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Resplendent models of o-minimal expansions of RCOF

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Abstract

In this paper, the author gives a characterization of resplendent models of the axioms, formulated by van den Dries, of restricted analytic real fields.

1 Introduction.

In classical model theory, we usually investigate properties of first order theories T, using their models. The properties that we are interested in are, for example, those concerning the existence of special types of models of T, such as prime models, saturated models and compact models, and so on. Of course, such models as listed above do not always exist. A saturated model of a complete T exists under the assumption of G.C.H., but it does not exist in general without such assumptions. However, if we replace the definition of saturation by a weaker version, we can sometimes show its existence without set theoretic assumptions. Especially, every theory has a recursively saturated model. J.P. Ressayre shows the following important fact on recursive saturation, which states that resplendence and recursively saturation coincide for countable structures.

1 Fact. (J.P. Ressayre(1972)[5]) For each countable structure M of finite language, M is resplendent if and only if M is recursively saturated.

It is not hard to show the existence of a recursively saturated model. From the fact above, we know that a resplendent model also exists for any countable theory. Resplendence seems a useful property to be studied. In Ressayer's proof of the only if part of the fact above, he finds some consistent sentence $\varphi(P)$ with a new unary predicate P such that if a structure has a solution of P, then the structure is recursively saturated. There are some works aiming to get a more concrete $\varphi(P)$, when the axioms are specified. For example, P. D'Aquino, J.F. Knight and S. Starchenko find a characterization of recursively saturated model in the theory of real closed field([1]). Moreover, the author and A. Tsuboi found a characterization of recursively saturation in an o-minimal effectively model complete theory of real closed fields with a finite number of functions. This can be applied to A. J. Wilkie's exponential fields([8]). However, we cannot apply this result to van den Dries's restricted analytic field because the restricted analytic field is not a constructive object. The author considered a constructive fragment of theories for restricted analytic fields and find a characterization of recursive saturation for models of such theories.

2 Preliminaries and basic facts.

Let L be a finite language, M an L-structure, T an L-theory(not necessarily complete). Let L_{or} be the language $\{+, \cdot, 0, 1, <\}$ of ordered rings, $RCOF$ the theory of real closed fields, PA the theory of first order arithmetic. Let $Th(M) := \{\phi : \phi$ is an L-sentence, $M \models \phi\}$ be a theory of M, $\mathrm{Diag}_{el}(M) := \{\phi : \phi$ is an $L(M)$-sentence, $M \models \phi\}$ an elementary diagram of M.

2 Definition. (1). We say that M is **resplendent** if for any new relational symbol $R \notin L$ and any $L(M) \cup \{R\}$-sentence $\phi(R)$ if $\mathrm{Diag}_{el}(M) \cup \{\phi(R)\}$ is consistent, then there is an interpretation R^M on M such that $(M, R^M) \models \phi(R)$.

(2).We say that M is **recursively saturated** if every recursive type(with finite parameters) is realized in M.

3 Fact. (J.P. Ressayre(1972)[5]) For each countable structure M of finite language, M is resplendent if and only if M is recursively saturated.

In Ressayer's proof of the only if part of the fact above, he finds some consistent sentence $\varphi(P)$ with a new unary predicate P such that if a structure has a solution of P, then the structure is recursively saturated. By the meaning of $\varphi(P)$ in Ressayer's proof, we can construct a model of arithmetic from a solution of $\varphi(P)$.

4 Question. If a theory T naturally involves some arithmetic structure, then $\varphi(P)$ can be taken as a natural form under T.

Next fact is an answer in the case of $T = RCOF$ for this question.

5 Definition. Let K be an orderd filed. We call an ordered subring $Z \subset K$ an **integer part** if it satisfies $\forall x \in K, \exists! n \in Z$ $s.t.$ $n \leq x < n + 1$.

