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A characterization of n-dependent theories

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1 Introduction

The notion of n-dependent property, a generalization of dependent property (NIP), were introduced by Shelah in [1] (2009). Only few basic properties of the n-dependent property are known, although Shelah showed an interesting result on definable groups for 2-dependent theories [2]. In this article, we show a characterization of n-dependent theories by using counting types over finite sets:

Theorem 1. *Let $\varphi(x, y_1, ..., y_n)$ be an L-formula. φ is n-dependent if and only if there is a constant $\epsilon > 0$ such that $|S_\varphi(\Pi_i A_i)| < 2^{k^{n-\epsilon}}$ for sufficiently large $k \in \omega$ and for all $|A_i| = k$.*

Then we see boolean combinations of n-dependent theories are again n-dependent as a corollary of the characterization. This characterization gives a partial answer for a conjecture on the number of types in n-dependent theories by Shelah in [1]:

Conjecture 2 (S. Shelah). Let $\varphi(x, y_1, ..., y_n)$ be an L-formula and let $m = len(x)$. φ is n-dependent if and only if $|S_\varphi(\Pi_i A_i)| < 2^{mk^{n-1}}$ for all $|A_i| = k$.

(Note that Shelah's conjecture is immediately false where $n = 1$, and you can check that it is also false where $n \neq 1$ with a little discussion. So I think we should replace "mk^{n-1}" by something like "$\beta(\log k)k^{n-1}$" (β depends on n, φ), to make a sense.)

One of the most important results in this article is a generalization of Sauer-Shelah lemma, a famous combinatorial lemma, discussed in section 3. One will notice that the characterization and (generalized) Sauer-Shelah lemma are two sides of the same coin. This report is a partial result of a study with A. Chernikov and D. Palacin on n-dependent theories.

2 Preliminaries

When we discuss on model theoretic topic, we will use ordinal notation in model theory: $\varphi(x)$, $\psi(y)$..., M, N,..., A, B..., are used for formulas, models and subsets of models, except $x, y, ..$ and $a, b, ...$ are used for tuples of variables and elements, respectively. We work under the big model of a complete L-theory T, so every model and set of elements are contained in it.

When we discuss on combinatorial situation, we will use $X, Y, ...$ for (universal) sets, $V, W, ...$ for subsets of $X, Y, ...$ and $v, w, ...$ for elements in $V, W, ...$.

First of all, we give a definition of n-dependence.

Definition 3. 1. Let $\varphi(x, y_1, ..., y_n)$ be an L-formula. The formula φ is said to be <u>n-independent</u> if there are sets A_i $(1 \leq i \leq n)$ such that for every disjoint subsets X and $Y \subset \Pi_i A_i$ there is a tuple b satisfies $\models \bigwedge_{(a_1,...,a_n)\in X} \varphi(b, a_1, ..., a_n) \wedge \bigwedge_{(a_1,...,a_n)\in Y} \neg\varphi(b, a_1, ..., a_n)$. <u>$n$-dependence</u> is defined by the negation of n-independence.

2. Let T be an L-theory. T is said to be n-dependent (or, have n-dependent property) if every formula $\varphi(x, y_1, ..., y_n)$ is n-dependent.

Note that $\varphi(x, y)$ is 1-dependent if and only if $\varphi(x, a)$ is independent for some a. It is immediate that n-dependence implies $(n + 1)$-dependence, so n-dependent property is a generalization of NIP.

Definition 4. Let $\varphi(x, y_1, ..., y_n)$ be an L-formula and A_i $(1 \leq i \leq n)$ a set of parameters. Let $B \subset \Pi_i A_i$.

1. A φ-types over B is a maximal consistent set of formulas $\varphi(x, a_1, ...a_n)$ and $\neg\varphi(x, a_1, ..., a_n)$ with $(a_1, ...a_n) \in B$.

2. $S_\varphi(B)$ is the set of all φ-types over B.

For the proof of the main result, we'll use a graph theoretic fact, as bellow.

Definition 5. Let $n \geq 1$ be a natural number. An n-partite n-hypergarph (V, E) is an n-uniform hypergraph satisfying the following:

- V is a disjoint union of sets V_i $(1 \leq i \leq n)$.
- If $E(v_1, ..., v_n)$ holds then $v_i \in V_i$.

