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RELATION AMONG DESIGNS ON COMPACT HOMOGENEOUS SPACES

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ABSTRACT. We show that designs on real Grassmannian manifolds can be obtained from a sequence of antipodal spherical designs. Our method is based on a study of relations among designs on compact homogeneous spaces.

1. DESIGNS ON REAL GRASSMANNIAN MANIFOLDS FROM SEQUENCES OF SPHERICAL DESIGNS

Let us denote by

$$\mathcal{G}^{\mathbb{R}}_{m,n} := \{m\text{-dimensional subspaces of real vector space } \mathbb{R}^n\},$$
$$S^d := \{\text{Unit vectors in } \mathbb{R}^{d+1}\}.$$

The purpose of this paper is to show that t-designs on a real Grassmannian manifold $\mathcal{G}^{\mathbb{R}}_{m,n}$ can be obtained from a sequence of antipodal spherical t-designs $Y_1, \dots, Y_{n-1}$ where Y_i is an antipodal spherical t-design on S^i.

The concept of spherical designs on S^d were introduced by Delsarte–Goethals–Seidel [4] in 1977 as follows: For a fixed $t \in \mathbb{N}$, a finite subset X of S^d is called a *spherical t-design on S^d* if

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^d|}\int_{S^d} f d\mu_{S^d} \tag{1.1}$$

for any polynomial f of degree at most t. Note that the left hand side and the right hand side in (1.1) are the averaging values of f on X and that on S^d, respectively. A spherical t-design X on S^d is called *antipodal* if for any $x \in X$, the vector $-x$ is also in X.

We also remark that any $(t+1)$-design on S^d is also a t-design on S^d. The development of spherical designs until 2009 can be found in Bannai–Bannai [2]. We remark that if we define designs on rank one compact symmetric spaces in a similar way to that on sphere, then the theory of designs on rank one compact symmetric spaces are parallel to the theory of spherical designs (see Bannai–Hoggar [3] more details).

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The concept of designs on real Grassmannian manifolds were introduced by Bachoc–Coulangeon–Nebe [1] in 2002. To state the definition of t-designs on $\mathcal{G}^{\mathbb{R}}_{m,n}$, we only need to explain what is our interested functional space on $\mathcal{G}^{\mathbb{R}}_{m,n}$.

Let us denote by $SO(n)$ the special orthogonal group of size n, then the real Grassmannian space $\mathcal{G}^{\mathbb{R}}_{m,n}$ can be regarded as a compact symmetric space $SO(n)/S(O(m) \times O(n-m))$. By Peter–Weyl's theorem, the irreducible decomposition of the L^2-functional space $\mathcal{L}^2(\mathcal{G}^{\mathbb{R}}_{m,n})$ on $\mathcal{G}^{\mathbb{R}}_{m,n}$ can be written by

$$\mathcal{L}^2(\mathcal{G}^{\mathbb{R}}_{m,n}) = \widehat{\bigoplus} H^{\nu}_{m,n}$$

where ν (which describes the highest weight of $H^{\nu}_{m,n}$ as an irreducible $SO(n)$-representation) runs all sequences $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ consists of non-negative even integers with $\nu_1 \geq \cdots \geq \nu_m \geq 0$ (see [1] for more details).

In [1], the definition of t-design on $\mathcal{G}^{\mathbb{R}}_{m,n}$ as follows: a finite subset X of $\mathcal{G}^{\mathbb{R}}_{m,n}$ is called a *t-design on $\mathcal{G}^{\mathbb{R}}_{m,n}$* if

$$\frac{1}{|X|}\sum_{x \in X} f(x) = \frac{1}{|\mathcal{G}^{\mathbb{R}}_{m,n}|}\int_{\mathcal{G}^{\mathbb{R}}_{m,n}} f \, d\mu_{\mathcal{G}^{\mathbb{R}}_{m,n}} \tag{1.2}$$

for any function f in $\bigoplus_{\sum \nu_i \leq t} H^{\nu}_{m,n}$.

In this paper, our concern is in constructions of t-designs on $\mathcal{G}^{\mathbb{R}}_{m,n}$. Let us fix an antipodal spherical t-design Y_i on S^i for each $i = 1, \dots, n-1$. We give an algorithm to construct t-designs on $\mathcal{G}^{\mathbb{R}}_{m,n}$ by $Y_1, \dots, Y_{n-1}$ as follows.

(i) We identify S^1 with $SO(2)$. Then Y_1 is a finite subset of $SO(2)$. Let us put $X_2 := Y_1$ as a finite subset of $SO(2)$.

(ii) Let us fix an isomorphism $SO(3)/SO(2) \simeq S^2$. Then we obtain the fiber bundle $SO(3) \to S^2$. The base space and fiber space of this map are S^2 and $SO(2)$, respectively. We take a "product" X_3 of Y_2 and X_2 in $SO(3)$.