6 Fact. (P. D ' Aquino, J.F. Knight and S. Starchenko (2010)[1]) For a countable ordered field K, the followings are equivalent:

- K is a recursively saturated model of $RCOF$;
- K has a non-archimedean integer part whose the non-negative part satisfies PA.

3 Background.

In this section, we introduce the previous investigation(A. Tsuboi and T.(2013)[8]). Firstly, we show a characterization of recursively saturated model of o-minimal expansion of the theory $RCOF$ as like Fact 6. Secondly, we will construct recursively saturated models by using nonstandard analysis.

3.1 o-minimal analogue

In the proof of Fact 6, we use o-minimality and quantifier elimination of the theory $RCOF$.

7 Question. Are there any analogue for o-minimal expantion of $RCOF$?

To answer the question above, we introduce definitions of o-minimality and weak form of quantifier elimination.

8 Definition. (o-minimal) We say that a theory T is **o-minimal** if for any model M of T and any definable set $A \subset M$ (with parameters from M), A can be described some finite union of open intervals and points.

9 Example. The following theories are o-minimal.

- The theory of real closed field:$RCOF$.
- $T_{exp} = Th(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$.
- $T_{an} = Th(\mathbb{R}, +, \cdot, 0, 1, <, (f_i)_i)$.
  Where $(f_i)_i$ is an enumeration of all analytic functions defined on closed box.

Next definition is a weak form of quantifier elimination.

10 Definition. We say that a theory T is **model complete** if every L-formula $\phi(\bar{x})$ is equivalent to some existential L-formula $\psi(\bar{x})$ modulo T:

$$\forall \phi(\bar{x}) \exists \psi(\bar{x}), T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

11 Example. $RCOF$, T_{exp} and T_{an} are model complete.

This definition is not sufficient to prove Fact. 6. We need an effective version of model completeness. Since $RCOF$ is recursively axiomatized, we can effectively obtain an equivalent existential formula $\psi(\bar{x}))$ for above setting. In general, a decidable and model complete theory has same property.

12 Definition. We say that a theory T is **effectively model complete** if there is a effective procedure finding an existential L-formula $\psi(\bar{x})$ which equivalent to any given L-formula $\phi(\bar{x})$ modulo T.

A. Macintyre and A. J. Wilkie defined the effectively model completeness for finding a decidability result of T_{exp}.

13 Fact. (A. Macintyre and A. J. Wilkie (1996)[4]) T_{exp} is effectively model complete.

Lastly we will define a notion of definably approximation which means a relevance of an integer part and additional functions, e.g. an exponential function.

14 Definition. Let R be a real closed ordered field with an integer part Z and let $Q \subset R$ be the quotient field of Z. Suppose that N (the nonnegative part of Z) satisfies PA. Finally, let $E : R^n \to R$ be a continuous function. We say that E is **Z-definably approximated** if there exists a continuous function $F : N \times Q^n \to Q$ such that

- F is definable in the ordered field Q;
- $\{F(m, \bar{x}) : m \in N\}$ converges uniformly to $E(\bar{x})$ on closed bounded subsets of Q. More precisely, for all closed bounded boxes $B \subset Q^n$ and $\varepsilon > 0$, there exists $n_0 \in N$ such that, for all $n \in N$ with $n \geq n_0$ and all $\bar{x} \in B$, $R \models |E(\bar{x}) - F(n, \bar{x})| < \varepsilon$.

Then we can state an answer of the question above.

15 Theorem. (A. Tsuboi(2013)[8]) Let L be a language $L_{or} \cup \{f_1, ..., f_k\}$, T an o-minimal and effectively model complete L-theory extended from $RCOF$. Let R be a model of T. R is a recursively saturated if there is an integer part $Z \subset R$ such that

- the non-negative part of Z satisfies PA, $Z \neq \mathbb{Z}$ and
- each f_i is Z-definably approximated.

