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Burgers type equation models on connected graphs and their application to open channel hydraulics

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1 Introduction

Flows in water delivery systems, such as open channel networks, pipe networks and pore structures are described with the cross-sectionally averaged 1-D models. Water wave propagations in open channels in particular have been modeled using the shallow water theory that assumes the incompressibility of fluids and hydrostatic pressure distribution (Szymkiewicz, 2010). In this theory, an open channel network is identified with a connected graph that consists of a finite number of reaches attached via junctions (Bapat, 2010; Yoshioka *et al.*, 2012). The 1-D shallow water equations (1-D SWEs), a system of non-linear hyperbolic partial differential equations (PDEs) describing the balance laws of mass and momentum in the stream direction (Unami and Alam, 2012), have served as one of the most successful models for water flows in open channel networks. As well as the 1-D SWEs, several reduced mathematical models have also been applied to both in theoretical and practical analysis. Major examples are the diffusion wave models and kinematic wave models (Singh, 1996; Yen and Tsai, 2001; Tsai, 2002; Santillana and Dawson, 2010), both of which are derived with neglecting the temporal and/or momentum flux terms in the 1-D SWEs while maintaining the complete mass conservation property. Although the reduced models are not capable of reproducing some important transient phenomena involving discontinues water surface profiles, they are recognized as useful alternatives to the 1-D SWEs because of the simplicity.

This paper focuses on one of the diffusion wave models, the Burgers type equation model (BTE model). The BTE model is a non-degenerate parabolic PDE having a nonlinear advection term. The model is considered as one of the useful minimal models to characterize water wave propagations. Typical dependent variable of the model is water depth or its fluctuation. Motikawa (1957) analyzed propagations of small traveling waves on water surface using a BTE model derived on the basis of the asymptotic expansion of the 1-D SWEs. Yu and Kevorkian (1992) analyzed a BTE model for the dynamics of roll waves in open channels, followed up on by Noble (2007) and Baker *et al.* (2010). Oey (2005) developed a BTE model for water flows in narrow and shallow areas of coastal zones and applied it to numerical analysis of flows involving wet and dry interfaces. Odai *et al.* (2006) and Odai and Kubo (2007) developed an analytical solution method for the BTE models of water depth in inclined channels with uniform rectangular cross-sections utilizing the Cole-Hopf transformation (Hopf, 1950; Cole, 1951). Application of the Cole-Hopf transformation to a BTE model leads to the heat equation whose analytical solution is available for simplified cases (Salsa, 2009). Nasseri and Attarnejad, (2010) developed a variational method to solve a class of nonlinear PDEs including a BTE model.

Many researches have been carried out for the BTE models in single open channels based on the well-established 1-D framework. However, no approach has been made for those in open channel networks due to the difficulties in handling singularities encountered at junctions. Nevertheless, some researches discussed similar BTE models on connected graphs. Bressloff *et al.* (1997) developed a nonlinear parabolic PDE of road traffic dynamics whose resolution is reduced to solving a BTE model on a connected graph. They transformed the model to an easily solved integro-differential equation. The authors numerically solved the BTE models on connected graphs using FEMs (Yoshioka *et al.*, 2013a-b). Since typical water delivery system consists of a number of reaches presenting a network structure, to reveal mathematical properties of the BTE models on connected graphs contributes to improving understandings of the water wave propagations in the domains.

The objective of this paper is to carry out mathematical and numerical analyses of a BTE model on connected graphs. The mathematical analysis focuses on the model on a star-shaped connected graph defined later. A weak formulation of the model that consistently and implicitly takes an internal boundary condition (IBC) into its formulation is introduced. Unique solvability of steady and unsteady problems of the model is proven under certain constraints. An energy estimate and a maximum principle are presented for unsteady problems. The numerical analysis is carried out to further investigate behaviour of solutions to the model.

2 Preliminaries

2.1 Locally one-dimensional open channel network

Analysis of transport phenomena in open channel networks is typically based on PDEs defined on a connected graph extending over the 3-D space or the horizontal 2-D plane. A connected graph consists of a collection of a finite number of reaches attached via volume-less nodes. A reach is identified with a Jordan curve

