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INTRODUCTION TO THE CONSTRUCTION OF ADDITIVE INVARIANTS IN O-MINIMAL VALUED FIELDS

by

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Abstract. — In this note we give a brief description of the key points of Hrushovski-Kazhdan style motivic integration for certain type of non-archimedean o-minimal fields, namely polynomial-bounded T-convex valued fields. These include canonical homomorphisms between the Grothendieck semirings of various categories of definable sets that are associated with the usual VF-sort and the RV-sort of the language $\mathcal{L}_{\mathrm{TRV}}$, the groupifications of some of these homomorphisms, which may be described explicitly and are understood as generalized Euler characteristics, and topological zeta functions associated with (germs of) definable continuous functions in an arbitrary polynomial-bounded o-minimal field, which are shown to be rational.

Towards the end of the introduction of [**8**] three hopes for the future of the theory of motivic integration are mentioned. In [**12**] we have investigated one of them: integration, or rather, since we will not consider general volume forms, additive invariants, in o-minimal valued fields. The prototype of such valued fields is $\mathbb{R}((t^{\mathbb{Q}}))$, the power series field over $\mathbb{R}$ with exponents in $\mathbb{Q}$. One of the cornerstones of the methodology of [**8**] is C-minimality, which is the right analogue of o-minimality for algebraically closed valued fields and other closely related structures that epitomizes the behavior of definable subsets of the affine line. It, of course, fails in an o-minimal valued field, mainly due to the presence of a total ordering. The construction of additive invariants in [**12**] is thus carried out in a different framework, which affords a similar type of normal forms for definable subsets of the affine line, a special kind of weak o-minimality; this framework is van den Dries and Lewenberg's theory of T-convex valued fields [**6, 4**].

For a description of the ideas and the main results of the Hrushovski-Kazhdan style integration theory, we refer the reader to the original introduction in [**8**] and also the introductions in [**12, 13**]. There is also a quite comprehensive introduction to the same materials in [**9**] and, more importantly, a specialized version that relates the Hrushovski-Kazhdan style integration to the geometry and topology of Milnor fibers over the complex field. The method expounded there is featured in [**12**] as well. In fact, since much of the work there is closely modeled on that in [**8, 11, 12, 9**], the reader may simply substitute the term "theory of polynomial-bounded T-convex valued fields" for "theory

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of algebraically closed valued fields" or more generally "V-minimal theories" in those introductions and thereby acquire a quite good grip on what the results of [**12**] look like.

Let $(K, \mathrm{val} : K \longrightarrow \Gamma)$ be a valued field, where val is the valuation map, and $\mathcal{O}$, $\mathcal{M}$, $\Bbbk$ the corresponding valuation ring, its maximal ideal, and the residue field. Let

$$\mathrm{RV}(K) = K^\times/(1 + \mathcal{M})$$

and $\mathrm{rv} : K^\times \longrightarrow \mathrm{RV}(K)$ be the quotient map. Note that, for each $a \in K$, val is constant on the subset $a + a\mathcal{M}$ and hence there is a naturally induced map vrv from $\mathrm{RV}(K)$ onto the value group Γ. The situation is illustrated in the following commutative diagram

$$\begin{tikzcd}
\mathcal{O} \smallsetminus \mathcal{M} \arrow[r, hook] \arrow[d, two heads, "\text{quotient}"'] & K^\times \arrow[d, two heads, "\mathrm{rv}"] \arrow[dr, two heads, "\mathrm{val}"] & \\
\Bbbk^\times \arrow[r, hook] & \mathrm{RV}(K) \arrow[r, two heads, "\mathrm{vrv}"] & \Gamma
\end{tikzcd}$$

where the bottom sequence is exact. This structure may be expressed by a two-sorted first-order language $\mathcal{L}_{\mathrm{TRV}}$, where K is referred to as the VF-sort and RV is taken as a new sort. On the other hand, for the main construction in [**12**], K could carry any extra structure that amounts to a polynomial-bounded o-minimal expansion T of the theory of real closed fields (henceforth abbreviated as RCF); this is what the letter "T" stands for in $\mathcal{L}_{\mathrm{TRV}}$. In fact, there is essentially no loss of generality if we take $K = \mathbb{R}(\!(t^{\mathbb{Q}})\!)$, which we shall do in the remainder of this note.

