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# REGARDING 2-CHAINS WITH 1-SHELL BOUNDARIES IN ROSY THEORIES

SUNYOUNG KIM
(JOINT WORK WITH BYUNGHAN KIM AND JUNGUK LEE)
YONSEI UNIVERSITY, KOREA

## 1. INTRODUCTION

In [6], Hrushovski developed connections between amalgamation properties and definable groupoids for a stable theory: if a stable theory $T$ fails the 3-uniqueness property, then there exists a definable groupoid. J. Goodrick and A. Kolesnikov constructed such groupoid in [5]. Furthermore J. Goodrick, B. Kim, and A. Kolesnikov developed homology groups $H_n$ associated to a family of amalgamation functors and computed the group $H_2$ for strong types in stable theories. In particular, they showed that if $T$ has $n$-CA based on $A=\operatorname{acl}(A)$ for $n\geqslant 3$, then $H_{n-2}=0$ for $p\in S(A)$, thus $H_1(p)=0$ holds for any simple $T$.

In this article, we work with amenable families of functors and corresponding homology groups from [3],[4] to show $H_1(p)=0$ holds for a rosy $T$, where $p$ is a Lascar type and classify all the possible 2-chains with a 1-shell boundary in a nontrivial amenable collection of functors.

This article is only intended to present a summary of the results from [7],[8] and we do not include all the details of the proofs.

## BASIC DEFINITIONS

In this section, we recall the basic definitions and facts which are established in [3],[4]. Throughout, $s$ denotes some finite set of natural numbers. A subset $X\subseteq\mathcal{P}(s)$ is called *downward closed* if whenever $u\subseteq v\in X$, then $u\in X$. Then as an ordered (by inclusion) set, $X$ is a category. Before defining an amenable family of functors, we introduce some notations. We fix a category $\mathcal{C}$. Given a functor $f: X\to\mathcal{C}$ and $u\subseteq v\in X$, $f_v^u:=f(\iota_{u,v})\in\operatorname{Mor}_{\mathcal{C}}(f(u),f(v))$ where $\iota_{u,v}$ is the single inclusion map in $\operatorname{Mor}(u,v)$.

**Definition 1.1.** (1) Let $X$ be a downward closed subset of $\mathcal{P}(s)$ and let $t\in X$. The symbol $X|_t$ denotes the set

$$\{u\in\mathcal{P}(s\backslash t)\mid t\cup u\in X\}\subseteq X.$$

(2) For $s,t$, and $X$ as above, let $f: X\to\mathcal{C}$ be a functor. Then the *localization* of $f$ at $t$ is the functor $f|_t: X|_t\to\mathcal{C}$ such that

$$f|_t(v)=f(t\cup v),$$

and $(f|_t)_v^u=f_{v\cup t}^{u\cup t}$, for any $u\subseteq v\in X|_t$.

(3) Let $X\subseteq\mathcal{P}(s)$ and $Y\subseteq\mathcal{P}(t)$ be downward closed subsets, where $s$ and $t$ are finite sets of natural numbers. Let $f: X\to\mathcal{C}$ and $g: Y\to\mathcal{C}$ be functors. We say $g$ is a *permutation* of $f$ if there is a bijection (not necessarily order-preserving) $\sigma: s\to t$ such that $Y=\{\sigma(u): u\in X\}$ and for $v\subseteq w\in Y$, $g(w)=f(\sigma^{-1}(w))$ and $(g)_w^v=f_{\sigma^{-1}(w)}^{\sigma^{-1}(v)}$. In this case we write $g=f\circ\sigma^{-1}$.

We say that $f$ and $g$ are *isomorphic* if there are an order-preserving bijection $\tau: s\to t$ such that $Y=\{\tau(u): u\in X\}$ and a family of morphisms $\{h_u: f(u)\to g(\tau(u))\mid u\in X\}$ from $\operatorname{Mor}(\mathcal{C})$ such that for any $u\subseteq v\in X$,

$$h_v\circ f_v^u=g_{\tau(v)}^{\tau(u)}\circ h_u.$$

For example $f$ and $f\circ\sigma^{-1}$ are isomorphic when $\sigma$ is order-preserving.

**Remark 1.2.** It easily follows that for a downward closed $X \subseteq \mathcal{P}(s)$ and $t \in X$, we have

$$X|_t = X \cap \mathcal{P}(s\backslash t) \text{ iff } X = \bigcup\{\mathcal{P}(u) |\ t \subseteq u \in X\};$$

and in that case $X|_t = \bigcup\{\mathcal{P}(u\backslash t) |\ t \subseteq u \in X\}$.