16 Corollary. Let R be a countable model of T_{exp}. R is recursively saturated if and only if there is an integer part $Z \subset R$ such that

- the non-negative part of Z satisfies PA, $Z \neq \mathbb{Z}$ and
- $\exp(x)$ is Z-definably approximated.

Since T_{an} is a non-constructive object, we can not consider effective model completeness of T_{an}. For application, we need to consider a constructive sub-theory of T_{an}.

3.2 natural construction of recursively saturated real closed fields

In previous arguments, we give a characterization of recursively saturated model of a fixed theory. We do not consider applications of a given characterization. In this subsection, we will construct a recursively saturated models by using nonstandard analysis. We can easily construct a recursively saturated model by adding ideal elements, but our construction, showed below, is adding elements simultaneously.

17 Question. Is there a "natural" construction of recursively saturated model of $RCOF$?

Next theorem is an answer of the question above.

18 Definition. Let K be an ordered field and K^* an elementary extension of K. We call following sets **finite part** and **infinitismal part** respectively:

- $F_K := \{x \in K^* : \exists q \in K \text{ s.t. } |x| < |q|\}$
- $I_K := \{x \in K^* : \forall q \in K^\times \text{ s.t. } |x| < |q|\}$.

19 Theorem. (A. Tsuboi and T.(2013)[8]) Let K be an ordered field with an integer part Z satisfying PA. If $F_K \neq K^*$, the quotient field $R := F_K / I_K$ satisfies $RCOF$. Moreover, if $Z \neq \mathbb{Z}$,then R is recursively saturated.

Similarly, we can construct a recursively saturated model of T_{exp}. Let $\mathbb{Q}^*$ be an ω_1-saturated elementary extension of $\mathbb{Q}$. Let $(Q^*, Q) \equiv (\mathbb{Q}^*, \mathbb{Q})$ where $Q \neq \mathbb{Q}$. Then $\mathbb{R} \cong F_{\mathbb{Q}}/I_{\mathbb{Q}} \equiv F_Q/I_Q$. Let $\phi_{\mathbb{Z}}(x)$ be a defining formula of $\mathbb{Z}$ in $\mathbb{Q}$.(by J.Robinson) Let $Z := \phi_{\mathbb{Z}}(Q)$ and $Z^* := \phi_{\mathbb{Z}}(Q^*)$. Fix $n^* \in Z^* - Z$ and define $e(x) := \sum_{k=0}^{n^*} \frac{1}{k!} x^k$. Define $\exp^* : F_Q/I_Q \to F_Q/I_Q$ by $\exp^*(x + I_Q) := e(x) + I_Q$. Then $(\mathbb{R}, \exp) \cong (F_{\mathbb{Q}}/I_{\mathbb{Q}}, \exp^*) \equiv (F_Q/I_Q, \exp^*)$ holds. In $(F_Q/I_Q, \exp^*)$, $\exp^*$ is approximated in its integer part $\cong Z$.

20 Example. $(F_Q/I_Q, \exp^*)$ is a recursively saturated model of T_{exp}.

4 Results.

We will review a definition of the restricted analytic field.

21 Definition. Let $L_{an} = L_{or} \cup \{f_i\}_i$ where f_i is a function symbol, $\mathbb{R}_{an} = (\mathbb{R}, +, \cdot, 0, 1, <, (f_i)_i)$ where $(f_i)_i$ is an enumeration of all analytic functions defined on closed box, and $T_{an} = Th(\mathbb{R}_{an})$.

22 Theorem. T_{an} is model complete and o-minimal.

For application of our theorem15, we need a good fragment of T_{an}. Let F be a class of restricted analytic functions. Then $L_{an}|F$ is $L_{or} \cup F$ and $T_{an}|F$ is restriction of T_{an} to $L_{an}|F$. It is easy to show that every complete subtheory of o-minimal theory is o-minimal, i.e. $T_{an}|F$ is o-minimal(for any F). For a subtheory of T_{an}, A. Gabrièlov finds a condition of F whether $T_{an}|F$ is model complete.