Let VF_* and $\mathrm{RV}[*]$ be two categories of definable sets that are respectively associated with the VF-sort and the RV-sort. In VF_*, the objects are definable subsets of products of the form $\mathrm{VF}^n \times \mathrm{RV}^m$ and the morphisms are definable bijections. On the other hand, for technical reasons (particularly for keeping track of ambient dimensions), $\mathrm{RV}[*]$ is formulated in a somewhat complicated way and is hence equipped with a gradation by ambient dimension. The main construction of the Hrushovski-Kazhdan theory is a canonical homomorphism from the Grothendieck semiring $\mathbf{K}_+ \mathrm{VF}_*$ to the Grothendieck semiring $\mathbf{K}_+ \mathrm{RV}[*]$ modulo a semiring congruence relation $\mathrm{I_{sp}}$ on the latter. In fact, it turns out to be an isomorphism. This construction has three main steps.

Step 1. First we define a lifting map $\mathbb{L}$ from the set of objects in $\mathrm{RV}[*]$ into the set of objects in VF_*. Next we single out a subclass of isomorphisms in VF_*, which are called special bijections. Then we show that for any object A in VF_* there is a special bijection T on A and an object $\boldsymbol{U}$ in $\mathrm{RV}[*]$ such that $T(A)$ is isomorphic to $\mathbb{L}(\boldsymbol{U})$. This implies that $\mathbb{L}$ hits every isomorphism class of VF_*. Of course, for this result alone we do not have to limit our means to special bijections. However, in Step 3 below, special bijections become an essential ingredient in computing the semiring congruence relation $\mathrm{I_{sp}}$.

Step 2. For any two isomorphic objects $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ in $\mathrm{RV}[*]$, their lifts $\mathbb{L}(\boldsymbol{U}_1), \mathbb{L}(\boldsymbol{U}_2)$ in VF_* are isomorphic as well. This shows that $\mathbb{L}$ induces a semiring homomorphism from $\mathbf{K}_+ \mathrm{RV}[*]$ into $\mathbf{K}_+ \mathrm{VF}_*$, which is also denoted by $\mathbb{L}$.

Step 3. A number of classical properties of integration can already be (perhaps only partially) verified for the inversion of the homomorphism $\mathbb{L}$ and hence, morally, this third step is not necessary. For applications, however, it is much more satisfying to have a precise description of the semiring congruence relation induced by $\mathbb{L}$. The basic notion used in the description is that of a blowup of an object in $\mathrm{RV}[*]$, which is essentially a restatement of the trivial fact that there is an additive translation from $1 + \mathcal{M}$ onto $\mathcal{M}$. We then show that, for any objects $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ in $\mathrm{RV}[*]$, there are isomorphic blowups $\boldsymbol{U}_1^\sharp$, $\boldsymbol{U}_2^\sharp$ of them if and only if $\mathbb{L}(\boldsymbol{U}_1)$, $\mathbb{L}(\boldsymbol{U}_2)$ are isomorphic. The "if" direction essentially contains a form of Fubini's Theorem and is the most technically involved part of the construction.

The inverse of $\mathbb{L}$ thus obtained is called a Grothendieck homomorphism. If the Jacobian transformation preserves integrals, that is, the change of variables formula holds, then it may be called a motivic integration; only a very primitive case of this notion is considered in [**12**]. When the semirings are formally groupified, this Grothendieck homomorphism is recast as a ring homomorphism, which is denoted by $\int$.

The Grothendieck ring $\mathbf{K}\,\mathrm{RV}[*]$ may be expressed as a tensor product of two other Grothendieck rings $\mathbf{K}\,\mathrm{RES}[*]$ and $\mathbf{K}\,\Gamma[*]$, where $\mathrm{RES}[*]$ is essentially the category of definable sets over $\mathbb{R}$ (as a model of the theory T) and $\Gamma[*]$ is essentially the category of definable sets over $\mathbb{Q}$ (as an o-minimal group), and both are graded by ambient dimension. This results in various retractions from $\mathbf{K}\,\mathrm{RV}[*]$ into $\mathbf{K}\,\mathrm{RES}[*]$ or $\mathbf{K}\,\Gamma[*]$ and, when combined with the canonical homomorphism $\int$, yields various (generalized) Euler characteristics

$$\int\!\!\!\!\!\!\mathbb{G} : \mathbf{K}\,\mathrm{VF}_* \xrightarrow{\sim} \mathbb{Z}^{(2)} := \mathbb{Z}[X]/(X + X^2),$$

which is actually an isomorphism, and

$$\int\!\!\!\!\!\!\mathbb{R}^{g}, \int\!\!\!\!\!\!\mathbb{R}^{b} : \mathbf{K}\,\mathrm{VF}_* \longrightarrow \mathbb{Z}.$$

