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Dividing and Forking – A Proof of the Equivalence –

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1 Introduction and Preliminaries

Let T be a simple complete theory. Then the two notions forking and dividing are equivalent. (See [1].) The usual proof of this fact needs Erdös-Rado theorem, a basic result in combinatorial set theory. Erdös-Rado theorem is a theorem on uncountable cardinals, extending Ramsey's theorem. So it is somewhat strange to use such a theorem for proving the fact when the language is countable. In this article, we present a new proof that will only use compactness and a Ramsey-type argument.

We explain the notations in this article and recall some basic definitions. L is a language and T is a complete theory formulated in L. Although the countability of L is not necessary, we assume that L is countable for simplicity. We fix a big saturated model $\mathcal{M}$ of T and we work in $\mathcal{M}$. Small subsets of $\mathcal{M}$ are denoted by $A, B, \ldots$. Finite tuples in $\mathcal{M}$ are denoted by $a, b, \ldots$. Variables are $x, y, \ldots$. Formulas are denoted by φ, $\psi, \ldots$. If all the free variables of φ are contained in x, φ is sometimes written as $\varphi(x)$. For a set A, $L(A)$ is the language L augmented by the names (constants) for $a \in A$. For simplicity of the notation, we write $\varphi \in L(A)$, if φ is a formula in $L(A)$. In general, a formula $\varphi \in L(A)$ has the form $\psi(x, a)$, where $\psi(x, y)$ is an L-formula with xy free and a is the list of parameters (from A) appearing in φ. a will be called the parameters of φ.

A sequence $\{a_i : i \in \omega\}$ is called an indiscernible sequence over A, if for any strictly increasing $f : \omega \to \omega$, there is an automorphism σ of $\mathcal{M}$ that extends the mapping $id_A \cup \{\langle a_i, a_{f(i)}\rangle\}_{i\in\omega}$. We say that $\{a_i : i \in \omega\}$ starts with a, if $a_0 = a$.

Definition 1. A formula $\varphi(x, a)$ divides over A if there is an indiscernible sequence $\{a_i : i \in \omega\}$ starting with a such that $\{\varphi(x, a_i) : i \in \omega\}$ is inconsistent.

A set Φ of formulas is said to be k-inconsistent, if every subset $\Psi_0 \subset \Phi$ of size k is inconsistent. If $\varphi(x, a)$ divides over A, by the indiscernibility of $\{a_i : i \in \omega\}$, there is some $k \in \omega$ such that $\{\varphi(x, a_i) : i \in \omega\}$ is k-inconsistent. In this case we say that $\varphi(x, a)$ k-divides over A.

Definition 2. A formula $\varphi(x, a)$ forks over A if it is covered by a finite number of dividing formulas, more precisely, if there is a finite number of formulas $\psi_i(x, b_i)$ $(i = 1, \ldots, n)$ with the following properties:

1. $\mathcal{M} \models \forall x[\varphi(x,a) \to \bigvee_{i=1,\dots,n} \psi_i(x,b_i)]$;

2. Each $\psi_i(x,b_i)$ divides over A.

T is called simple if there is a bound for the length of a dividing sequence of complete types. The simplicity of T is equivalent to the finiteness of the rank defined below:

Definition 3. Let $\Sigma(x)$ be a set of formulas with parameters (with x free). Let $\Phi(x,y)$ be a finite set of L-formulas and let $k \in \omega$. The rank $D(\Sigma(x), \Phi(x,y), k)$ is defined by:

1. $D(\Sigma(x), \Phi(x,y), k) \geq 0$ if $\Sigma(x)$ is consistent;

2. $D(\Sigma(x), \Phi(x,y), k) \geq \alpha + 1$ if there is a and $\varphi \in \Phi$ such that $D(\Sigma(x) \cup \{\varphi(x,a)\}, \Phi(x,y), k) \geq \alpha$ and such that $\varphi(x,a)$ k-divides over the parameter set of Σ;

3. $D(\Sigma(x), \Phi(x,y), k) \geq \delta$ (a limit ordinal) if $D(\Sigma(x), \Phi(x,y), k) \geq \alpha$ for any $\alpha < \delta$.

In the same manner,

2 Simple theories

In what follows, T is a simple complete theory. Let us begin with the following lemma. A proof here is essentially the same as the one presented in Ziegler's book [3].

Lemma 4. *Let $\varphi(x) \in L(A)$. Then $\varphi(x)$ does not fork over A.*

Proof. For simplicity we assume $A = \emptyset$. Suppose otherwise and choose $\psi_i(x,b)$ ($i = 1,\dots,n$) and $k \in \omega$ such that

1. each $\psi_i(x,b)$ k-divides over $\emptyset$;

2. $\forall x(\varphi(x) \to \bigvee_{i=1,\dots,n} \psi_i(x,b))$ holds.