23 Theorem. (A. Gabrièlov(1996)[3]) Let F be a class of restricted analytic functions closed under derivation. Then $T_{an}|F$ is model complete.

This proof is not prefer an effective version because it is a geometric. Since a proof of J.Denef and L.van den Dries (1988)[2] is algorithmic, we based on it. This proof of the model completeness of T_{an} depends on following two basic facts for analytic functions.

- Wierstrass's preparation theorem,
- van den Dries's preparation theorem

In the first subsection, we will give an outline of effective proofs. We will give a coding of restricted analytic functions and statements of an effective form of facts above. Moreover, we give a condition of a set F such that $T_{an}|F$ is eventually effective model complete. In the second subsection, we will give a characterization of recursively saturated model of $T_{an}|F$ for some F and a construction of recursively saturated model of it.

4.1 effective proof of basic facts

We fix notations.

- O_n : a ring of n-ary analytic functions on neighborhood of 0;

- $R[Y]$: a polynomial ring of a new variable Y with coefficients from a ring R;
- We use multi-index notations: if $\bar{i} = (i_1, ..., i_n)$, then $\bar{x}^{\bar{i}} = x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$;
- For a function $f(\bar{x}) = \sum_{\bar{i}} a_{\bar{i}} \bar{x}^{\bar{i}} \in O_n$ and a tuple of positive reals $\bar{e}$,

$$||f||_{\bar{e}} := \begin{cases} \sum_{\bar{i}} |a_{\bar{i}} \bar{e}^{\bar{i}}| & \text{if it convergences} \\ \infty & \text{otherwise} \end{cases};$$

- $|\bar{x}| \leq |\bar{e}|$ means $\wedge_i |x_i| \leq |e_i|$.

We will define a coding of restricted analytic functions to prove effective results.

24 Definition. (coding of real) Let $(a^n)^n$ be a recursive sequence of rational numbers. We say that a real $\alpha \in \mathbb{R}$ is **coded** by $(a^n)^n$ if $\forall n, |\alpha - a^n| < 2^{-n}$.

25 Definition. (coding of restricted analytic function) Let $(a_{\bar{i}}^n)_{\bar{i}}^n$ be a recursive multi-indexed sequence of rational numbers and $\bar{e}, b, M$ are positive rational numbers. We say that a restricted analytic function $f(\bar{x}) = \sum_{\bar{i}} \alpha_{\bar{i}} \bar{x}^{\bar{i}} \in O_n$ is **coded** by a code $C = ((a_{\bar{i}}^n)_{\bar{i}}^n; \bar{e}, b; M)$ if $||f||_{b\bar{e}} < M$, $\alpha_{\bar{i}}$ is coded by $(a_{\bar{i}}^n)^n$, $b > 1$ and $dom(f) = \{\bar{x} : |\bar{x}| \leq |\bar{e}|\}$.

For a code $C = ((a_{\bar{i}}^n)_{\bar{i}}^n; \bar{e}, b; M)$, let $a_{\bar{i}}^n, (C)\bar{e}(C), b(C)$ and $M(C)$ denote components $a_{\bar{i}}^n, \bar{e}, b$ and M of C respectively.

26 Example. Let π_n be n decimal digits of π and M a sufficiently large poditivr number. Then the restricted sine function $\sin(\pi x)|[-1,1]$ can be coded by $((\frac{1-(-1)^{i+1}}{2\cdot(2i+1)!} \cdot \pi_n^i)_i^n, 1, 2, M)$.

Remark: Let $f \in O_n$ and $g_1, ..., g_n \in O_m$ be coded by $C, D_1, ..., D_n$ respectively. If $M(D_i) \leq \bar{e}(C)_i (i < n)$, then $f(g_1, ..., g_n)$ can be coded by some $G = C_{com}(C, D_1, ..., D_n)$.

To state the Wierstarss's preparation, we define the regularity of an analytic function.