**Definition 1.3.** Let $\mathcal{A}$ be a non-empty collection of functors $f : X \to \mathcal{C}$ for various non-empty downward-closed subsets $X \subseteq P(s)$ for all finite sets s of natural numbers. We say that $\mathcal{A}$ is *amenable* if it satisfies all of the following properties:

(1) (Closed under isomorphisms and permutations) If $f : X \to \mathcal{C}$ is in $\mathcal{A}$, then every functor $g : Y \to \mathcal{C}$ which is either a permutation of $f$ or is isomorphic to $f$ is also in $\mathcal{A}$.
(2) (Closed under restrictions and unions) Given a functor $f : X \to \mathcal{C}$, $f \in \mathcal{A}$ if and only if for every $u \in X$, we have that $f \upharpoonright \mathcal{P}(u) \in \mathcal{A}$.
(3) (Closed under localizations) Suppose that $f : X \to \mathcal{C}$ is in $\mathcal{A}$. Then for any $t \in X$, $f|_t : X|_t \to \mathcal{C}$ is also in $\mathcal{A}$.
(4) (Extensions of localizations are localizations of extensions.) Let $f : X \to \mathcal{C}$ be in $\mathcal{A}$, and let $t \in X \subseteq \mathcal{P}(s)$ be such that $X|_t = X \cap \mathcal{P}(s\backslash t)$ (see Remark 1.2). Suppose that the localization $f|_t : X|_t \to \mathcal{C}$ has an extension $g : Z \to \mathcal{C}$ in $\mathcal{A}$ for some $(X|_t \subseteq) Z \subseteq \mathcal{P}(s\backslash t)$. Then there is a functor $g_0 : Z_0 \to \mathcal{C}$ in $\mathcal{A}$ such that $Z_0 = \{u \cup v : u \in Z, v \subseteq t\}$, $f \subseteq g_0$, and $g_0|_t = g$.

**Definition 1.4.** Let $B \in \mathrm{Ob}(\mathcal{C})$ and suppose $f(\varnothing) = B$. We say that $f$ is *over* $B$ and we let $\mathcal{A}_B$ denote the set of all functors $f \in \mathcal{A}$ that are over $B$.

Let $\mathcal{A}$ be a non-empty amenable collection of functors mapping into the category $\mathcal{C}$.

**Definition 1.5.** Let $n \geqslant 0$ be a natural number. A (*regular*) *n-simplex* in $\mathcal{C}$ is a functor $f : \mathcal{P}(s) \to \mathcal{C}$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set $s$ is called the *support* of $f$, or supp$(f)$.

Let $S_n(\mathcal{A}; B)$ denote the collection of all regular $n$-simplices in $\mathcal{A}_B$. Then put $S(\mathcal{A}; B) := \bigcup_n S_n(\mathcal{A}; B)$ and $S(\mathcal{A}) := \bigcup_{B \in \mathrm{Ob}(\mathcal{C})} S(\mathcal{A}; B)$.

Let $C_n(\mathcal{A}; B)$ denote the free abelian group generated by $S_n(\mathcal{A}; B)$; its elements are called *n-chains* in $\mathcal{A}_B$, or $n$-chains over $B$. Similarly, we define $C(\mathcal{A}; B) := \bigcup_n C_n(\mathcal{A}; B)$ and $C(\mathcal{A}) := \bigcup_{B \in \mathrm{Ob}(\mathcal{C})} C(\mathcal{A}; B)$.

If $c$ is an $n$-chain in the form $\sum_{1 \leqslant i \leqslant k} n_i f_i$, where the $f_i$'s are distinct $n$-simplices and the $n_i$'s are nonzero integers, then we define the *length* of $c$ as $|c| = |n_1| + \cdots + |n_k|$ and the *support* of $c$ as the union of the supports of $f_i$'s.

Of course $c$ can be sometimes written as $(c+g)-g$, but $|c|$ and the support of $c$ are always uniquely computed in its standard form.

We use $a, b, c, \ldots, f, g, h, \ldots, \alpha, \beta, \ldots$ to denote simplices and chains. Now we will define the boundary operators and the homology groups.

**Definition 1.6.** If $n \geqslant 1$ and $0 \leqslant i \leqslant n$, then the $i$-th *boundary operator* $\partial_n^i : C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$ is defined so that if $f$ is a regular $n$-simplex with domain $\mathcal{P}(s)$ with $s = \{s_0 < \cdots < s_n\}$, then

$$\partial_n^i(f) = f \upharpoonright \mathcal{P}(s\backslash\{s_i\})$$

and extended linearly to a group map on all of $C_n(\mathcal{A}; B)$.

If $n \geqslant 1$ and $0 \leqslant i \leqslant n$, then the boundary map $\partial_n : C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$ is defined by the rule

$$\partial_n(c) = \textstyle\sum_{0 \leqslant i \leqslant n} (-1)^i \partial_n^i(c).$$

We write $\partial^i$ and $\partial$ for $\partial_n^i$ and $\partial_n$, respectively, if $n$ is clear from context.

**Definition 1.7.** The kernel of $\partial_n$ is denoted $Z_n(\mathcal{A}; B)$, and its elements are called $(n-)$*cycles*. The image of $\partial_{n+1}$ in $C_n(\mathcal{A}; B)$ is denoted $B_n(\mathcal{A}; B)$. The elements of $B_n(\mathcal{A}; B)$ are called $(n-)$*boundaries*.