(Von Below, 1986). A node is a point that represents an intersection of reaches or an end point: here the former is referred to as a junction and the latter as a boundary. This paper focuses on a connected graph that consists of a finite number of straight reaches meeting at a junction J (**Figure 1**), which is hereafter referred to as a star graph Ω. The junction attaches m inflowing reaches (R_1 through R_m) and n outflowing reaches (R_{m+1} through R_{m+n}). The i th reach of Ω is denoted by R_i. The length of R_i is $L_i < \infty$. A 1-D abscissa is defined in a reach and that in R_i is denoted by x_i. The reach R_i is thus identified with the 1-D interval $(0, L_i)$. The junction J can be regarded as the downstream-ends of the inflowing reaches ($x_i \to L_i - 0$ for $1 \le i \le m$) as well as the upstream-ends (origins) of the outflowing reaches ($x_i \to +0$ for $m+1 \le i \le m+n$). The union set of the reaches of Ω is denoted by $\Omega_\mathrm{R} = \bigcup_{i=1}^{m+n} \mathrm{R}_i$. The union set of the upstream boundaries of the inflowing reaches of Ω is denoted by Γ_I and that of the downstream boundaries of the outflowing reaches by Γ_O. The boundary Γ of Ω is therefore decomposed as $\Gamma = \Gamma_\mathrm{I} \cup \Gamma_\mathrm{O}$.

2.2 Functional settings

This section defines the functional settings used in in this paper. Let $C^0(\Omega)$ be the set of continuous function in the star graph Ω as

$$C^0(\Omega) = \left\{ u \middle| u \in \prod_{i=1}^{m+n} C^0(\mathrm{R}_i),\ u_{x_{i_1} \to L_{i_1} - 0} = u_{x_{i_2} \to +0} = u_\mathrm{J}\ (1 \le i_1 \le m,\ m+1 \le i_2 \le m+n) \right\} \tag{1}$$

where the subscript J represents the value at the junction. Denote the usual Sobolev space in Ω_R by $L^p(\Omega_\mathrm{R})$ ($1 \le p < \infty$) equipped with the norm

$$\|u\|_{L^p} = \left(\sum_{i=1}^{m+n} \int_{\mathrm{R}_i} u^p \mathrm{d}x_i \right)^{\frac{1}{p}}. \tag{2}$$

The space $L^\infty(\Omega_\mathrm{R})$ is accordingly defined with the norm

$$\|u\|_{L^\infty} = \operatorname{ess\,sup} \left\{ |u| \middle| u \in L^\infty(\Omega_\mathrm{R}),\ x \in \Omega_\mathrm{R} \right\}, \tag{3}$$

and for a continuous function $u \in C^0(\Omega)$ which can be replaced by

$$\|u\|_{L^\infty} = \operatorname{ess\,sup} \left\{ |u| \middle| u \in L^\infty(\Omega),\ x \in \Omega \right\}. \tag{4}$$

Let the usual H^1 Hilbert space in Ω_R be $H^1(\Omega_\mathrm{R})$ equipped with the norm

$$\|u\|_{H^1} = \sqrt{\sum_{i=1}^{m+n} \int_{\mathrm{R}_i} u^2 \mathrm{d}x_i + \sum_{i=1}^{m+n} \int_{\mathrm{R}_i} \left(\frac{\partial u}{\partial x_i} \right)^2 \mathrm{d}x_i} = \sqrt{\langle u, u \rangle + \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right\rangle} \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega_\mathrm{R})$. Let $X(\Omega)$ be the intersection space $C^0(\Omega) \cap H(\Omega_\mathrm{R})$. Closure of $X(\Omega)$ in the space of infinitely differentiable functions $C_0^\infty(\Omega_\mathrm{R})$ is defined as

$$X_0(\Omega) = \left\{ u \middle| u \in X(\Omega),\ u_\Gamma = 0 \right\}. \tag{6}$$

The space of the functions of $X(\Omega)$ that vanishes on Γ_I is denoted by $X_1(\Omega)$. The spaces $X(\Omega)$, $X_0(\Omega)$ and $X_1(\Omega)$ are Hilbert spaces equipped with the norm (5) (Mugnolo, 2007). The space $X_0(\Omega)$ is identified with its dual $X_0^{-1}(\Omega)$ in this paper. The trace theorem for functions in finite 1-D intervals shows that the value u_J at the junction for $u \in X(\Omega)$ is justified as a trace in an L^2 sense. There exists a positive coefficient C_G that satisfies the Gagliardo-Nirenberg inequality (Mugnolo, 2007;

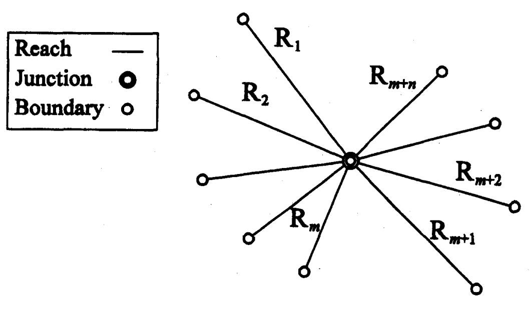

Figure 1. Schematic sketch of the star graph Ω.