27 Definition. (regularity) We say that a restricted analytic function $f(x_1, ..., x_n) \in O_n$ is **regular of order** p **with respect to** x_n if $f(0, 0, ..., x_n) = c \cdot x_n^p + o(x_n^p)$ where $c \neq 0$.

28 Fact. (Wierstarss's preparation) Let $\Phi \in O_n$ be regular of order p with respect to x_n. There exists unique unit $Q \in O_n$ and unique $R \in O_{n-1}[x_n]$ regular of order p with respect to x_n such that $R = \Phi Q$.

29 Lemma. (Effective Wierstarss's preparation) There exist recursive functions $C_{WQ}(C, n)$, $C_{WR}(C, n)$ which map from pairs of a code and a natural number to codes such that the followings holds: for any given $\Phi \in O_n$ which is regular of order p with respect to x_n and coded by C, $Q \in O_n$ and $R \in O_{n-1}[x_n]$ are obtained by the Wierstarss's preparation; then for any sufficiently large $n \in \mathbb{N}$, Q, R are coded by$C_{WQ}(C, n)$, $C_{WR}(C, n)$ respectively.

Unfortunately, there is no effective procedure finding sufficiently large n. This problem deduce to check $\forall X, R(X) = \Phi(X)Q(X)$. Next, we will state the van den Dries's preparation and an effective form of this.

30 Fact. **(van den Dries's preparation)** Let $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_m), m > 0$ and $\Phi(X, Y) \in O_{n+m}$. There exist $d \in \mathbb{N}$, $a_{\bar{i}}(X) \in O_n$ and units $u_{\bar{i}}(X, Y) \in O_{n+m}$ ($|\bar{i}| < d$) such that:

$$\Phi(X, Y) = \sum_{|\bar{i}|<d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y).$$

31 Lemma. **(Effective van den Dries's preparation)** Let $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_m), m > 0$. There exist recursive functions $C_{vA}(C, d, n, \bar{i}), C_{vU}(C, d, n, \bar{i})$ such that the followings holds: for any given $\Phi(X, Y) \in O_{n+m}$ be coded by C, for any sufficiently large $d \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\Phi(X, Y) = \sum_{|\bar{i}|<d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y)$, where $a_{\bar{i}}(X), u_{\bar{i}}(X, Y)$ are coded by $C_{vA}(C, d, n, \bar{i}), C_{vU}(C, d, n, \bar{i})$ respectively and each $u_{\bar{i}}$ is a unit.

There is a problem how to find d, n effectively. This problem deduce to check $\forall XY, \Phi(X, Y) = \sum_{|\bar{i}|<d} a_{\bar{i}}(X) Y^{\bar{i}} u_{\bar{i}}(X, Y)$. Then we will give a condition of a set F such that $T_{an}|F$ is eventually effective model complete and a definition of eventually effective model complete.

32 Definition. We say that a set S of codes **closed** if it is closed under C_{com}, C_{vA}, C_{vU}, C_{WQ}, C_{WR} and contains codes of bounded polynomial functions. Let $F_S = \{f \in \cup_n O_n : f$ is coded by some element of $S\}$.

33 Definition. We say that an L-theory T is **eventually effectively nearly model complete** if there is an effective procedure, for any given formula L-formula $\phi(x)$, finding recursive enumeration of boolean combinations of existential L-formulas $\{\psi_n(x)\}_{n\in\omega}$ such that $T \models \phi(x) \to \psi_m(x)$ for any m and $T \models \phi(x) \leftarrow \bigwedge_{m<n} \psi_m(x)$ for any sufficiently large n.

We obtain a weak form of the effective model completeness for some fragment of T_{an}.

34 Theorem. **(T. 2013)** Let S be a r.e. closed set of codes, $L = L_{an}|F_S$. Then $T_{an}|F_S = Th(\mathbb{R}_{an}|F_S)$ is eventually effectively nearly model complete.