For the construction of topological zeta functions we shall need to introduce the simplest volume form, namely the constant Γ-volume form 1, into the various categories above. This modification has no bearing on the collection of objects in these categories, but does trim down the collection of morphisms. The resulting categories are denoted by $\mathrm{vol}\,\mathrm{VF}[*]$, $\mathrm{vol}\,\mathrm{RV}[*]$, etc. For example, given (a, a') and (b, b') in $\mathbb{Q}^2$, the subsets $\mathrm{val}^{-1}(a, a')$ and $\mathrm{val}^{-1}(b, b')$ of $\mathbb{R}((t^{\mathbb{Q}}))^2$ are isomorphic in VF_* but are not isomorphic in $\mathrm{vol}\,\mathrm{VF}[*]$ unless $a + a' = b + b'$. Another change in $\mathrm{vol}\,\mathrm{VF}[*]$ is that morphisms may ignore a subset whose dimension is smaller than the ambient dimension. Thus $\mathrm{vol}\,\mathrm{VF}[*]$ is also graded by ambient dimension; this is why we have changed the position of "$*$" in the notation. The semiring congruence relation $\mathrm{I_{sp}}$ is now homogeneous and we have a canonical isomorphism of *graded* Grothendieck rings

$$\int : \mathbf{K}\,\mathrm{vol}\,\mathrm{VF}[*] \longrightarrow \mathbf{K}\,\mathrm{vol}\,\mathrm{RV}[*]/\,\mathrm{I_{sp}}\,.$$

Moreover, if we do restrict our attention to a special type of objects, namely those objects whose images under val are (in effect) bounded from both sides, then there are two natural homomorphisms of graded rings

$$\int\!\!\!\!\!\!\mathbb{R}^{\pm} : \mathbf{K}\,\mathrm{vol}\,\mathrm{VF}^{\diamond}[*] \longrightarrow \mathbb{Z}[X].$$

Now let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a definable non-constant continuous function sending 0 to 0, or more generally a germ at 0 of such functions. For example, f could be a polynomial function or a subanalytic function if it is allowed by the theory T. Unlike in the complex case, there is no Milnor fibration of f (and hence there is no consensus on what the monodromy of f should be). Nevertheless we may still define the positive and the negative *Milnor fibers* of f at 0:

$$M_+ = B(0, \epsilon) \cap f^{-1}(\delta) \quad \text{and} \quad M_- = B(0, \epsilon) \cap f^{-1}(-\delta),$$

where $0 < \delta \ll \epsilon \ll 1$ and $B(0, \epsilon)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius ϵ. By o-minimal trivialization (see [**5**, §9]), the (embedded) definable homeomorphism types of M_+ and M_- are well-defined (of course M_+ and M_- are not necessarily homeomorphic, definably or not; indeed M_- may be empty while M_+ is not). By T-convexity, f may be lifted in a unique way to a definable continuous function $f^\uparrow : \mathcal{O}^n \longrightarrow \mathcal{O}$. The Milnor fibers of f with (thickened) formal arcs attached to each point are defined as

$$\widetilde{M}_+ = \{a \in \mathcal{M}^n : \operatorname{rv}(f^\uparrow(a)) = \operatorname{rv}(t)\},$$
$$\widetilde{M}_- = \{a \in \mathcal{M}^n : \operatorname{rv}(f^\uparrow(a)) = -\operatorname{rv}(t)\}.$$