Berkolaiko and Kuchment, 2012)

$$\|u\|_{L^\infty} \le \sqrt{C_G}\,\|u\|_{H^1} \tag{7}$$

with $C_G > 0$. Let $L^p(0,T;H)$ with a finite $T > 0$ denote the space of temporally L^p class functions from $(0,T)$ into a Hilbert space H. The space $L^p(0,T;H)$ $(1 \le p < \infty)$ is equipped with the norm

$$\|u\|_{L^p(0,T;H)} = \left(\int_0^T \|u\|_H^p \,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{8}$$

Similarly, the space $L^\infty(0,T;H)$ is equipped with the norm

$$\|u\|_{L^\infty(0,T;H)} = \operatorname{ess\,sup}\left\{\|u\|_H \,\middle|\, t \in (0,t)\right\}. \tag{9}$$

3 Burgers type equation (BTE) model

3.1 Model description

Water wave propagations in open channels are reasonably characterized with a BTE model, a parabolic PDE having a nonlinear advection term. Typical form of the model is

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{M+1}Vh^{M+1} - D\frac{\partial h}{\partial x}\right) = \frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{10}$$

with the unit width discharge of water q defined as

$$q = q(h) = \frac{1}{M+1}Vh^{M+1} - D\frac{\partial h}{\partial x} \tag{11}$$

where the dependent variable $h = h(t,x)$ represents the water depth its fluctuation, $V > 0$ and $D > 0$ are the model parameters assumed as reach-distributed constant and $M \ge 0$ is another model parameter related with friction laws (Singh, 1996). For example, $M = 1$ in Onizuka and Odai (1998), and $M = 0.666$ in Mizumura (2010). In this paper, these parameters are assumed to be bounded as in the literatures. The BTE model (10) with the particular choice of $M = 0$ loses the nonlinearity and is regarded as a solute transport equation of a contaminant in which h, V and D are understood as the concentration of the contaminant, the fluid velocity and the dispersion coefficient, respectively (Yoshioka *et al.*, in press).

3.2 Internal boundary condition (IBC)

A major mathematical and numerical difficulty in solving the BTE model (10) on a connected graph is the existence of junctions, which require the use of appropriate IBCs so that the problem is well-posed. The IBCs are also referred to as the Kirchhoff's conditions or the transmissive conditions in the literatures (Lumer, 1980; Pokornyi and Borovskikh, 2004). Influences of the IBCs on properties of solutions to PDEs on connected graphs have extensively been studied, in particular for the spectral theory (Carlson, 2009), solvability and multiplicity theory (Von Below, 1986; Lubary, 1998), semi-group theory (Mugnolo, 2007) and relations with stochastic processes (Friedlin and Sheu, 2000). The authors used an analytical approach for parabolic PDEs on connected graphs based on the weak forms that implicitly incorporate the IBCs to investigate mathematical properties of the PDEs and to develop efficient numerical methods for solving them (Yoshioka *et al.*, 2012).

In this paper, a similar analytical method is presented to deal with the BTE model (10) consistently on the star graph Ω. The model (10) on Ω shall be understood as a weak form so that the junction J in Ω is consistently dealt with. The weak form of (10) is given by

$$\sum_{i=1}^{m+n}\int_{\mathrm{R}_i}\left[w\frac{\partial h}{\partial t} - \frac{\partial w}{\partial x_i}\left(\frac{1}{M+1}V_i h^{M+1} - D_i\frac{\partial h}{\partial x_i}\right)\right]\mathrm{d}x_i = \sum_{i=1}^{m+n}\int_{\mathrm{R}_i}\left(w\frac{\partial h}{\partial t} - \frac{\partial w}{\partial x_i}q_i(h)\right)\mathrm{d}x_i = 0 \tag{12}$$

where the value of h is directly specified on Γ_{I} (Dirichlet boundary condition) and the free outflow condition

$$D_i\frac{\partial h}{\partial x_i} = 0 \tag{13}$$

is assumed on Γ_{O} (Neumann boundary condition). No boundary term is embodied in (12). Hereafter, the weak form (12) is regarded as the BTE model. According to Cecchi *et al.* (1996) and Clark *et al.* (2011), the 1-D counterpart of (12) is well-posed and has a unique weak solution with H^1 regularity. The parameters $V_i > 0$ and $D_i > 0$ are constant in each reach, and M is a constant in the entire Ω. It is assumed that V_i satisfies the constraint

$$\Delta V = \sum_{j=m+1}^{m+n} V_i - \sum_{j=1}^{m} V_i = 0, \tag{14}$$

which is understood as a balance law of V_i around the junction J. The constraint (14) for the linear case of contaminant transport ($M = 0$) means physically that mass conservation of water in terms of the discharges is satisfied at J (Oppenheimer, 2000; Yoshioka *et al.*, in press). The constraint (14) is essential in order to guarantee the energy estimate and the maximum principle of the BTE model as shown in the later sections.

