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Equivariant definable homotopy extensions

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Abstract

Let G be a definably compact definable group, X a definable G set and Y a definable closed G subset of X. We prove that a pair (X, Y) admits an equivariant definable homotopy extension.

1 Introduction

In this paper we consider equivaraint definable homotopy extensions in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field R. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([8]).

Definable set and definable maps are studied in [3], [4], and see also [9]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, ...)$ and definable maps are assumed to be continuous unless otherwise stated.

2 Preliminaries

Let R be a real closed field.

A *structure* $\mathcal{N}$ is given by the following data.

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1. A set R is called the *universe* or *underlying set* of $\mathcal{N}$.

2. A collection of *functions* $\{f_i | i \in I\}$, where $f_i : R^{n_i} \to R$ for some $n_i \geq 1$.

3. A collection of *relations* $\{R_j | j \in J\}$, where $R_j \subset R^{m_j}$ for some $m_j \geq 1$.

4. A collection of distinguished elements $\{c_k | k \in K\} \subset R$, and each c_k is called a *constant.*

Any (or all) of the sets I, J, K may be empty.

We say that f (resp. L) is m-place function (resp. m-place relation) if $f : R^m \to R$ (resp. $L \subset R^m$).

A *term* is a finite string of symbols obtained by repeated applications of the following three rules:

1. Constants are terms.

2. Variables are terms.

3. If f is an m-place function of $\mathcal{N}$ and $t_1, \ldots, t_m$ are terms, then the concatenated string $f(t_1, \ldots, t_m)$ is a term.

A *formula* is a finite string of symbols $s_1 \ldots s_k$, where each s_i is either a variable, a function, a relation, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (,), or comma ,. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms t_1 and t_2, $t_1 = t_2$ and $t_1 < t_2$ are formulas.

2. If R is an m-place relation and $t_1, \ldots, t_m$ are terms, then $R(t_1, \ldots, t_m)$ is a formula.

3. If ϕ and ψ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If ϕ is a formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset X of R^n is *definable* (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in R$ such that $X = \{(a_1, \ldots, a_n) \in R^n | \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \text{ is true in } \mathcal{N}\}$.

For any $-\infty \leqq a < b \leqq \infty$, an open interval $(a,b)_R$ means $\{x \in R | a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a,b]_R$ means $\{x \in R | a \leqq x \leqq b\}$. We call $\mathcal{N}$ *o-minimal* (*order-minimal*) if every definable subset of R is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [10], and a definable map is a semialgebraic map [10]. In particular, the semialgebraic category is a special case of the definable one.

The topology of R is the interval topology and the topology of R^n is the product topology. Note that R^n is a Hausdorff space.

The field $\mathbb{R}$ of real nubmers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} | x$ is algeraic over $\mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

Fact 2.1. *(1) The characteristic of a real closed field is* 0.

(2) For any cardinality $\kappa \geqq \aleph_0$, *there exist* 2^κ *many non-isomorphic real closed fields whose cardinality are* κ.

(3) In a general real closed field, even for a C^∞ *function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a* C^∞*function* f *in one varianble, the result that* $f' > 0$ *implies* f *is increasing does not hold.*

Definition 2.2. Let $X \subset R^n$, $Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \to Y$ is a *definable map* if the graph of f ($\subset R^n \times R^m$) is definable.

(2) A definable map$f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definition 2.3. A group G is a *definable group* if G is definable and the group operations $G \times G \to G, G \to G$ are definable.

As in the field $\mathbb{R}$, for any real closed field R, we can define the n-th general linear $G(n,R)$, the n-th orthogonal group $O(n)$.

Let G, G' be definable groups. A group homomorphism $f : G \to G'$ is a *definable group homomorphism* if f is definable. A definable group homomorphism $f : G \to GL(n,R)$ is called a *definable G representation*. A definable group homomorphism $f : G \to O(n)$ is called a *definable orthogonal*

G representation and R^n with the orthogonal action induced from an orthogonal G representation is called a *definable orthogonal G representation space.*

Definition 2.4. (1) A G invariant definable subset of a definable orthogonal G representation space is a *definable G set.*

Let X, Y be definable G sets.