Then we choose $n_1,\dots,n_m \leq n$ and $b_1,\dots,b_m$ (copies of b) such that

3. $\psi_{n_i}(x,b_i)$ k-divides over $\{b_j : j < i\}$, for each $i = 1,\dots,n$;

4. $\varphi(x) \wedge \bigwedge_{i=1,\dots,m} \psi_{n_i}(x,b_i)$ is consistent, and its $D(*, \{\psi_i : i = 1,\dots,n\}, k)$-rank is minimum among such.

By moving the b_i's, we can assume that each $\psi_{n_i}(x,b_i)$ k-divides over $\{b\} \cup \{b_j : j < i\}$. By conditions 2 and 4, there is $n_{m+1} \leq n$ such that

$$\varphi(x) \wedge \bigwedge_{i=1,\dots,m} \psi_{n_i}(x,b_i) \wedge \psi_{n_{m+1}}(x,b) \text{ is consistent.}$$

Since $\psi_{n_{m+1}}(x,b)$ divides, by letting $b_{m+1} = b$, we have

$$D\big(\varphi(x) \wedge \bigwedge_{i=1,\dots,m} \psi_{n_i}(x,b_i), \Psi, k\big) > D\big(\varphi(x) \wedge \bigwedge_{i=1,\dots,m+1} \psi_{n_i}(x,b_i), \Psi, k\big)$$

where $\Psi = \{\psi_i : i = 1,\dots,n\}$. This contradicts our choice of n_i ($i \leq m$) and b_i ($i \leq m$) (condition 4). □

Remark 5. 1. Let $A \subset B$ and $p(x) \in S(A)$. Then there is an extension $q(x) \in S(B)$ of $p(x)$ such that $q(x)$ does not divide over A. This can be shown as follows: Let $\Gamma(x) = p(x) \cup \{\neg\varphi(x) \in L(B) : \varphi(x)$ does not divide over $A\}$. Then $\Gamma(x)$ is consistent, since otherwise we would have $p(x) \vdash \varphi_1(x) \vee \cdots \vee \varphi_n(x)$, for some φ_i dividing over A. So $p(x) \in S(A)$ forks over A, contradicting the above lemma. Choose $a \models \Gamma$, and let $q(x) = \text{tp}(a/B)$. Then, clearly $q(x)$ does not divide over A.

2. Suppose that $\text{tp}(a/Abc)$ does not divide over A and that $\text{tp}(b/Ac)$ does not divide over A. Then $\text{tp}(ab/Ac)$ does not divide over A: Let $\varphi(x, y, c) \in \text{tp}(ab/Ac)$. Let $I = \{c_i : i \in \omega\}$ be an arbitrary indiscernible sequence with $c_0 = c$. Since $\text{tp}(b/Ac)$ does not divide over A, there is b' (a copy of b over Ac) such that I is Ab'-indiscernible. For an A-automorphism $\sigma : b' \mapsto b$, $\sigma(I)$ is an Ab-indiscernible sequence. Notice then that $J = \{b\sigma(c_i) : i \in \omega\}$ is an A-indiscernible sequence with $b\sigma(c_0) = bc$. Since $\text{tp}(a/Abc)$ does not divide over A, there is a' (a copy of a over A) such that $a' \models \bigwedge_{d\in J} \varphi(x, d)$. So $\sigma^{-1}(a') \models \bigwedge_{i\in\omega} \varphi(x, b', c_i)$. In particular, $\{\varphi(x, y, c_i) : i \in \omega\}$ is satisfiable.

Lemma 6. *For each non-algebraic type $p(x) \in S(A)$, there is an A-indiscernible sequence $J = \{b_i : i \in \omega\}$ in p such that $\text{tp}(J \setminus \{b_0\}/Ab_0)$ does not divide over A.*

Proof. First we inductively choose a_i's realizing p such that, for each $i \in \omega$,

$$\text{tp}(a_i/A_i) \text{ does not divide over } A,$$

where $A_i = A \cup \{a_j\}_{j<i}$. Then, by an iterative use of Remark above, $\text{tp}(\{a_j\}_{j>0}/Aa_0)$ does not divide over A. Similarly we can show that $\text{tp}(\{a_j\}_{j>i}/Aa_i)$ does not divide over A, for each i.