4.2 main results

Similarly to a proof of Theorem 15, we will show the main theorem.

35 Theorem. **(revisited A. Tsuboi(2013) : modified by T.)** Let L be a language $L_{or} \cup \{f_i\}_{i\in\mathbb{N}}$, T an o-minimal and eventually effectively nearly model complete L-theory extended from $RCOF$. Let R be a model of T. Then R is a recursively saturated if there is an integer part $Z \subset R$ such that:

- the non-negative part of Z satisfies PA, $Z \neq \mathbb{Z}$ and
- each f_i is Z-definably approximated by a Σ_{k_0}-formula where k_0 does not depend on i.

We fix L, T, R and Z as in Theorem 35, and prove a series of lemmas before proving the theorem. Let N be the non-negative part of Z, Q the quotient field of Z in R. Choose k_0 such that every $f_i(\bar{x})(i \in \omega)$ is Z-definably approximated by a Σ_{k_0}-formula. To prove Theorem 35, we need following lemmas proved in [8].

36 Lemma. ([8]) Every L-term (i.e., every term constructed from $+, \cdot$ and the f_i's) is Z-definably approximated by Σ_{k_0}-formulas.

37 Lemma. **([8] modified by T.)** Let $\varphi(\bar{x})$ be a boolean combination of existential L-formulas. Then we can effectively find an L-formula $\varphi_0(\bar{x})$ and an L_{or}-formula $\varphi'(\bar{x})$ such that

- $R \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi_0(\bar{x}))$;
- $R \models \varphi_0(\bar{b}) \iff Q \models \varphi'(\bar{b})$, for all $\bar{b} \in Q$.

The formula φ' obtained in Lemma 37 is a Σ_{k_0+5}-formula.

38 Lemma. **([8] modified by T.)** Let $\varphi(\bar{x})$ and $\psi(\bar{x})$ be boolean combinations of existential L-formulas such that $R \models \forall \bar{x}(\varphi \to \psi)$. Let φ' and ψ' be the formulas obtained in Lemma 37. Then $Q \models \forall x(\varphi' \to \psi')$.

39 Lemma. ([8]) For any $\bar{a} \in R$, $\mathrm{dcl}(\bar{a})$ is a bounded subset of R.

Proof. (Proof of Theorem35) Let $\Sigma(x, \bar{a}) = \{\varphi_i(x, \bar{a}) : i \in \omega\}$ be a (non-algebraic) recursive type with $\bar{a} \in R$. We can assume that $\varphi_{i+1}(x, \bar{a}) \to \varphi_i(x, \bar{a})$ holds in R. Since other cases can be treated similarly, we assume that $\bar{a} \in R \setminus \mathrm{dcl}(\emptyset)$ and that elements in $\bar{a}$ are mutually non-algebraic. For each $\varphi_i(x, \bar{a}) \in \Sigma$, let $\theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1) = \theta_i(u_0, u_1, v_{00}, \ldots, v_{0,k-1}, v_{1k}, \ldots, v_{1,k-1})$ be the formula

$$\forall x\bar{y}\left(u_0 < x < u_1 \land \bigwedge_{j<k} v_{0j} < y_j < v_{1j} \ \to \ \varphi_i(x, \bar{y})\right),$$

where k is the length of $\bar{a}$. Notice that $\exists u_0 u_1 (u_0 < u_1 \land \bigwedge_{j<k} v_{0j} < a_j < v_{1j} \land \theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1))$ is satisfiable in R. (We can use the cell decomposition theorem to see this.) We can assume that θ_i is an boolean combination of existential formulas by the eventually effective nearly model completeness assumption.