Since $\partial_n \circ \partial_{n+1} = 0$, $B_n(\mathcal{A}; B) \subseteq Z_n(\mathcal{A}; B)$ and we can define simplicial homology groups relative to $\mathcal{A}_B$.

**Definition 1.8.** The $n$th *(simplicial) homology group* of $\mathcal{A}$ over $B$ is

$$H_n(\mathcal{A}; B) := Z_n(\mathcal{A}; B)/B_n(\mathcal{A}; B).$$

**Remark/Definition 1.9.** Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sigma$ induces an automorphism $\sigma^* : C_n(\mathcal{A}, B) \to C_n(\mathcal{A}, B)$ as follows: Let $c = \sum_i n_i f_i \in C_n(\mathcal{A}, B)$, where each $n$-simplex $f_i$ with $s_i := \operatorname{supp}(f_i) = \{s_{i,0} < \cdots < s_{i,n}\}$. Let $\sigma_i := \sigma \restriction s_i$ and let $t_i := \sigma_i(s_i) = \{t_0 < \cdots < t_n\}$. We define

$$\sigma^*(c) := \sum_i n_i (-1)^{|\sigma_i|} f_i \circ \sigma_i^{-1}$$

(see Definition 1.1(3)) with $|\sigma_i| := |\sigma_i'|$ (so $= 0$ or $1$), where $\sigma_i' \in \operatorname{Sym}(n+1)$ such that for $j \leqslant n$, $\sigma_i(s_{i,j}) = t_{\sigma_i'(j)}$.

Now $\sigma^*$ commutes with $\partial$, i.e.,

$$\partial(\sigma^*(c)) = \sigma^*(\partial(c)).$$

This can be inductively shown after verifying when $\sigma$ is a transposition.

Next we define the amalgamation properties. Notice that for $n = \{0, \ldots, n-1\}$, we use $\mathcal{P}^-(n)$ is $\mathcal{P}(n) \setminus \{n\}$.

**Definition 1.10.** Let $\mathcal{A}$ be a non-empty amenable family of functors into a category $\mathcal{C}$ and let $n \geqslant 1$.

(1) $\mathcal{A}$ has *$n$-amalgamation* if for any functor $f : \mathcal{P}^-(n) \to \mathcal{C}, f \in \mathcal{A}$, there is an $(n-1)$-simplex $g \supseteq f$ such that $g \in \mathcal{A}$.

(2) $\mathcal{A}$ has *$n$-complete amalgamation* or $n$-CA if $\mathcal{A}$ has $k$-amalgamation for every $k$ with $1 \leqslant k \leqslant n$.

(3) $\mathcal{A}$ has *strong 2-amalgamation* if whenever $f : \mathcal{P}(s) \to \mathcal{C}$, $g : \mathcal{P}(t) \to \mathcal{C}$ are simplices in $\mathcal{A}$ and $f \restriction \mathcal{P}(s \cap t) = g \restriction \mathcal{P}(s \cap t)$, then $f \cup g$ can be extended to a simplex $h : \mathcal{P}(s \cup t) \to \mathcal{C}$ in $\mathcal{A}$.

**Definition 1.11.** An amenable family of functors $\mathcal{A}$ is called *non-trivial* if it is non-empty and satisfies the strong 2-amalgamation property.

It easily follows that any non-trivial amenable family of functors contains an $n$-simplex for each $n \geqslant 1$. In the rest of the paper, we shall only work with a non-trivial amenable family $\mathcal{A}$ of functors into $\mathcal{C}$.

**Definition 1.12.** If $n \geqslant 1$, an *$n$-shell* is an $n$-chain $c$ of the form

$$\pm \sum_{0 \leqslant i \leqslant n+1} (-1)^i f_i,$$

where $f_0, \ldots, f_{n+1}$ are $n$-simplices such that whenever $0 \leqslant i < j \leqslant n+1$, we have $\partial^i f_j = \partial^{j-1} f_i$.

**Remark/Definition 1.13.** The boundary of an $(n+1)$-simplex is an $n$-shell, and the boundary of any $n$-shell is 0. Note that $\mathcal{A}$ has $(n+2)$-amalgamation iff any $n$-shell is a boundary of an $(n+1)$-simplex. For an $(n+1)$-chain $c$ having an $n$-shell boundary, $|c|$ is always an odd integer.

Now we introduce a weaker notion than 3-amalgamation: $\mathcal{A}$ has *weak 3-amalgamation* over $B$ if any 1-shell over $B$ is the boundary of a 2-chain over $B$ of length $\leqslant 3$.

The details of the following fact and corollaries can be found in [3],[4].

**Fact 1.14.** If $\mathcal{A}$ has $(n+1)$-CA for some $n \geqslant 1$, then

$$H_n(\mathcal{A}; B) = \{[c] : c \text{ is an } n\text{-shell over } B \text{ with support } n+2\}.$$

Since $\mathcal{A}$ already has 2-amalgamation, we have that $H_1(\mathcal{A}; B)$ is trivial iff any 1-shell over $B$ is the boundary of some 2-chain over $B$.