We say (V, E) has size k if $|V_i| = k$ for all i. An n-partite n-hypergraph (V, E) is said to be complete if there is no n-partite n-hypergraph (V, E') with $E' \supsetneq E$. If $n = 1$, the n-hypergraph (V, E) is just a set V and a subset $E \subset V$, and it is complete if $E = V$.

Let G be an n-partite n-hypergraph of size k. If G is complete, then it has k^n edges, and immediately contains copies of complete n-partite n-hypergraphs of size $< k$. The following fact shows that there is ϵ (not depending on the choice of G) such that if G has $k^{n-\epsilon}$ edges then it contains a copy of complete n-partite n-hypergraph of size d.

Fact 6 (Erdös[3]). *Let $d, n > 1$ be natural numbers. Then for sufficiently large $k > n_0$, the following condition holds: Let (V, E) be an n-partite n-hyper graph of size k. If $|E| \geq k^{n-\epsilon}$ with $\epsilon = d^{1-n}$ then (V, E) contains a copy of a complete n-partite n-hypergarph of size d.*

Remark 7. Fact 6 given in [3] doesn't hold where $n = 1$, because $k^{n-\epsilon} = k^0 = 1$. So we replace the lower bound $k^{n-\epsilon}$ by $dk^{n-\epsilon}$, then the fact holds for all $n \geq 1$. This replacement is necessary for our main lemma to include Sauer-Shelah lemma. But it make the inequation in the main lemma more complex.

Our characterization of n-dependent property is related to a combinatorial proposition, called Sauer-Shelah lemma. To explain this lemma, we need to introduce some notions in combinatorics. Most of the following is proved in Hang Q. Ngo's online note [4] and [5].

Definition 8. Let X be a set.

1. A set system $\mathcal{H}$ on X is a subset of the power set $\mathcal{P}(X)$ of X.
2. $\mathcal{H} \cap V := \{W \cap V : W \in \mathcal{H}\}$ for $V \subset X$.
3. We say $V \subset X$ is shuttered by $\mathcal{H}$ if $\mathcal{H} \cap V = \mathcal{P}(V)$.

Definition 9. Let X be an infinite set and $\mathcal{H}$ a set system on X.

1. $\pi_{\mathcal{H}}(k) := \max\{|\mathcal{H} \cap V| : |V| = k\}$. The function $\pi_{\mathcal{H}} : \mathbb{N} \to \mathbb{N}$ is called a shutter function.
2. VC-dimension (Vapnik-Chervonenkis dimension): $VC(\mathcal{H}) = \max\{k : \pi_{\mathcal{H}}(k) = 2^k\}$.

Fact 10 (Sauer-Shelah lemma). *Let $\mathcal{H}$ be a set system on an infinite set X. Suppose that $VC(\mathcal{H}) = d < \infty$. Then for $n > d$,*

$$\pi_{\mathcal{H}}(k) \leq \sum_{i=1}^{d} \binom{k}{i} \leq \left(\frac{ek}{d}\right)^{d} = O(2^{d \log_2(k)}).$$

By using Sauer-Shelah lemma, we have the following:

Fact 11. *Let $\varphi(x, y)$ be an L-formula. $\varphi(x, y)$ is dependent if and only if there is d such that for all $k > d$, $|S_{\varphi}(A)| \leq \left(\frac{ek}{d}\right)^{d} = O(2^{d \log_2(k)})$, where $|A| = k$.*

One of elegant proofs of Sauer-Shelah lemma is given by Shifting technique, as below.

Fact 12. *Let X be a finite set and $\mathcal{H}$ a set system on X. Then we can find a set system $\mathcal{G}$ on X such that*

- $|\mathcal{H}| = |\mathcal{G}|$,
- *if $V \subset X$ is shuttered by $\mathcal{G}$ then V is shuttered by $\mathcal{H}$,*
- *$\mathcal{G}$ is closed under taking subset.*

3 A generalization of Sauer-Shelah lemma

In this section, we prove an inequation like Sauer-Shelah lemma. There may be better bound for our inequation, but still it is useful enough to apply to n-dependent theories.

We'll generalize the notions in the previous section to higher dimension. Suppose $n \geq 1$. Let X_i $(1 \leq i \leq n)$ be sets of size $m \in \omega \cup \{\omega\}$ and let $X = \Pi_i X_i$. Let $\mathcal{H}$ be a set system on X. (Note that $|X| = m^n$, and if X is shuttered by $\mathcal{H}$ then $|\mathcal{H}| = 2^{m^n}$.)