The BTE model (12) implicitly assumes an IBC at the junction J. The IBC is not embodied in (12) and is referred to as the implicit IBC (Yoshioka *et al.*, 2013), which is equivalent to the conventional ones for the solution h if its sufficient regularity is guaranteed. By (14), a representation formula for the IBC is obtained as

$$\sum_{i=1}^{m} q_i(h)\Big|_{x_i \to L_i - 0} - \sum_{i=m+1}^{m+n} q_i(h)\Big|_{x_i \to +0} = \sum_{i=1}^{m} D_i \frac{\partial h}{\partial x_i}\bigg|_{x_i \to L_i - 0} - \sum_{i=m+1}^{m+n} D_i \frac{\partial h}{\partial x_i}\bigg|_{x_i \to +0} = 0. \tag{15}$$

The IBC (15) describes a mass conservation law of water at the junction J for a nonlinear case ($M \geq 1$) and that of a contaminant for the linear case ($M = 0$). Each partial derivatives in (15) is understood in the sense of a trace because the space of differentiable functions $C^1\left(\overline{\mathrm{R}_i}\right)$ is dense in $H^1\left(\mathrm{R}_i\right)$ (Salsa, 2009). The IBC (15) is satisfied in a strong sense for the solution h in $H^2\left(\Omega_{\mathrm{R}}\right)$.

4 Solvability of BTE model on connected graph

The objective of this section is to prove unique existence of the weak solution h under the homogenous Dirichlet boundary conditions. The parameter M is assumed to equal to or larger than 1. The proofs presented in this section can also be applied to the problems with other boundary conditions, such as the homogenous Neumann boundary condition (13). Here, (12) is rewritten in the abstract form

$$\left\langle w, \frac{\partial}{\partial t} h \right\rangle + a(w,h) + b(w,h,h) = 0 \tag{16}$$

with the bi-linear form

$$a(w,h) = \sum_{i=1}^{m+n} \int_{\mathrm{R}_i} D_i \frac{\partial w}{\partial x_i} \frac{\partial h}{\partial x_i} \mathrm{d}x_i \tag{17}$$

and the (non-linear) operator form

$$b(w,u,v) = -\sum_{i=1}^{m+n} \int_{\mathrm{R}_i} \frac{1}{M+1} V_i \frac{\partial w}{\partial x_i} u^M v \mathrm{d}x_i\ . \tag{18}$$

Here also considers the steady counterpart

$$a(w,h) + b(w,h,h) = \int_\Omega w f \mathrm{d}x = \langle w, f \rangle \tag{19}$$

with a source f, which is independent of the solution h.

4.1 Steady problem

This section proves unique solvability of the steady problem (19) for $h \in X_0(\Omega)$ with $w \in X_0(\Omega)$. The proofs presented in this section are inspired by Boules (1990) who presented unique solvability of a BTE model ($M = 1$) in a 1-D interval. Define the upper and lower bounds of V_i as

$$\overline{V} = \max_{1 \leq i \leq m+n} V_i \ \text{ and } \ \underline{V} = \min_{1 \leq i \leq m+n} V_i, \tag{20}$$

respectively. Similarly, define the upper and lower bounds of D_i as

$$\overline{D} = \max_{1 \leq i \leq m+n} D_i \ \text{ and } \ \underline{D} = \min_{1 \leq i \leq m+n} D_i, \tag{21}$$

respectively. Several important properties of the bi-linear form a and the operator form b are presented. The bi-linear form a satisfies

$$|a(w,h)| = \left| \sum_{i=1}^{m+n} \int_{\mathrm{R}_i} D_i \frac{\partial w}{\partial x_i} \frac{\partial h}{\partial x_i} \mathrm{d}x_i \right| \leq \overline{D} \left\| \frac{\partial w}{\partial x} \right\|_{L^2} \left\| \frac{\partial h}{\partial x} \right\|_{L^2} \leq \overline{D} \|w\|_{H^1} \|h\|_{H^1} \tag{22}$$

and

$$a(u,u)=\sum_{i=1}^{m+n}\int_{\mathrm{R}_i} D_i\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x_i \geq \underline{D}\left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 \geq \alpha\|u\|_{H^1}^2 \tag{23}$$