**Corollary 1.15.** Assume that $T$ has $n$-CA over $A = \mathrm{acl}(A)$ for some $n \geqslant 3$. Then $H_{n-2}(p) = 0$ for $p \in S(A)$.

**Corollary 1.16.** If $T$ is simple, then $H_1(p) = 0$ for any complete type $p$ in $T$.

From now on, we work with a large saturated model $\mathcal{M} = \mathcal{M}^{\mathrm{eq}}$ whose theory $T$ is rosy. Recall that $T$ is *rosy* if there is a ternary independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on the small sets of $\mathcal{M}$ satisfying the basic independence properties [1],[2]. We take $\mathop{\smile\!\!\!\!\!\!\mid}$ here as thorn-independence.

Now fix an algebraically closed set $B = \mathrm{acl}(B)$, and let $\mathcal{C}_B$ denote the category of all small subsets of $\mathcal{M}$ containing $B$ and morphisms are elementary maps over $B$ (i.e., fixing $B$ pointwise). For a functor $f : X \to \mathcal{C}_B$ and $u \subseteq v \in X$, we write $f_v^u(u) := f_v^u(f(u)) \subseteq f(v)$. We now fix $p(x) \in S(B)$ where the tuple $x$ may possibly have an infinite arity.

**Definition 1.17.** A *closed independent functor in* $p$ is a functor $f : X \to \mathcal{C}_B$ such that:

(1) $X$ is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$; $f(\varnothing) \supseteq B$; and for $i \in s$, $f(\{i\})$ is of the form $\mathrm{acl}(Cb)$, where $b(\models p)$ is independent with $C = f_{\{i\}}^{\varnothing}(\varnothing)$ over $B$.

(2) For all non-empty $u \in X$, we have
$$f(u) = \mathrm{acl}(B \cup \bigcup_{i \in u} f_u^{\{i\}}(\{i\}));$$
and $\{f_u^{\{i\}}(\{i\}) |\ i \in u\}$ is independent over $f_u^{\varnothing}(\varnothing)$.

Let $\mathcal{A}(p)$ denote all closed independent functors in $p$.

**Fact 1.18.** $\mathcal{A}(p)$ is a non-trivial amenable family of functors.

## 2. Main result : $H_1(p) = 0$ in rosy theories

We have $H_1(p) = 0$ for any Lascar strong type which follows from the fact that Lascar distances are finite in rosy theories. Meanwhile the same holds for a simple $T$ due to 3-amalgamation and Fact 1.14. For given $f : X \to \mathcal{C}_B$ in $\mathcal{A}(p)_B$ (so $f(\varnothing) = B$), and $u = \{i_0 < \cdots < i_k\} \in X$, we write $f(u) = [a_0, \ldots, a_k]$, where $a_j \models p$, $f(u) = \mathrm{acl}(B, a_0 \ldots a_k)$, and $\mathrm{acl}(a_j B) = f_u^{\{i_j\}}(\{i_j\})$. Thus $\{a_0, \ldots, a_k\}$ is independent over $B$.

**Theorem 2.1.** If $B = M$ is a model, then $\mathcal{A}(p)$ has weak 3-amalgamation over $M$. Therefore $H_1(p) = 0$.

**Definition 2.2.** Let a set $B$ and tuples $a$, $b$ be such that $a \equiv_B b$. By *the Lascar distance* over $B$ of $a$ and $b$, denoted by $d_B(a, b)$, we mean the smallest natural number $n$ such that there are tuples $a = a_0, \ldots, a_n = b$, where for each $a_i a_{i+1} (i < n)$ begins some $B$-indiscernible sequence.

**Theorem 2.3.** Suppose that the strong type $p$ is the Lascar (strong) type. Then $H_1(p) = 0$.

*Proof.* For notational simplicity we may assume $B$ to be $\varnothing$. Given a 1-shell $f = a_{12} - a_{02} + a_{01}$ where each $a_{ij} : \mathcal{P}(\{i,j\}) \to \mathcal{C}_B$ is a 1-simplex in $S_1(\mathcal{A}(p))$, we want to find a 2-chain $g$ such that $\partial g = f$. Again there is no harm to assume that $a_{01}(\{1\}) = [b] = a_{12}(\{1\})$ and $a_{02}(\{2\}) = [c] = a_{12}(\{2\})$, and $a_{01}(\{0\}) := [d]$, $a_{02}(\{0\}) := [d']$. By the extension axiom, we can assume that $\{b, c, d, d'\}$ is independent. Let $d, d' \models p$ such that $d(d, d') = n$. So we have $d = d_0, \ldots, d_n = d'$, where $d_i d_{i+1} (i < n)$ begins an indiscernible sequence. Assume that $bc \mathop{\smile\!\!\!\!\!\!\mid}_{dd'} d_i d_{n-1}$ so $bc \mathop{\smile\!\!\!\!\!\!\mid} d_0 \cdots d_n$.

