数理解析研究所講究録
第 1888 巻 2014 年 83-86

Graph axiom and model companion

Takehiro Naito
Graduate school of Pure and Applied
Sciences, University of Tsukuba

1 Introduction

It is known that several theories have model companions. For example, the field axioms, the linear order axioms and the graph axioms, all have a model companion.
On the other hand, several theories have no model companion :

Theorem 1. [Kikyo and Shelah, 2002][3] If T is a model complete theory with the strict order property, then the theory of the models of T with an automorphism has no model companion.

Theorem 2. [Kikyo, 1997] Axioms of graph containing an automorphism have no model companion.

In this paper, we focus on theories in the graph language, and present two examples, both have no model companion. These two examples are derived from a discussion with A. Tsuboi.

2 Notations and Preliminaries

Before starting, I remark some elementary facts.

Definition 3. Let R be a binary relation. An R-structure G will be called a directed graph (or a digraph in short), if R is irreflexive. A digraph G is called a graph, if it is also symmetric.

We are interested those theories T such that :

1 T is $\forall\exists$-axiomatizable ;

2 T dose not have a model companion.

It is known that such a theory T exist [1]. Here, we will construct an R-theory T extending (di)graph axioms.

Definition 4. Let M be a model of T. M will be called an existentially closed model (EC model in short) of T if, for any $N \models T$ extending M, the following statement holds:

$$N \models \varphi(\bar{a}),\ \bar{a} \in M \Rightarrow M \models \exists \bar{x}\varphi(\bar{x}) \quad (\varphi : L\text{-formula})$$

We use the following fact :

Fact 5. Suppose that an $\forall\exists$-theory T has a model companion S. Then S is characterized by the following property :

$$M \models S \Leftrightarrow M : \text{an EC models of } T.$$

3 Two examples without a model companion

An example of digraph. Let T be the following set of R-sentences :

1 R gives a digraph structure, i.e., $\forall x \neg R(x,x)$
(It is irreflective.)

2 $\forall x \exists y D(x,y) \wedge \forall x \exists y D(y,x)$, where $D(x,y) = R(x,y) \wedge \neg R(y,x)$.
(Every x has a successor and a predecessor.)

3 $\forall xyz[D(x,y) \wedge D(x,z) \to y = z]$, $\forall xyz[D(y,x) \wedge D(z,x) \to y = z]$

4 $\forall x \forall y[D(x,y) \to \forall z(E(x,z) \leftrightarrow E(y,z))]$, where $E(x,y) = R(x,y) \wedge R(y,x)$.

Axioms 1,2 and 3 express that $D(x,y)$ defines a 1-1 function and that an D-orbit of a point forms an infinite line or a cycle. Of course, there is a possibility that the line is a cycle. We can consider that a cycle is a line as extending cyclically. We call those lines D-path. The relation E is a symmetric relation between x and y. If we take x and y on D-path, and x and z have a relation E, then y and z have a relation E.

Cleary T is a consistent $\forall\exists$-theory. For example, let $M = \mathbb{Z}$ and define $R(a,b) \Leftrightarrow b = a+1$. Then (M,R) is a model of T.

Proposition 6. T does not have a model companion.

Proof. Suppose otherwise and let S be a model companion of T. We consider the following set:

$$\begin{aligned}\Gamma(x,y) = \quad & \{\neg D^n(x,y),\ \neg D^n(y,x)\ :\ n \in \omega\} \\ & \cup \{\forall z(E(x,z) \leftrightarrow E(y,z))\},\end{aligned}$$

where $D^n(x,y) = \exists z_1 \cdots z_n(D(x,z_1) \wedge \cdots \wedge D(z_n,y))$.

Claim A. $\Gamma(x,y)$ is inconsistent with S.

If this set is consistent with S, there would be a model $M \models S$ and $a,b \in M$ realizing Γ. Let $N = M \sqcup \mathbb{Z}$ (disjoint union) and let

$$\begin{aligned}R^N = R^M \cup & \{(c,c+1) : c \in \mathbb{Z}\} \\ & \{(a',c),(c,a') : a' \text{ and } a \text{ are connected by a } D\text{-path } ;\ c \in \mathbb{Z}\}.\end{aligned}$$

Then $N \models T$ and $N \models E(a,c) \wedge \neg E(b,c)$. Since M is an EC model of T, there must be $c' \in M$ with $M \models E(a,c') \wedge \neg E(b,c')$. This contradicts the choice of a,b. Thus Γ is inconsistent. (End of proof of claim A)

So there is n such that

$$\bigwedge_{i \le n}(\neg D^i(x,y) \wedge \neg D^i(y,x)) \rightarrow \exists z(E(x,z) \wedge \neg E(y,z)).$$

We can choose $d,e \in M$ such that $D^{n+1}(d,e)$ and $\neg D^i(d,e)$ $(i \le n)$, because

$$\begin{aligned}& N \models D(0,1) \wedge \cdots \wedge D(n+1,n+2) \\ \Rightarrow & \exists d, a_0, \cdots, a_n, e \in M,\ M \models D(d,a_0) \wedge \cdots \wedge D(a_n,e).\end{aligned}$$

Then (d,e) gives a counterexample to the condition 4. Thus, T dose not have a model companion. □

An example of graph. Let T be the following set of R-sentences :

1 Graph axioms;

2 There is no cycle i.e.

$$\neg\exists x_0 \cdots x_n \left(\bigwedge_{i\neq j} x_i \neq x_j \wedge R(x_0, x_1) \wedge \cdots \wedge R(x_n, x_0)\right)$$
$$(n = 1, 2, \cdots)$$

Proposition 7. T does not have a model companion.

Proof. Suppose otherwise and let S be a model companion of T. We consider the following set:

$$\Gamma(x, y) = \{\neg R^n(x, y) \ : \ n \in \omega\}$$

Claim A. $\Gamma(x, y)$ is consistent with S.

If this set is inconsistent, so there is n such that

$$S \models \forall xy(R(x, y) \vee \cdots \vee R^n(x, y)).$$

Let M be a $\aleph_0$-saturated model of S, $M \sqcup \mathbb{Z}$ be the same one as in the proof of Proposition 6. We can choose $d, e \in M$ such that $R^m(d, e)$, where $m(> n)$ is big enough, because M is an EC model of T. On the other hand, we can find a path of length at most n connecting d and e. This means M has a cycle. A contradiction to condition 2. (End of proof of claim A)

Thus there is $a, b \in M$ such that $M \models \Gamma(a, b)$. Let $N = M \sqcup \{c\}$ (disjoint union) and let

$$R^N = R^M \cup \{(a, c),\ (c, a),\ (b, c),\ (c, b)\}$$

Then $N \models T$ and, since M is an EC model of T, there is a c' such that

$$M \models R(a, c') \wedge R(b, c').$$

This contradicts the choice of a, b. Thus, T dose not have a model companion.
□

References

[1] Wilfrid Hodges, Model Theory, Cambridge University Press, 2008.

[2] David Marker, Model Theory : An Introduction, Springer, 2002.

[3] The strict order property and generic automorphisms, Hirotaka Kikyo and Saharon Shelah, Journal of Symbolic Logic, 2002.