for $u\in X(\Omega)$ with a some positive constant α, showing that it is bounded and coercive (Mugnoro, 2007). The operator form b is also bounded for $u\in X_0(\Omega)$. In fact, according to the Sobolev embedding theorem (Tartar, 2000), the inequality

$$|b(w,u,v)|=\left|\sum_{i=1}^{m+n}\int_{\mathrm{R}_i} V_i\frac{\partial w}{\partial x_i}u^M v\mathrm{d}x_i\right| \leq \beta\|w\|_{H^1}\|u\|_{H^1}^M\|v\|_{H^1} \tag{24}$$

with $\beta=\overline{V}(C_G)^{\frac{M}{2}}>0$ holds. Define the operator $\lambda(u,v)\in X_0(\Omega)$ via the inner product

$$b(w,u,v)=\langle\lambda(u,v),w\rangle, \tag{25}$$

and denote $\lambda(u,u)$ by $\lambda(u)$. The norm of $\lambda(u)$ is given by

$$\|\lambda(u)\|=\sup\left\{\frac{\langle\lambda(u),w\rangle}{\|w\|_{H^1}}\middle| w\in X_0(\Omega),\ w\neq 0\right\}\leq\overline{V}(C_G)^{\frac{M}{2}}\|u\|_{H^1}^M=\beta\|u\|_{H^1}^M, \tag{26}$$

showing that $\lambda(u)$ is bounded for $u\in X_0(\Omega)$. Furthermore, $\lambda(u)$ is continuous. In fact, for $u_1,u_2\in X_0(\Omega)$ the inequality

$$\|\lambda(u_1)-\lambda(u_2)\|=\sup\left\{\frac{\langle\lambda(u_1)-\lambda(u_2),w\rangle}{\|w\|_{H^1}}\middle| w\in X_0(\Omega),\ w\neq 0\right\}\leq\overline{V}\sum_{j=1}^{M-1}\|u_1\|_{L^\infty}^j\|u_2\|_{L^\infty}^{M-j-1}\|u_1-u_2\|_{L^\infty} \tag{27}$$

is satisfied because $\overline{V}\sum_{j=1}^{M-1}\|u_1\|_{L^\infty}^j\|u_2\|_{L^\infty}^{M-j-1}$ is bounded.

Before starting the main proof, a uni-variate function $g=g(r)$ with $r\geq 0$ and M, defined by

$$g(r)=r\left(\alpha-\beta r^M\right) \tag{28}$$

is introduced. The function g satisfies

$$g(0)=g(r_0)=0 \text{ with } r_0=\left(\frac{\alpha}{\beta}\right)^{\frac{1}{M}}, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}g(r_1)=0 \text{ with } r_1=\left(\frac{\alpha}{\beta(M+1)}\right)^{\frac{1}{M}} \text{ and } g(r_1)=\frac{M}{\beta^{\frac{1}{M}}}\left(\frac{\alpha}{M+1}\right)^{\frac{M+1}{M}}. \tag{30}$$

The function g is strictly positive in $(0,r_0)$ and attains its maximal at $r=r_1<r_0$. g monotonically increases and decreases in $(0,r_1)$ and in (r_1,r_0), respectively. It follows that the equation

$$r\left(\alpha-\beta r^M\right)=\overline{g} \tag{31}$$

with the constant $\overline{g}$ satisfying

$$0<\overline{g}<g(r_1). \tag{32}$$

has two positive solutions r_{m} and r_{M} such that

$$0<r_{\mathrm{m}}<r_1<r_{\mathrm{M}}<r_0. \tag{33}$$

Here firstly proves the following theorem.

Theorem 1. *There exists a unique solution $u\in X_0(\Omega)$ to the linear problem*

$$a(w,v)+b(w,h,v)=\langle w,f\rangle \tag{34}$$

with a fixed $h\in X_0(\Omega)$ such that the inequality

$$\|h\|_{H^1} < r_{\rm m} \tag{35}$$

holds where $w \in X_0(\Omega)$ *is the weight,* $f \in X(\Omega)$ *is a sufficiently regular source term such that*

$$\|f\| = \overline{g} = \sup\left\{\frac{|\langle f, w\rangle|}{\|w\|_{L^2}}\middle| w \in X_0(\Omega),\ w \neq 0\right\} \leq g(r_{\rm m}) \tag{36}$$

with

$$K = \frac{\beta M r_{\rm m}^{M-1}\|f\|}{\left(\alpha - \beta r_{\rm m}^M\right)^2} < 1. \tag{37}$$

Note that $r_{\rm m}$ *depends only on* f, α, β *and* M.