*Claim.* There are $e_i \models p$ $(i < n)$ such that $d_i d_{i+1} \mathop{\smile\!\!\!\!\!\!\mid} e_i$ and $e_i d_i \equiv e_i d_{i+1}$.

*Proof of Claim.* Let $I = \langle d_i d_{i+1} \cdots \rangle$ be an indiscernible sequence having a sufficiently large length. Due to the extension axiom, we can choose $e_i' \equiv d_i$ with $e_i' \mathop{\smile\!\!\!\!\!\!\mid} I$. Since there are only boundedly many types over $e_i'$, one can find $d_j, d_{j'} (j < j')$ with $e_i'dj \equiv e_i'd_{j'}$. Due to the indiscernibility of $I$, there is a map $f$ that maps $d_id_{i+1}$ to $d_jd_{j'}$. Then $e_i := f(e_i')$ satisfies $e_id_i \equiv e_id_{i+1}$ as desired. $\dashv$

Again by extension we suppose $bc \downarrow_{d_id_{i+1}} e_i$, so that each $\{b, d_i, e_i\}, \{b, d_{i+1}, e_i\}$ is independent. Also each $\{b, c, e_{n-1}\}, \{c, d_n, e_{n-1}\}$ is independent $(*)$.

There is $g_0 := g_0^+ - g_0^-$, where $g_0^+, g_0^-$ are 2-simplices with support $u := \{0, 1, 3\}$ such that $g_0^+(u) = [d_0, b, e_0]$ and $g_0^-(u) = [d_1, b, e_0]$; $\partial^0 g_0^+ = \partial^0 g_0^-$; $\partial^1 g_0^+ = \partial^1 g_0^-$ (this follows from the above Claim); and $g_0^+$ extends $a_{01}$ (i.e., $\partial^2 g_0^+ = a_{01}$). Hence $\partial g_0 = a_{01} - \partial^2 g_0^-$.

Similarly, we can find $g_i := g_i^+ - g_i^-$ $(0 < i < n-1)$, where each $g_i^+, g_i^-$ is a 2-simplex with support $u$ such that $g_i^+(u) = [d_i, b, e_i]$ and $g_i^-(u) = [d_{i+1}, b, e_i]$; $\partial^0 g^+ = \partial^0 g^-$; $\partial^1 g^+ = \partial^1 g^-$; and $\partial^2 g_i^+ = \partial^2 g_{i-1}^-$. Therefore we have

$$\partial(g_0 + \cdots + g_{n-2}) = a_{01} - \partial^2 g_{n-2}^-.$$

Put $g_{n-1} := g_{n-1}^+ - a_{023} + a_{123}$, where $a_{j23}$ is a 2-simplex with support $\{j, 2, 3\}$ extending $a_{j2}$ such that $a_{023}(\{0, 2, 3\}) = [d_n, c, e_{n-1}], a_{123}(\{1, 2, 3\}) = [b, c, e_{n-1}]$. Also $g_{n-1}^+$ is a 2-simplex with $g_{n-1}^+(\{0, 1, 3\}) = [d_{n-1}, b, e_{n-1}]$ extending $\partial^2 g_{n-2}^-$. Moreover again by $(*)$, we have $\partial^1 g_{n-1}^+ = \partial^1 a_{023}$. Thus it follows

$$\partial g_{n-1} = \partial^2 g_{n-1}^+ - a_{02} + a_{12} = \partial^2 g_{n-2}^- - a_{02} + a_{12}.$$

Therefore $g := g_0 + \cdots + g_{n-1}$ satisfies $\partial g = f$ as desired. □

## 3. Classification

In this section, we classify 2-chains having 1-shell boundaries using two operations, the crossing operation and the renaming support operation.

**Remark/Definition 3.1.** Suppose that an $n$-chain $c = \sum_i n_i f_i$ is given where each $f_i$ is an $n$-simplex. Assume that $j \in \text{supp}(c) \setminus \text{supp}(\partial(c))$. In this case we say $c$ has a *vanishing support* (in its boundary). Given $k \notin s := \text{supp}(c)$, we let $\sigma$ be a map sending $j$ to $k$ while fixing numbers in $s \setminus \{j\}$. Now as in 1.9, $\partial(\sigma^*(c)) = \sigma^*(\partial(c)) = \partial(c)$.