(iii) Repeat the previus step. That is, for finite subsets Y_i of S^i and X_i of $SO(i)$, respectively, we take a product X_{i+1} in $SO(i+1)$ through a fiber bundle $SO(i+1) \to S^i$ (Note that such "a product" is not unique since the fiber bundle is not unique).

(iv) We get a finite subset X_n in $SO(n)$. We can find an isomorphism $SO(n)/S(O(m) \times O(n-m)) \simeq \mathcal{G}^{\mathbb{R}}_{m,n}$ such that the induced surjection $SO(n) \to \mathcal{G}^{\mathbb{R}}_{m,n}$ is $2:1$ on X_n. Then the image of X_n in $\mathcal{G}^{\mathbb{R}}_{m,n}$ is a t-design.

Our method is based on the following two theorems:

- As a product of designs on S^d and $SO(d)$, we have a design on $SO(d+1)$.
- For a center-invariant design on $SO(n)$, we can find a nice projection $SO(n) \to \mathcal{G}^{\mathbb{R}}_{m,n}$ such that the image of the design is a design on $\mathcal{G}^{\mathbb{R}}_{m,n}$, where we say that a finite subset X of $SO(n)$ is center-invariant if $-X$ is also in $SO(n)$ for even n. If n is odd, any finite subset of $SO(n)$ is center-invariant because $SO(n)$ is center-free.

To state our idea, we have to define designs on the special orthogonal group $SO(d)$. We give a definition of designs on compact Lie groups and its homogeneous spaces. Our constructions above is also stated for more general cases.

In this paper, we will state our general results but omit the details of our construction of designs on $\mathcal{G}^{\mathbb{R}}_{m,n}$. The full detail will be reported elsewhere.

2. Relation among designs on compact homogeneous spaces

2.1. Definitions of designs on compact homogeneous spaces. Let G be a compact Lie group and K a closed subgroup of G. We write G/K for the quotient space of G by K. In this subsection, for a finite-dimensional representation (ρ, V) of G, we introduce the concept of *ρ-designs on G/K*.

It is well known that the closed subgroup K of G is also a compact Lie group and the quotient space G/K has the unique C^∞-manifold structure such that the quotient map $G \to G/K$ is a C^∞-submersion. The C^∞-manifold G/K is called a homogeneous space of G by K. Let us denote by $\mu_{G/K}$ the left G-invariant Haar measure on G/K with $\mu_{G/K}(G/K) = 1$. For simplicity, let us put $\Omega := G/K$ and $\mu := \mu_{G/K}$.

Let us put $C^0(\Omega)$ to the space of $\mathbb{C}$-valued continuous functions on $\Omega = G/K$. Note that any continuous function on Ω is L^1-integrable since Ω is compact. For a finite-dimensional complex representation (ρ, V) of G, we shall define subspace $\mathcal{H}^{\rho}_{\Omega}$ of $C^0(\Omega)$ as follows (cf. [7, Chapter I, §1]): Let us denote by $V^\vee$ the dual space of V, i.e. $V^\vee$ is the vector space consisted of all $\mathbb{C}$-linear maps from V to $\mathbb{C}$. We write

$$(V^\vee)^K := \{\, \varphi \in V^\vee \mid \varphi \circ (\rho(k)) = \varphi : V \to \mathbb{C} \text{ for any } k \in K \,\},$$

and define a $\mathbb{C}$-linear map $\Phi : V \otimes (V^\vee)^K \to C^0(G/K)$ by

$$\Phi(v \otimes \psi)(gK) := \langle \rho(g^{-1})v, \psi \rangle \quad \text{for } v \in V,\ \psi \in (V^\vee)^K \text{ and } g \in G.$$

One can observe that $\Phi(v \otimes \psi)$ is well-defined as a C^∞-function on G/K. Thus we define the functional space $\mathcal{H}^\rho_\Omega$ by

$$\mathcal{H}^\rho_\Omega := \Phi(V \otimes (V^\vee)^K). \tag{2.1}$$

We give two easy observations for $\mathcal{H}^\rho_\Omega$ as follows:

Observation 2.1.
- *For two finite-dimensional representations (ρ_1, V_1) and (ρ_2, V_2) of G, we have $\mathcal{H}^{\rho_1 \oplus \rho_2}_\Omega = \mathcal{H}^{\rho_1}_\Omega + \mathcal{H}^{\rho_2}_\Omega$ for $\Omega = G/K$.*
- *If representations (ρ_1, V_1) and (ρ_2, V_2) of G are isomorphic from each other, then $\mathcal{H}^{\rho_1}_\Omega = \mathcal{H}^{\rho_2}_\Omega$ for $\Omega = G/K$. In particular, $\mathcal{H}^{\rho_1 \oplus \rho_2}_\Omega = \mathcal{H}^{\rho_1}_\Omega$.*