Proof of Theorem 1. The set

$$D(r_{\rm m}) = \left\{h \middle| h \in X_0(\Omega),\ \|h\|_{H^1} < r_{\rm m}\right\} \tag{38}$$

is a compact, convex subset of $X_0(\Omega)$. By (23) and (24), the left hand side of (34) is bounded from zero as

$$a(v,v) + b(v,h,v) \geq \alpha\|v\|_{H^1}^2 - \beta\|v\|_{H^1}^2\|h\|_{H^1}^M = \left(\alpha - \beta\|h\|_{H^1}^M\right)\|v\|_{H^1}^2 \geq \left(\alpha - \beta r_{\rm m}^M\right)\|v\|_{H^1}^2, \tag{39}$$

showing that it is coercive. Application of the Lax-Milgram Theorem (Atkinson and Han, 2009) to (34) leads to that the solution $v \in X_0(\Omega)$ exists and is uniquely determined.

Here secondly proves unique solvability of (19) under the stated conditions.

Theorem 2. (19) *has a unique solution* $h \in X_0(\Omega)$ *under the stated conditions.*

Proof of Theorem 2. Denote $\Phi(h)$ by the map identified with the inverse of the operator A_h, which is defined via the inner product as

$$\langle A_h v, w\rangle = a(w,v) + b(w,h,v), \tag{40}$$

namely,

$$\Phi(h) = (A_h)^{-1} f. \tag{41}$$

The operator norm of A_h satisfies

$$\|A_h v\| = \sup\left\{\frac{|a(w,v)+b(w,h,v)|}{\|w\|_{H^1}}\middle| w \in X(\Omega),\ w \neq 0\right\} \geq \frac{a(v,v) - |b(v,h,v)|}{\|v\|_{H^1}} \geq \left(\alpha - \beta r_{\rm m}^M\right)\|v\|_{H^1}. \tag{42}$$

Since A_h is bijective by the definition, it is an open map. Application of the open-mapping theorem (Okamoto and Nakamura, 1997a) to A_h yields the estimate

$$\left\|(A_h)^{-1}\right\| \leq \frac{1}{\alpha - \beta r_{\rm m}^M}, \tag{43}$$

which further leads to

$$\left\|(A_{h_1})^{-1} - (A_{h_2})^{-1}\right\| \leq \frac{1}{\left(\alpha - \beta r_{\rm m}^M\right)^2}\left\|A_{h_1} - A_{h_2}\right\|. \tag{44}$$

Application of a differential formula to A_h yields

$$\begin{aligned}\left\|A_{h_1} - A_{h_2}\right\| &= \sup\left\{\frac{|a(w,v)+b(w,h_1,v)-a(w,v)-b(w,h_2,v)|}{\|v\|_{H^1}\|w\|_{H^1}}\middle| v,w \in X(\Omega),\ v,w \neq 0\right\} \\ &\leq \overline{V} C_{\rm G}^{\frac{1}{2}} \sum_{j=0}^{M-1}\left\|h_1^j h_2^{M-j-1}\right\|_{L^\infty}\|h_1 - h_2\|_{H^1}\end{aligned}. \tag{45}$$

By (35), (45) results in

$$\left\|A_{h_1} - A_{h_2}\right\| \leq \beta M r_{\rm m}^{M-1}\|h_1 - h_2\|_{H^1}, \tag{46}$$

showing that A_h is continuous. Consequently, Φ is a contraction map that maps $X(\Omega)$ onto $X(\Omega)$. This is because, by (36) and (39), $\|v\|_{H^1}$ is bounded from away as

$$\|v\|_{H^1} \le \frac{\|f\|^*}{\alpha - \beta r_{\mathrm{m}}^M} \le \frac{g(r_{\mathrm{m}})}{\alpha - \beta r_{\mathrm{m}}^M} = r_{\mathrm{m}} \tag{47}$$

and by (37), (44) and (45), Φ satisfies the inequality

$$\|\Phi(h_1) - \Phi(h_2)\| < \|h_1 - h_2\|_{H^1}. \tag{48}$$

According to the Leray-Schauder fixed point theorem (Okamoto and Nakamura, 1997b), the solution $h \in X_0(\Omega)$ to (19) exists and is uniquely determined under the stated conditions.

Here finally proves the following theorem.

Theorem 3. *(19) does not have any solutions such that*

$$r_{\mathrm{m}} < \|h\|_{H^1} < r_{\mathrm{M}}. \tag{49}$$

Proof of Theorem 3. Substituting $w = h$ into (19) yields

$$a(h,h) + b(h,h,h) = \langle w, f \rangle, \tag{50}$$

leading to the inequality

$$\left(\alpha - \beta \|h\|_{H^1}^M\right)\|h\|_{H^1} \le \|f\|^*. \tag{51}$$

Substituting (36) into (51) yields

$$\left(\alpha - \beta \|h\|_{H^1}^M\right)\|h\|_{H^1} \le g(r_{\mathrm{m}}). \tag{52}$$

Since r_{m} and r_{M} are the solutions to (31), the assumption (49) contradicts with the inequality (52) showing that the statement is true.