Now let $\Gamma(\{x_i : i \in \omega\})$ be the following set of $L(A)$-formulas:

$$\bigcup_{i\in\omega} p(x_i) \cup \bigcup_{i\in\omega,\ F\subset\omega\setminus i} \{\neg\varphi(x_F, x_i) : \varphi(x_F, a_0) \text{ divides over } A\},$$

where $x_F = x_{i_0}, ..., x_{i_k}$ if $F = \{i_0 < \cdots < i_k\}$. Clearly Γ is realized by $I = \{a_i : i \in \omega\}$. Moreover, since each a_i realizes p, any infinite subsequence of I realizes Γ. In other words, Γ has the subsequence property. So there is an A-indiscernible sequence $J = \{b_i : i \in \omega\}$ realizing Γ. It is clear that $\text{tp}(J \setminus \{b_0\}/Ab_0)$ does not divide over A. □

Lemma 7. *Suppose that $\varphi(x, a)$ divides over A. Let $p(x) = \text{tp}(a/A)$ and choose an A-indiscernible sequence $J = \{b_i : i \in \omega\}$ in p having the property described in Lemma 6. Then $\{\varphi(x, b_i) : i \in \omega\}$ is inconsistent.*

Proof. Choose k such that $\varphi(x, a)$ k-divides over A, and choose an A-indiscernible sequence $I = \{a_i : i \in \omega\}$ such that $\{\varphi(x, a_i) : i \in \omega\}$ is k-inconsistent. By moving J by an A-automorphism, we may assume that $b_0 = a_0$. Since $\text{tp}(J \setminus \{b_0\}/b_0)$ does not divide, there is $\{b_i' : i > 0\}$ (a copy of $J \setminus \{b_0\}$ over Ab_0) such that

$$a_i\{b_i' : i > 0\} \equiv_A a_j\{b_i' : i > 0\} \equiv_A J$$

holds for any i, j. Moreover, by Ramsey's theorem, we can assume that I is indiscernible over $A\{b_i' : i > 0\}$.

Claim A. *$\Phi(x) = \{\varphi(x, b_i) : i \in \omega\}$ is k-inconsistent.*

Suppose otherwise and let $\alpha = D(\Phi(x), \varphi(x, y), k)$. Since J is an indiscernible sequence, we have $J \equiv_A J \setminus \{b_0\} \equiv_A \{b'_i : i > 0\}$. So we have

$$\alpha = D(\{\varphi(x, b_i) : i > 0\}, \varphi(x, y), k).$$

However, $\varphi(x, a_0)$ divides over $A\{b'_i : i > 0\}$, so we must have $D(\{\varphi(x, a_0)\} \cup \{\varphi(x, b_i) : i > 0\}, \varphi(x, y), k) < \alpha$. This is a contradiction. □

Proposition 8. *For $i = 1, ..., m$, let $\varphi_i(x, a)$ be a formula that divides over A. Then $\bigvee_{i=1,...,m} \varphi_i(x, a)$ divides over A.*

Proof. Choose J as in Lemma 7, then for each i there is k_i such that $\{\varphi_i(x, b) : b \in J\}$ is k_i-inconsistent. Let $k = \max\{k_1, ..., k_m\}$. Then $\{\bigvee_{i=1,...,m} \varphi_i(x, b) : b \in J\}$ is mk-inconsistent. Hence $\bigvee_{i=1,...,m} \varphi_i(x, a)$ divides over A. □

Lemma 9. *Suppose that $p(x) \in S(A)$ does not divide (fork) over $A_0 \subset A$. For any $B \supset A$, there is $a \models p$ such that $\mathrm{tp}(a/B)$ does not divide over A_0.*

Proof. Let $\Psi(x)$ be the following set of $L(B)$-formulas:

$$p(x) \cup \{\neg\varphi(x) \in L(B) : \varphi(x) \text{ divides over } A_0\}.$$

$\Psi(x)$ is consistent, since otherwise we would have that $p(x)$ forks over A_0. Let $a \models \Psi(x)$. Then it follows that $\mathrm{tp}(a/B)$ does not divide over A_0. □

Proposition 10 (Symmetry). *$\mathrm{tp}(a/Ab)$ does not divide over $A \Rightarrow \mathrm{tp}(b/Aa)$ does not divide over A.*

Proof. First we inductively choose a_i's such that

- $a_i \models \mathrm{tp}(a/Ab)$;
- $\mathrm{tp}(a_i/A \cup \{a_j : j < i\})$ does not divide over A.

This process can be done by an iterative use of Lemma 9. As in the proof of Lemma 7, by compactness, we can assume that $I = \{a_i : i \in \omega\}$ is an A-indiscernible sequence satisfying the conditions

1. $\mathrm{tp}(\{a_i : i > 0\}/Aa_0)$ does not divide over A $(i \in \omega)$;
2. By letting $q_a(x) = \mathrm{tp}(b/Aa)$, b is a common solution of $q_{a_i}(x)$ $(i \in \omega)$.

Thus $q_a(x)$ does not divide over A, by Lemma 7. □

References

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[3] Martin Ziegler, "A Course in Model Theory", Cambridge University Press (2012).

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