For a finite-dimensional complex representation (ρ, V) of G, we define (weighted) ρ-designs on $\Omega = G/K$ as follows:

Definition 2.2. *Let X be a finite subset of Ω and $\lambda : X \to \mathbb{R}_{>0}$ be a positive function on X. We say that (X, λ) is an weighted ρ-design on (Ω, μ) if*

$$\sum_{x \in X} \lambda(x) f(x) = \int_\Omega f d\mu$$

for any $\mathcal{H}^\rho_\Omega$. Furthermore, if λ is constant on X, then X is said to be an ρ-design on (Ω, μ) with respect to the constant λ.

Let us consider the cases where any constant function on Ω is in ρ. Then for any weighted ρ-design (X, λ) on (Ω, μ), we have $\sum_{x \in X} \lambda(x) = 1$. In particular, if X is an ρ-design on (Ω, μ) with respect to a positive constant λ, then $\lambda = \frac{1}{|X|}$.

Remark 2.3. *The concept of ρ-designs on (Ω, μ) is a special cases of averaging sets on a topological finite measure space (Ω, μ) (see* [6] *for the definition of averaging sets). In particular, by* [6, Main Theorem], *if $\Omega = G/K$ is connected, then ρ-designa on (Ω, μ) exists for any ρ.*

We also define multi-ρ-designs on (Ω, μ) as follows. Let us denote by Ω^N the direct product of N-times copies of Ω as a set. For a sequence $X = (x_1, \dots, x_N) \in \Omega^N$, we say that X is a multi-ρ-design on (Ω, μ) with respect to a positive constant λ if

$$\lambda \sum_{i=1}^N f(x_i) = \int_\Omega f d\mu \quad \text{for any } f \in \mathcal{H}.$$

We shall explain that multi-designs can be regard as weighted designs as follows. Let us denote by $\overline{X} = \{x_1, \dots, x_N\} \subset \Omega$. Note that

$|\overline{X}| < N$ if $x_1, \dots x_N$ are not distinct. For each element $\overline{x} \in \overline{X}$, we put

$$m(\overline{x}) := |\{\, i \mid x_i = \overline{x}\,\}|.$$

For any positive constant $\lambda > 0$, we define a positive function $\lambda_{\overline{X}}$ on $\overline{X}$ by

$$\lambda_{\overline{X}} : \overline{X} \to \mathbb{R}_{>0}, \quad \overline{x} \mapsto \lambda \cdot m(\overline{x}).$$

Then by the definition of multi-designs and weighted designs on (Ω, μ), we have the next proposition:

Proposition 2.4. *Let us fix $X \in \Omega^N$, a finite-dimensional G-representation ρ, and a positive constant λ as above. Then the following conditions on (X, ρ, λ) are equivalent:*

(i) *X is a multi-ρ-design on (Ω, μ) with respect to the constant λ.*
(ii) *$(\overline{X}, \lambda_{\overline{X}})$ is an weighted ρ-design on (Ω, μ).*

Let us denote the normalizer of K in G by

$$N_G(K) := \{\, g \in G \mid g^{-1}Kg = K\,\} \subset G.$$

Then $N_G(K)$ is a closed subgroup of G. We consider the right $N_G(K)$-action on $\Omega = G/K$ defined by:

$$\omega h := ghK \quad \text{for any } h \in N_G(K) \text{ and } \omega = gK \in G/K.$$

Then the following fundamental propoaition holds:

Proposition 2.5. *Let (ρ, V) be a finite-dimensional unitary representation of G. If Y is a ρ-design on Ω, then for any $g \in G$ and $h \in N_G(K)$, the subset*

$$gYh := \{\, gyh \mid y \in Y\,\} \subset G/K = \Omega$$

is also a ρ-design on Ω.

2.2. **Results for designs on compact homogeneous spaces.** Throughout this subsection, let us fix a finite-dimensional complex representation (ρ, V) of G. Recall that we defined a functional spaces $\mathcal{H}^{\rho}_{\Omega}$ and ρ-designs on G/K.

We also consider G and K as homogeneous spaces of G and K by the trivial subgroup of these, respectively. Then ρ-designs on G and $\rho|_K$-designs on K are defined in the sense of Definition 2.2. For simplicity, we use the terminology of "ρ-designs on K" for $\rho|_K$-designs on K.

The first main theorem of this section is the following:

Theorem 2.6. *Let Y be an ρ-design on G/K, and Γ an ρ-design on K. We fix a map $s : Y \to G$ such that $\pi \circ s = id_Y$. Let us put*

$$X(Y, s, \Gamma) := \{\, s(y)\gamma \mid y \in Y,\ \gamma \in \Gamma\,\} \subset G.$$

Then $X(Y, s, \Gamma)$ is an ρ-design on G.