Definition 13. 1. $\pi_{\mathcal{H},n}(k) := \max\{|\mathcal{H} \cap V| : V = \Pi_i V_i, V_i \subset X_i, |V_i| = k\}$.

2. VC$_n$-dimension: $VC_n(\mathcal{H}) = \max\{k : \pi_{\mathcal{H},n}(k) = 2^{k^n}\}$.

Lemma 14 (Main lemma). *1. (precise form) Let $n \geq 1$ and let $VC_n(\mathcal{H}) = d < \infty$. For sufficiently large k, we have*

$$\pi_{\mathcal{H},n}(k) \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left(\frac{ek^n}{D(k)}\right)^{D(k)} = O(2^{D(k)(\epsilon \log_2 k + \log_2(e/(d+1)))}),$$

where $D(k) = (d+1)k^{n-\epsilon} - 1$ and $\epsilon = (d+1)^{1-n}$. Especially, if $n = 1$ then $\epsilon = 1$ and $D(k) = d$, so we have Sauer-Shelah lemma.

2. *(simpler form 1) Let $VC_n(\mathcal{H}) = d < \infty$ and let $\epsilon = (d+1)^{1-n}$. There is β (depends only on d and n) such that $\pi_{\mathcal{H},n}(k) \leq 2^{\beta n^{k-\epsilon} \log k}$ for sufficiently large k.*

3. *(simpler form 2) Let $VC_n(\mathcal{H}) = d < \infty$. There is ϵ' (depends only on d and n) such that $\pi_{\mathcal{H},n}(k) \leq 2^{k-\epsilon'}$ for sufficiently large k.*

Proof. The simpler form is immediately shown from the precise form by taking $\beta > (d+1)\epsilon$ and $\epsilon' < \epsilon$. We'll show the first item. Let $X = \Pi_x X_i$ and $\mathcal{H}$ a set system on X. Let $V_i \subset X_i$ be a set of size k and let $\mathcal{H}_0 = \mathcal{H} \cap V$ with $V = \Pi_i V_i$. We'll check $|\mathcal{H}_0| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon}-1} \binom{k^n}{i}$. By the shifting technique in Fact 12, we can find $\mathcal{G}$ satisfying

- $|\mathcal{H}_0| = |\mathcal{G}|$,
- if $W \subset V$ is shuttered by $\mathcal{G}$ then V is shuttered by $\mathcal{H}_0$,
- $\mathcal{G}$ is closed under taking subset. (Hence if $W \in \mathcal{G}$ then W is shuttered by $\mathcal{G}$.)

Consider any subset $W \subset V$ with $W = \Pi_i W_i$ and $|W_i| = d+1$. Since $VC_n(\mathcal{H}) = d < \infty$, $\mathcal{G}$ cannot contain W, otherwise W is also shuttered by $\mathcal{H}_0$, hence by $\mathcal{H}$, contradicting to the assumption $VC_n(\mathcal{H}) = d$. Take an element $W' \in \mathcal{G}$. Then we have $W \not\subset W'$ because $\mathcal{G}$ is closed under taking subset. We may regards W' as an n-partite n-hypergraph of size k with verticies $V_1 \sqcup ... \sqcup V_n$ and edges W'. Then W' has no complete n-partite n-hyper subgraph of size $d+1$. So, by Fact 6 and Remark 7, the number $|W'|$ of edges must be bounded by $(d+1)k^{n-\epsilon}$ where $\epsilon = (d+1)^{1-n}$. Then we have

$$\mathcal{G} \subset \{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\},$$

and

$$|\mathcal{G}| \leq |\{W' \subset V : |W'| \leq (d+1)k^{n-\epsilon} - 1\}| \leq \sum_{i=0}^{(d+1)k^{n-\epsilon}-1} \binom{k^n}{i}.$$

The rest of the inequation is shown by a general inequation $\sum_{i=0}^{s} \binom{t}{i} \leq (et/s)^s$ for $t > s \in \mathbb{N}$.

