm{e}^{-rt}d\psi^*(t)}{\displaystyle\int_0^{T_0} \mathrm{e}^{-rt}(1-\psi^*(t))dt}. \tag{6.12}$$

6.3 Application of the proposed simulation scheme

The proposed simulation scheme can be applied to estimate the fair price of CDS through estimation of the probability of default $\psi(T)$ as the author discussed in Ref.[19]. The estimation procedure consists of two steps. The first step is to transform the original probability measure P to the equivalent martingale measure P^*, which is executed based upon a probability measure transformation discussed in Section 3. To avoid the so-called incompleteness, the minimal entropy principle[15] is applied. The second step is to transform the equivalent martingale measure P^* to the importance sampling measure Q, which is executed just as discussed in Section 4.

6.4 Numerical examples

Here, we give numerical examples in which the Lévy process Z is a variance gamma process.

The variance gamma process is a Lévy process in which $\sigma_B = q_B = 0$ and its Lévy measure is given as

$$m_Z(A) = \int_A \frac{p}{|u|}\exp\left(-\sqrt{\frac{2}{\sigma}}|u|\right)du, \tag{6.13}$$

where p and σ are positive parameters characterizing the VG process. The VG process is frequently used for modeling random variation of stock price or firm asset. It should also be mentioned that more general VG process has been studied including three parameters[10].

We calculate the CDS premium rate by approximating integrals in Eq.(6.12) by the use of the trapezoidal rule by a time mesh $T_i = i\Delta t,\ \Delta t = \frac{T_0}{M}$, i.e., its estimator with sample size N_M, denoted by $\hat{q}(N_M)$ is given as follows;

$$\begin{aligned} \hat{q}(N_M) &= \frac{(1-R)\displaystyle\sum_{i=1}^{M}\frac{\mathrm{e}^{-rT_i}+\mathrm{e}^{-rT_{i-1}}}{2}(\hat{\psi}^*(T_i;N_M)-\hat{\psi}^*(T_{i-1};N_M))}{\displaystyle\sum_{i=1}^{M}\frac{\mathrm{e}^{-rT_i}(1-\hat{\psi}^*(T_i;N_M))+\mathrm{e}^{-rT_{i-1}}(1-\hat{\psi}^*(T_{i-1};N_M))}{2}(T_i-T_{i-1})} \\ &= (1-R)\frac{M}{T_0}\frac{\displaystyle\sum_{i=1}^{M}(\mathrm{e}^{-rT_i}+\mathrm{e}^{-rT_{i-1}})(\hat{\psi}^*(T_i;N_M)-\hat{\psi}^*(T_{i-1};N_M))}{\displaystyle\sum_{i=1}^{M}\{\mathrm{e}^{-rT_i}(1-\hat{\psi}^*(T_i;N_M))+\mathrm{e}^{-rT_{i-1}}(1-\hat{\psi}^*(T_{i-1};N_M))\}}, \end{aligned} \tag{6.14}$$

where $T_i = i \cdot \frac{T_0}{M}$ $(i = 1, 2, \cdots, M)$.

Table 2 shows the estimated CDS premium rate with parameters

$$T_0 = 0.1, b = 0.099, p = 5.0, \sigma = 0.05, R = 0. \tag{6.15}$$

estimated q	crude MC ($N_M = 10^5 \times 10$)	proposed MC ($N_M = 10^4 \times 10$)	crude MC ($N_M = 10^8$)
maximum value	0.00169715	0.00127754	
average value	0.00111931	0.00113313	0.00117775
minimum value	0.00080056	0.00097704	

Table 2 CDS premium rate estimated by the proposed method compared with the crude Monte Carlo method[19]. (Parameters are set as Eq.(6.15))

Table 2 shows that the proposed simulation method with 10^4 samples can give quite good estimations of small CDS premium rate compared to the crude Monte Carlo method with 10^5 samples. Therefore, we can expect that the proposed method is effective in the case when the value of the credit derivative in interest is derived from the credit risk of many firms, since the probability of the conjunction of many defaults is regularly quite small, even if these defaults are considered to be correlated with each other.

Next, supposing a long term case compared to Table 2, we show estimated CDS premium rate in Table 3, where parameters are set as

$$T_0 = 1.0, b = 0.099, p = 3.0, \sigma = 0.1, R = 0, \tag{6.16}$$

in the same way as Table 2. Even though the accuracy of the default probability estimated by the proposed method is quite good for small T_i, the accuracy of the estimated premium rate is almost same as the crude Monte Carlo method. It is due to that the integral in Eq.(6.14) mainly depends on the large default probability in the supposed time interval.

estimated q	crude MC ($N_M = 10^4 \times 10$)	proposed MC ($N_M = 10^4 \times 10$)	crude MC ($N_M = 10^8$)
maximum value	0.00651128	0.00653323	
average value	0.00552585	0.00571317	0.00568284
minimum value	0.00494841	0.00495594	

Table 3 CDS premium rate estimated by the proposed importance sampling method compared with the crude Monte Carlo method. (Parameters are set as Eq.(6.16))

7 Conclusion

In this paper, we have briefly discussed a variance reduction technique realizing importance sampling in the Monte Carlo simulation based upon a probability measure transformation available for stochastic systems driven by Lévy processes. Two practical application have been shown for demonstrating the proposed simulation scheme which works quite well for estimating very small probability of risk event.

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