**Definition 3.2.** (1) *The crossing operation* (or *CR-operation*): Let $\alpha$ and $\beta$ be 2-simplices with $\text{supp}(\alpha) = \{i_0, i_1, i_2\}$, $\text{supp}(\beta) = \{i_1, i_2, i_3\}$ $(i_0 \neq i_3)$ such that $\alpha \restriction \mathcal{P}(\{i_1, i_2\}) = \beta \restriction \mathcal{P}(\{i_1, i_2\}) := \gamma$. Suppose that $\partial(\alpha + \epsilon\beta)$ ($\epsilon = 1$ or $-1$) has no term $\gamma$ (i.e., $\gamma$ is cancelled out). Now by strong 2-amalgamation there is a 3-simplex $\delta$ with $\text{supp}(\delta) = \{i_0, i_1, i_2, i_3\}$ such that $\delta \restriction \mathcal{P}(\{i_0, i_1, i_2\}) = \alpha$ and $\delta \restriction \mathcal{P}(\{i_1, i_2, i_3\}) = \beta$. We take $\alpha' := \delta \restriction \mathcal{P}(\{i_0, i_2, i_3\})$ and $\beta' := \delta \restriction \mathcal{P}(\{i_1, i_2, i_3\})$. Then it follows $\partial(\alpha + \epsilon\beta) = \partial(\alpha' + \epsilon\beta')$. Replacing $\alpha + \epsilon\beta$ by $\alpha' + \epsilon\beta'$ is called the *crossing operation.* Hence from a 2-chain $c$, if we obtain $c'$ by the CR-operation (applied to two terms in $c$) then $\partial(c) = \partial(c')$ and $|c'| \leqslant |c|$.

(2) *The renaming support operation* (or *RS-operation*): This is basically what is described in 3.1 with $n = 2$. So let $c = \sum_i n_i f_i$ ($f_i$ 2-simplices) be a 2-chain having a vanishing support, say $j \in \text{supp}(c) \setminus \text{supp}(\partial(c))$. Let $k \notin \text{supp}(c)$. Then as in Remark/Definition 3.1, we can change the support $j$ to $k$ and replace $c$ by some $c' := \sigma^*(c)$ so that $c$ and $c'$ have the same boundary. This replacement of $c$ by $c'$ is called the RS-operation. In general, if $d'$ is the result of $d$ by applying the RS-operation to a subsummand of $d$, then $\partial(d) = \partial(d')$ and $|d'| \leqslant |d|$.

**Remark/Definition 3.3.** (1) In general, the CR-operation is not symmetric. For example suppose that $c = f_0 - f_1 + f_2$ is given where $f_i$ is a 2-simplex with $\text{supp}(f_i) = \{0, 1, 2, 3\} \setminus \{i\}$ such that $f_i \restriction \mathcal{P}(\{k, 3\}) = f_j \restriction \mathcal{P}(\{k, 3\})$ $(\{i, j, k\} = \{0, 1, 2\})$. Now assume that by the CR-operation, $f_0 - f_1$ is replaced by $f_4 - f_2$ where $\text{supp}(f_4) = \{0, 1, 2\}$ and $\partial f_4 = \partial c$ so that $c$ is replaced by $(f_4 - f_2) + f_2 = f_4$. But $c$ is not obtained from $f_4$ using the CR-operation (unless $f_4$ is redundantly written as $f_4 - f_2 + f_2$).

(2) Now we say a 2-chain $c$ is *proper* if for any $c'$ obtained from $c$ by finitely many applications of the CR or RS-operation (to subsummands), we have $|c| = |c'|$. Among proper 2-chains, now the CR and RS-operations are symmetric. Moreover clearly any 2-chain is reduced to a proper 2-chain by finitely many applications of the two operations.

We call proper 2-chains $c$ and $c'$ are *equivalent* (written $c \sim c'$) if $c'$ is obtained from $c$ by finitely many applications of the CR or RS-operation to some subsummands. Hence if proper $c$ and $c'$ are equivalent then $\partial(c) = \partial(c')$ and $|c| = |c'|$.

Now we are ready to define the notions of two different types of 2-chain having a 1-shell boundary.

**Definition 3.4.** Let $\alpha$ be a 2-chain having a 1-shell boundary.

(1) We call $\alpha$ *renameable type* (or *RN-type*) if a subsummand of $\alpha$ has a vanishing support. If $\alpha$ is not an RN-type 2-chain (so $|\operatorname{supp}(\alpha)| = 3$) we call $\alpha$ *non-renameable* (*NR-*)type.

(2) $\alpha$ is said to be *minimal* if it is proper, and for any proper $\alpha'$ equivalent to $\alpha$ there does not exist a subsummand $\beta$ of $\alpha'$ such that $\partial(\beta) = 0$.

For the notational simplicity, given a simplex $f_i$ with $u = \{j_0, \ldots, j_k\} \subseteq \operatorname{supp}(f_i)$, we write $f_i^{j_0,\ldots,j_k}$ to denote $f_i \restriction \mathcal{P}(u)$. Also given a chain $c = \sum_{i \in I} n_i f_i$ (in its unique form), we write $c^{j_0,\ldots,j_k}$ to denote $\sum_{i\in J} n_i f_i$, where $J := \{i \in I \mid \operatorname{supp}(f_i) = \{j_0, \ldots, j_k\}\}$.

For the rest of this section, we fix a 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\operatorname{supp}(f_{jk}) = \{j < k\}$.