Remark 2.7. *Let G be a finite group, K a subgroup of G, and (ρ, V) a finite-dimensional complex representation of G. Then K itself is an ρ-design on K. Thus, by Theorem 2.6, for any ρ-design Y on G/K, the finite subset $X := \pi^{-1}(Y)$ of G is an ρ-design on G. This fact was already proved by T. Ito* [5].

The next corollary followed from Theorem 2.6 immediately:

Corollary 2.8. *For a fixed finite-dimensional complex representation (ρ, V) of G,*

$$N_G(\rho) \leq N_K(\rho) \cdot N_{G/K}(\rho),$$

where $N_\Omega(\rho)$ denotes the smallest cardinality of an ρ-design on Ω.

In the rest of this section, let us suppose $\dim K > 1$. Then the following theorem holds:

Theorem 2.9. *Let $X = (x_1, \dots, x_N) \in G^N$ be a multi-ρ-design on G. Then $Y := (\pi(x_1), \dots, \pi(x_N)) \in (G/K)^N$ is a multi-ρ-design on G/K.*

Hence, we obtain the following corollary, which gives an algorithm to make a ρ-design on G/K from an ρ-design on G with a certain condition:

Corollary 2.10. *Let X be an ρ-design on G and fix $p \in \mathbb{N}$. If $|X \cap \pi^{-1}(\pi(x))| = p$ for any $x \in X$, then $\pi(X)$ is an ρ-design on G/K with $|\pi(X)| = \frac{1}{p}|X|$.*

2.3. **Results for designs on a compact symmetric space.** When the assumption for X in Corollary 2.10? We give a reasonable sufficient condition for X and (G, K) in Theorem 2.14.

Throughout this subsection, we consider the following setting:

Setting 2.11. *G is a connected compact semisimple Lie group. $\tau : G \to G$ is an involutive homeomorphism on G such that* $\operatorname{Lie} G^\tau$ *contains no simple factor of* $\operatorname{Lie} G$*, where $G^\tau := \{ g \in G \mid \tau(g) = g \}$. K is a closed subgroup of G^τ with* $\operatorname{Lie}(K) = \operatorname{Lie}(G^\tau)$.

Then G/K becomes a compact symmetric space with respect to the canonical affine connection on G/K. Note that a connected compact Lie group G is semisimple if and only if the center of $\operatorname{Lie} G$ is trivial.

We denote the center of G by

$$Z_G := \{ g_0 \in G \mid g_0 g {g_0}^{-1} = g \text{ for any } g \in G \}.$$

Let us put

$$Z_K(G) := K \cap Z_G.$$

Since G is semisimple, Z_G and is finite, and hence $Z_K(G)$ too.

Definition 2.12. *Let X be a subset of G. For $p \in \mathbb{N}$, we say that X has p-multiplicity for $Z_K(G)$ if*

$$|X \cap xZ_K(G)| = p \quad \textit{for any } x \in X.$$

Since $x \in xZ_K(G)$ for any $x \in G$, we have $1 \leq |X \cap xZ_K(G)| \leq |Z_K(G)|$ for any subset X of G. Hence, if X has a p-multiplicity for $Z_K(G)$ then $1 \leq p \leq |Z_K(G)|$.

Proposition 2.13. *We consider a symmetric pair (G, K) in Setting 2.11. Let X be a finite subset of G with p-multiplicity for $Z_K(G)$. Then for any open neighberhood U of the unit of G, there exists $g \in U$ such that $|Xg \cap \pi^{-1}(y)| = p$ for any $y \in \pi(Xg)$.*

Recall that by Proposition 2.5, for any ρ-design X on G and any element g of G, the finite subset Xg is also an ρ-design on G. Therefore, by combining Corollary 2.10 with Proposition 2.13, we obtain the next theorem:

Theorem 2.14. *We consider a symmetric pair (G, K) in Setting 2.11 and fix a finite-dimensional complex representation ρ of G. Then for any ρ-design X on G with p-multiplicity for $Z_K(G)$ and any open neighborhood U of the unit of G, there exists $g \in U$ such that $Y := \pi(Xg)$ is an ρ-design on G/K with $|Y| = \frac{1}{p}|X|$.*

By applying Theorem 2.6 for $(G, K) = (SO(d), SO(d-1))$ $(d = 2, \dots, n)$ and Theorem 2.14 for $(SO(n), S(O(m) \times O(n-m)))$ with suitable ρ, we obtain our construction of designs on $\mathcal{G}^{\mathbb{R}}_{m,n}$ in Section 1.

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