□

Note that if $VC_n(\mathcal{H}) = \infty$ then $\pi_{\mathcal{H},n}(k) = 2^{k^n}$ for all k. So we have the following dichotomy:

Corollary 15. *Let $\mathcal{H}$ be a set system on $X = \Pi_{i=1}^n X_i$ with $|X_i| = \omega$. One of the following holds.*

1. $\pi_{\mathcal{H},n}(k) = 2^{k^n}$ *for all k.*

2. *There is $\epsilon' > 0$ such that for sufficiently large k, $\pi_{\mathcal{H},n}(k) < 2^{k^{n-\epsilon'}}$.*

4 Characterizing n-dependent property

First we recall the definition of n-dependent property.

Definition 16. 1. Let $\varphi(x, y_1, ..., y_n)$ be an L-formula. The formula φ is said to be <u>n-independent</u> if there are sets A_i $(1 \leq i \leq n)$ such that for every disjoint subsets X and $Y \subset \Pi_i A_i$ there is a tuple b satisfies $\models \bigwedge_{(a_1,...,a_n)\in X} \varphi(b, a_1, ..., a_n) \wedge \bigwedge_{(a_1,...,a_n)\in Y} \neg\varphi(b, a_1, ..., a_n)$. <u>$n$-dependence</u> is defined by the negation of n-independence.

Let $A = \Pi_i A_i$ be a set of parameters with A_i of size k and let $\varphi(x, y_1, ..., y_n)$ be an L-formula. We want to measure the size of the set $S_\varphi(A)$ of φ-types over A.

Definition 17. Let M be an ω-saturated model of T and let $\varphi(x, y_1, ..., y_n)$ be an L-formula.

1. For $p \in S_\varphi(\Pi_i M^{|y_i|})$, we define $(\Pi M)_p \subset \Pi_i M^{|y_i|}$ by $\{(a_1, ..., a_n) \in \Pi_i M^{|y_i|} : \varphi(x, a_1, ..., a_n) \in p\}$.

2. A set system $\mathcal{H}_\varphi$ on $\Pi_i M^{|y_i|}$ is the set $\{(\Pi M)_p : p \in S_\varphi(\Pi_i M^{|y_i|})\}$.

Remark 18. Let $A \subset \Pi_i M^{|y_i|}$. The following are immediate from the definitions.

1. $M_p \cap A = M_q \cap A$ if and only if $p|A = q|A$.

2. $|\mathcal{H}_\varphi \cap A| = |S_\varphi(A)|$.

3. φ is n-dependent if and only if $VC_n(\mathcal{H}) = d < \infty$.

With the above remark, we can calculate the number of types by counting $\mathcal{H}_\varphi \cap A$.

Theorem 19. *Let $\varphi(x, y_1, ..., y_n)$ be an L-formula. The following are equivalent.*

1. *φ is n-dependent.*

2. *For sufficiently large k, if $A = \Pi_i A_i$ with $|A_i| = k$, then $|S_\varphi(A)| \leq \sum_{i=0}^{D(k)} \binom{k^n}{i} \leq \left(\frac{ek^n}{D(k)}\right)^{D(k)} = O(2^{D(k)(\epsilon \log_2 k + \log_2(e/(d+1)))})$, where $D(k) = (d+1)k^{n-\epsilon} - 1$ and $\epsilon = (d+1)^{1-n}$. Especially, the case $n = 1$ implies the well know characterization of dependent property.*

3. *Let $\epsilon = (d+1)^{1-n}$. There is β such that for sufficiently large k, $|S_\varphi(A)| \leq 2^{\beta k^{n-\epsilon} \log_2 k}$ for all $A = \Pi_i A_i$ with $|A_i| = k$.*

4. *There is ϵ' such that for sufficiently large k, $|S_\varphi(A)| \leq 2^{n^{k-\epsilon}}$ for all all $A = \Pi_i A_i$ with $|A_i| = k$.*

Proof. Immediately shown from Lemma 14 and Remark 18. $\square$

Corollary 20. *n-dependent formulas are closed under taking boolean combinations.*

Proof. Let $\varphi(x, y_1, ..., y_n)$ and $\psi(x, y_1, ..., y_n)$ be n-dependent formulas. By the definition, the negation of n-dependent formula is n-dependent. On the other hand, $|S_{\varphi \wedge \psi}(A)| \leq |S_\varphi(A)| \times |S_\psi(A)| \leq 2^{k^{n-\epsilon'}} \times 2^{k^{n-\epsilon''}} \leq 2^{k^{n-\epsilon'''}}$ for some ϵ', ϵ'' and ϵ'''. So $\varphi \wedge \psi$ is n-dependent. $\square$

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