(2) A definable map $f : X \to Y$ is a *definable G map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$.

(3) A definable G map $f : X \to Y$ is a *definable G homeomorphism* if there exists a definable G map $h : Y \to X$ such that $f \circ h = id_Y$, $h \circ f = id_X$.

Definition 2.5. (1) A definable set $X \subset R^n$ is *definably compact* if for any definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X.

(2) A definable set $X \subset R^n$ is *definably connected* if there exist no definable open subsets U, V of X such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leqq x \leqq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

Theorem 2.6 ([7]). *For a definable set $X \subset R^n$, X is definably compact if and only if X is closed and bounded.*

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

Proposition 2.7. *Let $X \subset R^n$, $Y \subset R^m$ be definable set, $f : X \to Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).*

Theorem 2.8. *(1) (The intermediate value theorem) For a definable function f on a definably connected set X, if $a, b \in X$, $f(a) \neq f(b)$ then f takes all values between $f(a)$ and $f(b)$.*

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact set attains maximum and minimum.

(3) (Rolle's theorem) Let $f : [a,b]_R \to R$ *be a definable function such that* f *is differentiable on* $(a,b)_R$ *and* $f(a) = f(b)$*. Then there exists* c *between* a *and* c *with* $f'(c) = 0$.

(4) (The mean value theorem) Let $f : [a,b]_R \to R$ *be a definable function which is differentiable on* $(a,b)_R$*. Then there exists* c *between* a *and* c *with* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(5) Let $f : (a,b)_R \to R$ *be a differentiable definable function. If* $f' > 0$ *on* $(a,b)_R$*, then* f *is increasing.*

Example 2.9. *(1) Let* $\mathcal{N}$ *be* $(\mathbb{R}_{alg}, +, \cdot, <)$*. Then* $f : \mathbb{R}_{alg} \to \mathbb{R}_{alg}, f(x) = 2^x$ *is not defined([11])*。

(2) Let $\mathcal{N}$ *be* $(\mathbb{R}, +, \cdot, <)$*. Then* $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x$ *is defined but not definable, and* $h : \mathbb{R} \to \mathbb{R}, h(x) = \sin x$ *is defined but not definable.*

3 Equivariant definable homotopy extensions

Let X, Y be definable set and $f : X \to Y$ a definable map. We say that f is *definably proper* if for any definably compact subset C of Y, $f^{-1}(C)$ is a definably compact subset of X.

Let $A \subset R^n$, $S \subset R^m$ be definable sets, and let $f : S \to A$ be a definable map. We say that f is *definably trivial* if there exist a definable set $F \subset R^N$ for some $N \in \mathbb{N}$, and a definable map $h : S \to F$ such that $(f, h) : S \to A \times F$ is a definable homeomorphism. In this case, each fiber $f^{-1}(a)$ of f over a is definably homeomorphic to F.

In o-minimal expansions of real closed fields, the following five theorems are known.

Theorem 3.1. *(1) (Monotonicity theorem (e.g. 3.1.2, 3.1.6. [3])). Let* $f : (a,b)_R \to R$ *be a function with the definable graph. Then there exist finitely many points* $a = a_0 < a_1 < \cdots < a_k = b$ *such that on each subinterval* $(a_j, a_{j+1})_R$*, the function is either constant, or strictly monotone and continuous. Moreover for any* $c \in (a,b)_R$*, the limits* $\lim_{x \to c+0} f(x)$, $\lim_{x \to c-0} f(x)$ *exist in* $R \cup \{\infty\} \cup \{-\infty\}$.

(2) (Cell decomposition theorem (e.g. 3.2.11. [3])). For any definable subsets $A_1, \ldots, A_k$ *of* R^n*, there exists a cell decomposition of* R^n *partitioning each* $A_1, \ldots A_k$.

Let A be a definable subset of R^n and $f : A \to R$ a function with the definable graph. Then there exists a cell decomposition $\mathcal{D}$ of R^n partitioning A such that each $B \subset A, B \in \mathcal{D}$, $f|B : B \to R$ is continuous.