By the o-minimality, there exist minimum $\bar{b}_0$ and maximum $\bar{b}_1$ (in the lexicographic ordering) such that $\exists u_0 u_1 (u_0 < u_1 \land \bigwedge_{j<k} b_{0j} < a_j < b_{1j} \land \theta_i(u_0, u_1, \bar{b}_0, \bar{b}_1))$ holds in R. Therefore, $\bar{b}_0, \bar{b}_1 \in \mathrm{dcl}(\bar{a}) \cup \{\pm\infty\}$. Using Lemma 39, choose a sufficiently large integer n^* such that $\mathrm{dcl}(\bar{a}) < n^*$. We can choose $\bar{c}_0, \bar{c}_1 \in Q$ with $\sum_{j<k} |c_{0j} - c_{1j}| < 1/n^*$ such that $b_{0j} < c_{0j} < a_j < c_{1j} < b_{1j}$ $(j < k)$. Then $\exists u_0 u_1 (u_0 < u_1 \land \theta_i(u_0, u_1, \bar{c}_0, \bar{c}_1))$ holds in R regardless of the choice of $i \in \omega$.

For each θ_i, choose a formula θ_i' having the property described in Lemma 37. Namely, choose θ_i' such that

1. $R \models \forall u_0 u_1 \bar{v}(\theta_i \leftrightarrow \theta_{i,0})$;
2. $R \models \theta_{i,0}(q_0, q_1, \bar{r}, \bar{s}) \iff Q \models \theta_i'(q_0, q_1, \bar{r}, \bar{s})$, for any $q_0, q_1, \bar{r}, \bar{s} \in Q$.

In the present situation, $\exists u_0 u_1 (u_0 < u_1 \land \theta_{i,0}(u_0, u_1, \bar{c}_0, \bar{c}_1))$ holds in R. Since u_0, u_1 can be chosen from Q, $\exists u_0 u_1 (u_0 < u_1 \land \theta_i'(u_0, u_1, \bar{c}_0, \bar{c}_1))$ holds in Q. Then, by Lemma 38, $\{\theta_i'(u_0, u_1, \bar{c}_0, \bar{c}_1) : i \in \omega\}$ is a recursive Σ_{k_0+5}-type in Q. So, by the Σ_{k_0+5}-recursive saturation of Q, there exists $(d_1, d_2) \in Q^2$ such that $Q \models \bigwedge_{i\in\omega} \theta_i'(d_0, d_1, \bar{c}_0, \bar{c}_1)$. Hence, $\Sigma(x, \bar{a})$ is realized in R by any e between d_0 and d_1. □

40 Example. Let F_{sin} be a closed r.e. set contains a code of $sin(\pi x)|[-1,1]$. Let $T_{sin} = T_{an}|F_{sin}$.

41 Corollary. Let R be a model of T_{sin}. R is a recursively saturated if there is an integer part $Z \subset R$ such that :

- the non negative part of Z satisfies PA, $Z \neq \mathbb{Z}$ and
- each $f \in F_{sin}$ is Z-definably approximated by a Σ_{k_0}-formula where k_0 does not depend on f.

Finally, we will construct a recursive saturated model of T_{sin} by using nonstandard analysis. Let $\mathbb{Q}, \mathbb{Q}^*, Q, Q^*, Z, Z^*, n^*$ be in a construction of Example 20. For any $f \in F_{sin}$ coded by $((a^n_{\bar{i}})^n_{\bar{i}}; \bar{e}, b; M)$, define $f^* : F_Q/I_Q \to F_Q/I_Q$ by $f^*(x + I_Q) := \sum_{|\bar{i}|<n^*}(\lim_n a^n_{\bar{i}})x^{\bar{i}} + I_Q$. Then $(\mathbb{R}, F_{sin}) \cong (F_{\mathbb{Q}}/I_{\mathbb{Q}}, \{f^* : f \in F_{sin}\}) \equiv (F_Q/I_Q, \{f^* : f \in F_{sin}\})$. In $(F_Q/I_Q, \{f^* : f \in F_{sin}\})$, f^* is approximated in its integer part $\cong Z$.

42 Example. $(F_Q/I_Q, \{f^* : f \in F_{sin}\})$ is a recursively saturated model of T_{sin}.

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