It can be shown in an essentially similar way that the solution $h \in X_0(\Omega)$ to the weak form

$$a(w,h) + b'(w,h,h) + c(w,h) + d\langle w,h \rangle = \langle w, f \rangle \tag{53}$$

with a positive constant d and the operator form

$$b'(w,u,v) = -\sum_{i=1}^{m+n} \int_{\mathrm{R}_i} \frac{\partial w}{\partial x_i} \left(\frac{1}{j+1} \sum_{j=1}^{M} V_{i,j} u^j \right) v \mathrm{d}x_i \tag{54}$$

with a positive and bounded sequence $V_{i,j}$, has a unique solution $h \in X_0(\Omega)$ such that $\|h\|_{H^1}^2 < M^{-1} r_{\mathrm{m}}^M$. Note that the analysis carried out in this section does not assume the constraint (14), which on the other hand serves as a crucial condition that determines properties of solutions to the unsteady BTE model.

4.2 Unsteady problem

4.2.1 Energy estimate

Energy estimate of the problem (16) is obtained using the Faedo-Galerkin method, which ensures the global unique existence of the solution h.

Theorem 4. *The energy estimate of (16) is derived as*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|_{L^2}^2 + \alpha \|h\|_{H^1}^2 \le 0. \tag{55}$$

Proof of Theorem 4. Denote W by a separable base of linearly independent elements of $X_0(\Omega)$. Consider linearly independent elements h_k $(k = 1,2,3\ldots)$ in $L^2(0,T;X_0(\Omega))$ defined as

$$h_k = \sum_{j=1}^{k} p_j \omega_{jk}(t) \tag{56}$$

with an orthogonal sequence p_j $(j \ge 1)$ of elements of W and $\omega_{jk} \in C^1(0,T)$. Each ω_{jk} is determined from the ordinary differential equations (ODEs)

$$\left\langle p_{j_1}, \frac{\partial}{\partial t} h_k \right\rangle + a\left(p_{j_1}, h_k\right) + b\left(p_{j_1}, h_k, h_k\right) = 0, \quad 1 \le j_1 \le k \tag{57}$$

with the initial condition

$$h_k\left(t=0\right) = h_{k,0} \tag{58}$$

where $h_{k,0}$ is a orthogonal projection of $h_0 \in L^2\left(0,T;X\left(\Omega\right)\right)$ to W. The square matrix constructed from the coefficients of the first term of the left hand side of (57) is a Gram matrix. The Peano's existence theorem (Hartman, 2002) shows that the approximate solution h_k exists and is uniquely determined at least locally in $\left(0,T\right)$. Multiplying (57) by ω_{jk} and assembling it for $1 \le j \le k$ yields

$$\left\langle h_k, \frac{\partial}{\partial t} h_k \right\rangle + a\left(h_k, h_k\right) + b\left(h_k, h_k, h_k\right) = 0 . \tag{59}$$

By (14), the third term of the left hand side of (59) vanishes as

$$b\left(h_k, h_k, h_k\right) = -\frac{1}{M+1}\sum_{i=1}^{m+n} V_i \int_{\mathrm{R}_i} \frac{\partial h_k}{\partial x_i} h_k^{M+1} \mathrm{d}x_i = -\frac{1}{\left(M+1\right)\left(M+2\right)} \Delta V h_{k,\mathrm{J}}^{M+2} = 0 , \tag{60}$$

leading to the energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|h_k\right\|_{L^2}^2 \le -\alpha \left\|h_k\right\|_{H^1}^2 . \tag{61}$$

Integrating (61) from $t=0$ to $t=T>0$ yields the energy estimate of h_k as

$$\frac{1}{2}\left\|h_k\left(T\right)\right\|_{L^2}^2 + \int_0^T \alpha \left\|h_k\right\|_{H^1}^2 \mathrm{d}t \le \frac{1}{2}\left\|h_{k,0}\right\|_{L^2}^2 , \tag{62}$$

showing that h_k remains in a bounded set of $L^\infty\left(0,T;L^2\left(\Omega\right)\right)$. Since (62) leads to

$$\max_{0 \le t \le T}\left\{\left\|h_k\left(T\right)\right\|_{L^2}^2 + \int_0^t \alpha \left\|h_k\right\|_{H^1}^2 \mathrm{d}t\right\} < \infty , \tag{63}$$