**Definition 3.5.** Let $\alpha$ be a 2-chain having the boundary $f_{12} - f_{02} + f_{01}$. A subchain $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ ($b_i$ 2-simplex) of $\alpha$ is called a *chain-walk in $\alpha$ from $f_{01}$ to $-f_{02}$* if

(1) there are non-zero numbers $k_0, \ldots, k_{m+1}$ (not necessarily distinct) such that $k_0 = 1$, $k_{m+1} = 2$, and for $i \leqslant m$, $\operatorname{supp}(b_i) = \{k_i, k_{i+1}, 0\}$;

(2) each $\epsilon_i \in \{1, -1\}$; $(\partial\epsilon_0 b_0)^{0,1} = f_{01}$, $(\partial\epsilon_m b_m)^{0,2} = -f_{02}$; and

(3) for $0 \leqslant i < m$,

$$(\partial\epsilon_i b_i)^{0,k_{i+1}} + (\partial\epsilon_{i+1} b_{i+1})^{0,k_{i+1}} = 0.$$

Notice that such a representation is sensitive to its order, and a chain-walk can have distinct representations. Unless said otherwise a chain-walk is written in a form of a representation. A subchain of the chain-walk $\beta$ of a form $\beta' := \sum_{i=j}^{m'} \epsilon_i b_i$ for some $0 \leqslant j < m' \leqslant m$ is called a *section* of $\beta$. A chain-walk $\beta$ in $\alpha$ is called *maximal* (in $\alpha$) if it has the maximal possible length. We say $\alpha$ is *centered at* 0 if a (so every) maximal chain-walk in $\alpha$ from $f_{01}$ to $-f_{02}$ is, as a chain, equal to $\alpha$.

Now a *chain-walk in $\alpha$ from $-f_{02}$ to $f_{12}$*, and that $\alpha$ is *centered at* 2, and so on are similarly defined.

**Lemma 3.6.** Let $\alpha$ be a 2-chain with the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Let $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ be a chain-walk in $\alpha$, say from $-f_{02}$ to $f_{12}$. Assume there is a section $\beta' = \sum_{i=j}^{m'} \epsilon_i b_i$ of $\beta$ such that for $\operatorname{supp}(b_i) = \{2, k_i, k_{i+1}\}$, either $k_i \neq k_{m'+1}$ for all $i \in \{j, \ldots, m'\}$; or $k_i \neq k_j$ for all $i \in \{j+1, \ldots, m'+1\}$. Then by finitely many applications of the CR-operation to $\beta'$, we obtain a 2-simplex $c$ with $\operatorname{supp}(c) = \{2, k_j, k_{m'+1}\}$ and $\epsilon = 1$ or $-1$ so that $\beta'' := \sum_{i=0}^{j-1} \epsilon_i b_i + \epsilon c + \sum_{i>m'}^{m} \epsilon_i b_i$ is still a chain-walk from $-f_{02}$ to $f_{12}$.

**Theorem 3.7.** Let $\alpha$ be a minimal 2-chain with the boundary $f_{12} - f_{02} + f_{01}$.

(1) Assume $\alpha$ is of NR-type. Then $|\alpha| = 1$ or $|\alpha| \geqslant 5$. If $|\alpha| \geqslant 5$ then any chain-walk in $\alpha$ from $f_{01}$ to $-f_{02}$ is of the form $\sum_{i=0}^{2n} (-1)^i a_i$ which is as a chain equal to $\alpha$ such that $f_{12} = a_{2j}^{1,2}$ for some $1 \leqslant j \leqslant n-1$.

(2) $\alpha$ is of RN-type iff $\alpha$ is equivalent to a 2-chain

$$\alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n}$$

($n \geqslant 1$) which is a chain-walk from $f_{01}$ to $-f_{02}$ such that $\mathrm{supp}(a_{2n}) = \{0,1,2\}$ and $\partial^0 a_{2n} = f_{12}$, $\partial^1(a_{2n}) = -f_{02}$.

*Proof.* (1) This is easy to check.

(2) Here we give a brief sketch of the left to right.

($\Rightarrow$) Note that $|\alpha| \geqslant 3$.

*Claim 1.* There is $\alpha_1 \sim \alpha$ centered at 2 such that $|\mathrm{supp}(\alpha_1)| > 3$.

*Claim 2.* There is a 2-chain $\alpha_2 \sim \alpha_1$ such that $\alpha_2$ has a 1-simplex term $c$ (with the coefficient 1) such that $\mathrm{supp}(c) = \{0,1,2\}$, and $f_{12} - f_{02} = \partial^0(c) - \partial^1(c)$.

Then let us take a chain-walk $\gamma$ from $f_{01}$ to $-f_{02}$ in $\alpha_2$ terminating with $c$. By repeatedly applying the CR-operations to $\gamma$ (while $c$ unchanged), we obtain a desired $\alpha' \sim \alpha_2$ centered at 0 forming a chain-walk from $f_{01}$ to $-f_{02}$. Then we reorder the representation of the chain-walk $\alpha'$ if necessary. □

The following theorem is proved using the notions of directed graph theory which are not covered in this note.

**Theorem 3.8.** Let $\alpha$ be a minimal 2-chain having the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Then $\alpha$ is equivalent to a 2-chain which is a chain-walk from $f_{01}$ to $-f_{02}$ such that $\mathrm{supp}(\alpha') = \{0,1,2\}$.