(3) (Triangulation theorem (e.g. 8.2.9. [3])). Let $S \subset R^n$ be a definable set and let $S_1, S_2, \dots, S_k$ be definable subsets of S. Then S has a triangulation in R^n compatible with $S_1, \dots, S_k$.

(4) (Piecewise trivialization theorem (e.g. 8.2.9. [3])). Let $f : S \to A$ be a definable map between definable sets S and A. Then there is a finite partition $A_1, \dots, A_k$ of A into definable sets A_i such that each $f|f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ is definably trivial.

(5) (Existence of definable quotients (e.g. 10.2.18 [3])). Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.

Question 3.2. *Let X, Y be definable sets and A a definable subset of X.*

(1) (Extensions of definable maps) Let $f : A \to Y$ be a definable map. When does $f : A \to Y$ extend a definable map $F : X \to Y$?

(2) (Definable homotopy extensions) Let $f : X \to Y$ be a definable map and a definable homotopy $F : A \times [0,1]_R \to Y$ such that $F(x,0) = f(x)$ for any $x \in A$. When does a definable homotopy $H : X \times [0,1]_R \to Y$ exist such that $H(x,0) = f(x)$ for any $x \in X$ and $H|A \times [0,1]_R = F$?

Theorem 3.3 (Definable Tietze extension theorem [1]). *Let X, Y be definable sets, A a definable closed subset of X and $f : A \to R$ a definable function. Then there exists a definable function $F : X \to R$ such that $F|A = f$.*

Theorem 3.4 (Definable homotopy extension theorem [2]). *Let X, Y be definable sets and A a definable closed subset of X. For any definable map $f : X \to Y$ and for any definable homotopy $F : A \times [0,1]_R \to Y$ such that $F(x,0) = f(x)$ for any $x \in A$, there exists a definable homotopy $H : X \times [0,1]_R \to Y$ such that $H(x,0) = f(x)$ for any $x \in X$ and $H|A \times [0,1]_R = F$.*

To consider Question 3.2, we need to construct an obstruction theory in the definable category.

The following question is an equivariant version of Question 3.2.

Question 3.5. *Let G be a definable group, X, Y a definable G sets and A a definable G subset of X.*

(1) (Extensions of definable G maps) Let $f : A \to Y$ be a definable G map. When does $f : A \to Y$ extend a definable G map $F : X \to Y$?

(2) (Equivariant definable homotopy extensions) Let $f : X \to Y$ be a definable G map and an equivaraint definable homotopy $F : A \times [0,1]_R \to Y$ such that $F(x,0) = f(x)$ for any $x \in A$. When does an equivaraint definable homotopy $H : X \times [0,1]_R \to Y$ exist such that $H(x,0) = f(x)$ for any $x \in X$ and $H|A \times [0,1]_R = F$?

We have the following result.

Theorem 3.6 ([6]). *Let G be a definably compact definable group, X a definable G set and A a definable closed G subset of X. For any definable G map $f : X \to Y$ and for any equivaraint definable homotopy $F : A \times [0,1]_R \to Y$ such that $F(x,0) = f(x)$ for any $x \in A$, there exists an equivariant definable homotopy $H : X \times [0,1]_R \to Y$ such that $H(x,0) = f(x)$ for any $x \in X$ and $H|A \times [0,1]_R = F$.*

Theorem 3.6 is proved in the case where $R = \mathbb{R}$ ([5]).
To prove Theorem 3.6, we need the following results.

Theorem 3.7 ([6]). *Let G be a definably compact definable group and Y a definable closed G subset of a definable G set X. Then there exists a G invariant definable open neighborhood U of Y in X such that Y is a definable strong G deformation retract of both U and of the closure cl U of U in X.*

Proposition 3.8 ([6]). *Let G be a definably compact definable group and A, B disjoint definable closed G subsets of a definable G set X. Then there exists a G invariant definable map $f : X \to [0,1]_R$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.*

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