Following [**2, 3**], we attach topological (or motivic) zeta functions $Z^\pm(\widetilde{M}_\pm)(Y)$ to f (two to each one of $\widetilde{M}_+$ and $\widetilde{M}_-$, due to the lack of a canonical identification of certain graded ring with $\mathbb{Z}[X]$), which are power series in $\mathbb{Z}[X][\![Y]\!]$ whose coefficients are integrands of truncated (thickened) formal arcs. In [**12**], it is shown that these zeta functions are rational and their denominators are products of terms of the form $1 - (-X)^a Y^b$, where $b \geq 1$. Consequently, $Z^\pm(\widetilde{M}_\pm)(Y)$ attain limits $e^\pm(\widetilde{M}_\pm) \in \mathbb{Z}$ as $Y \to \infty$ and we have the equality

$$e^\pm(\widetilde{M}_\pm)X^n = -\int_{\mathbb{R}}^{\pm} [\widetilde{M}_\pm].$$

For certain purposes, the difference between model theory and algebraic geometry is somewhat easier to bridge if one works over the complex field, as is demonstrated in [**9**]; however, over the real field, although they do overlap significantly, the two worlds seem to diverge in their methods and ideas. Our results should be understood in the context of "o-minimal geometry" [**5, 7**]. This is reflected in our preference of the terminology "topological zeta function" since, in the literature of real algebraic geometry, "motivic zeta function" has already been constructed (see, for example, [**1**]), which is a much finer invariant as there are much less morphisms in the background. In general, the various Grothendieck rings considered in real algebraic geometry bring about lesser collapse of "algebraic data" and hence are more faithful in this regard, although the flip side of the story is that they are computationally intractable (especially when resolution of singularities is involved) and specializations are often needed in practice. For instance, the Grothendieck ring of real algebraic varieties may be specialized to $\mathbb{Z}[X]$, which is called the virtual Poincaré polynomial (see [**10**]). Still, our method does not seem to be suited for recovering invariants at this level, at least not directly (that the homomorphism $\int_{\mathbb{R}}^{\pm}$ has $\mathbb{Z}[X]$ as its codomain is merely a coincidence and is not an essential feature of the construction).

Similar constructions are available for other (closely related) categories of definable sets, in particular, for such categories with general volume forms, which we have not included in this paper for the sake of simplicity and brevity. For those constructions, one needs to add a section from the RV-sort into the VF-sort or at least a standard part map, that is, a section from the residue field into the VF-sort, since it is not conceptually correct to use the "counting measure" on the residue field anymore. We shall elaborate on this in a sequel.

References

[1] Georges Comte and Goulwen Fichou, *Grothendieck ring of semialgebraic formulas and motivic real Milnor fibres*, (2012), arXiv:1111.3181.

[2] Jan Denef and François Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Progress in Mathematics, vol. 201, Birkhäuser Basel, 2001, arXiv:math/0006050, pp. 327–348.

[3] Jan Denef and François Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, Journal of the American Mathematical Society **5** (1992), no. 4, 705–720.

[4] Lou van den Dries, *T-convexity and tame extensions II*, Journal of Symbolic Logic **62** (1997), no. 1, 14–34.

[5] ______, *Tame topology and o-minimal structures*, LMS Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, UK, 1998.

[6] Lou van den Dries and Adam H. Lewenberg, *T-convexity and tame extensions*, Journal of Symbolic Logic **60** (1995), no. 1, 74–102.

[7] Lou van den Dries and Chris Miller, *Geometric categories and o-minimal structures*, Duke Mathematical Journal **84** (1996), no. 2, 497–540.

[8] Ehud Hrushovski and David Kazhdan, *Integration in valued fields*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser, Boston, MA, 2006, math.AG/0510133, pp. 261–405.

[9] Ehud Hrushovski and François Loeser, *Monodromy and the Lefschetz fixed point formula*, arXiv:1111.1954, 2011.

[10] Clint McCrory and Adam Parusiński, *Virtual Betti numbers of real algebraic varieties*, C. R. Math. Acad. Sci. Paris **Ser. I 336** (2003), no. 9, 763–768, arXiv:math/0210374.

[11] Yimu Yin, *Special transformations in algebraically closed valued fields*, Annals of Pure and Applied Logic **161** (2010), no. 12, 1541–1564, arXiv:1006.2467.

[12] ______, *Integration in algebraically closed valued fields*, Annals of Pure and Applied Logic **162** (2011), no. 5, 384–408, arXiv:0809.0473v2.

[13] ______, *Integration in algebraically closed valued fields with sections*, Annals of Pure and Applied Logic **164** (2013), no. 1, 1–29, arXiv:1204.5979v2.

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