In the next section, we explore some of the consequences of this theorem.

## 4. Application : matrix expression

In this section, we introduce the notion of a matrix expression, which determines whether a given minimal 2-chain having a 1-shell boundary is of RN-type.

For the rest, we fix a minimal 2-chain $\alpha$ of length $2n+1$ with the 1-shell boundary $f_{12} - f_{02} + f_{01}$, and $\mathrm{supp}(\alpha) = \{0,1,2\}$. For $\{0,1,2\} = \{i,j,k\}$, $f_i'$ denotes $f_{jk}$ ($j < k$). Fix $I = \{0,1,2\}$ and $J = \{0,\ldots,n\}$. Also, we write $\epsilon a \in \alpha$ to denote that a 2-simplex term $\epsilon a$ is in $\alpha$.

**Definition 4.1.** Let $\sum_{j=0}^{2n}(-1)^j a_j$ be a representation of the given $\alpha$ which is a chain-walk from $f_2'$ to $-f_1'$. By a *matrix expression* of (the representation of) $\alpha$, we mean a function $M : I \times J \to J$ such that

(1) for each $i \in I$, $M \restriction \{i\} \times J : (\{i\}\times) J \to J$ is a permutation of $J$;
(2) for each $i \in I$, $M(i,0)$ is an index such that $f_i' = \partial^i a_{2M(i,0)}$;
(3) for each $i \in I$, $j \in J \backslash \{0\}$, $M(i,j)$ is an index such that $\partial^i a_{2j-1} = \partial^i a_{2M(i,j)}$.

Interpret $M(i,j)$ as an entry $m_{ij}$ of a matrix in the $(i+1)$th row and the $(j+1)$th column, then $M = (m_{ij})_{I,J}$ is a $3 \times (n+1)$ matrix.

Notice that matrix expressions are determined according to the *choices* of pairs of terms which cancel out each other.

**Theorem 4.2.** The following conditions are equivalent:

(1) $\alpha$ is of RN-type.
(2) There is a matrix expression $M$ for a representation $\alpha = \sum_{j=0}^{2n}(-1)^j a_j$ such that for some $0 \leqslant i_0 < i_1 \leqslant 2$, and non-empty $J_0 \subseteq \{1,\ldots,n\}$, $M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)$ as image set under the function $M$.

*Proof.* ($\Rightarrow$) Let $\alpha_1$ be a subchain of $\alpha = \sum_{j=0}^{2n}(-1)^j a_j$ such that $\partial^i\alpha_1 = 0$ for $i \in I\backslash\{i_*\}$ and $|\alpha_1| = 2m$, where $i_* \in \{0,1,2\}$ is a vanishing support. Let $a_{2M(i,j)} \in \alpha_1$ for each $i \in I\backslash\{i_*\}$ and $j \in J_0 := \{j \in J \mid -a_{2j-1} \in \alpha_1\}$. Then here $M$ satisfies Definition 4.1 and $M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)$, where $\{i_0, i_1, i_*\} = I$, as desired.

($\Leftarrow$) Suppose that $M(\{i_0\} \times J_0) = M(\{i_1\} \times J_0)$, say $J_1$, where $J_0 \subseteq \{1, \ldots, n\}$ and $0 \leqslant i_0 < i_1 \leqslant 2$. Let $\alpha_1 := \sum_{j \in J_1} a_{2j} + \sum_{j \in J_0} -a_{2j-1}$, a subsummand of $\alpha$. Then we have $\partial^{i_0}\alpha_1 = \partial^{i_1}\alpha_1 = 0$, so $\alpha_1$ has a vanishing support $i_*$, where $\{i_0, i_1, i_*\} = I$. □

We end this note by stating some consequences of Theorem 4.2 which can be proved by using permutations induced from matrix expressions.

For a matrix expression $M : I \times J \to J$, there is a triple $(\sigma_{01}, \sigma_{12}, \sigma_{02})$ of permutations of $J$ such that $\sigma_{ik}$ is a map sending the $(i+1)$th row to the $(k+1)$th row, i.e., $\sigma_{ik}(m_{ij}) = m_{kj}$ for $j \in J$, and $0 \leqslant i < k \leqslant 2$. Notice that $\sigma_{02} = \sigma_{12} \circ \sigma_{01}$.

**Theorem 4.3.** If $n$ is odd, then $\alpha$ is always of RN-type.

**Theorem 4.4.** Suppose that for $\alpha$ as in Definition 4.1, one of the following holds:

(1) $\partial^\ell a_{2j_0-1} = \partial^\ell a_{2j_1-1}$ for some $0 < j_0 < j_1 \leqslant n$ and $0 \leqslant \ell \leqslant 2$;

(2) $\partial^\ell a_{2j_0} = \partial^\ell a_{2j_1}$ for some $0 \leqslant j_0 < j_1 \leqslant n$ and $0 \leqslant \ell \leqslant 2$.

Then $\alpha$ is of RN-type.

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