h_k remains in a bounded set of $L^2\left(0,T;X_0\left(\Omega_{\mathrm{R}}\right)\right)$. Ascoli-Arzela theorem shows that there exists a subsequence $h_{k'}$ that converses to h weakly in $X_0\left(\Omega\right)$ and thus there exists a unique solution $h \in L^2\left(0,T;X_0\left(\Omega\right)\right) \cap L^\infty\left(0,T;L^2\left(\Omega_{\mathrm{R}}\right)\right)$ by (7). The classical compactness theorem (Temam, 1997) ensures that the convergence of h_k to h is also achieved in the space $L^2\left(0,T;L^2\left(\Omega_{\mathrm{R}}\right)\right)$ in a strong sense because the operator form b defines a continuous and bounded function $\lambda\left(h\right)$ for $h \in X_0\left(\Omega\right)$. The resulting solution h to (16) satisfies (55), which finishes the proof. Note also that the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|h\right\|_{L^2}^2 + \alpha \left\|h\right\|_{L^2}^2 \le 0 \tag{64}$$

follows from the energy estimate (55), showing that h approaches 0 in the entire Ω in the L^2 sense.

4.2.2 Maximum principle

Here a maximum principle of the BTE model (16) is presented.

Theorem 5. *(16) satisfies the following maximum principle for any* $T>0$*;*

$$\left\|h\right\|_{L^\infty\left(\Omega_{\mathrm{R}}\times\left(0,T\right)\right)} < H \text{ if } \left\|h\right\|_{L^\infty\left(\Omega_{\mathrm{R}}\right)} < H < \infty \tag{65}$$

Proof of Theorem 5. Define a non-negative functions f^+ and f^- for a generic function f as

$$f^+ = \max\left\{f,0\right\} \text{ and } f^- = -\min\left\{f,0\right\} , \tag{66}$$

respectively. Substituting $w=\left(h-H\right)^+ \in X_0\left(\Omega\right)$ and $u=v=h$ into b yields

$$b\left(\left(h-H\right)^+, h, h\right) = -\sum_{i=1}^{m+n} \int_{\mathrm{R}_i} \frac{1}{M+1} V_i \frac{\partial\left(h-H\right)^+}{\partial x_i} h^{M+1} \mathrm{d}x_i . \tag{67}$$

Application of the binomial theorem to h^{M+1} yields the polynomial expansion in terms of $h-H$ as

$$h^{M+1}=(h-H+H)^{M+1}=\sum_{j=0}^{M+1}\binom{M+1}{j}(h-H)^{j}H^{M+1-j}. \tag{68}$$

Substituting (68) into (67) leads to

$$b\left((h-H)^{+},h,h\right)=-\frac{1}{M+1}\sum_{j=0}^{M+1}H^{M+1-j}\binom{M+1}{j}\sum_{i=1}^{m+n}V_i\int_{\mathrm{R}_i}\frac{\partial(h-H)^{+}}{\partial x_i}(h-H)^{j}\,\mathrm{d}x_i\,. \tag{69}$$

By (66), the equality

$$\int_{\mathrm{R}_i}\frac{\partial(h-H)^{+}}{\partial x_i}(h-H)^{j}\,\mathrm{d}x_i=\int_{\mathrm{R}_i}\frac{1}{j+1}\frac{\partial}{\partial x_i}\left[(h-H)^{+}\right]^{j+1}\mathrm{d}x_i=\left[\frac{1}{j+1}\left[(h-H)^{+}\right]^{j+1}\right]_{x_i=0}^{x_i=L_i} \tag{70}$$

holds. Substituting (70) into (69) results in

$$b\left((h-H)^{+},h,h\right)=\frac{\Delta V}{M+1}\sum_{j=0}^{M+1}H^{M+1-j}\binom{M+1}{j}\frac{1}{j+1}\left[(h_1-H)^{+}\right]^{j+1}=0. \tag{71}$$

In addition, since

$$a\left((h-H)^{+},h\right)=\sum_{i=1}^{m+n}\int_{\mathrm{R}_i}D_i\frac{\partial}{\partial x_i}(h-H)^{+}\frac{\partial h}{\partial x_i}\mathrm{d}x_i=\sum_{i=1}^{m+n}\int_{\mathrm{R}_i}D_i\left[\frac{\partial}{\partial x_i}(h-H)^{+}\right]^{2}\mathrm{d}x_i\geq 0 \tag{72}$$

holds, substituting (71) and (72) into (16) obtains the estimate

$$\left\langle (h-H)^{+},\frac{\partial}{\partial t}h\right\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(h-H)^{+}\right\|_{L^2}^{2}=-a\left((h-H)^{+},h\right)\leq 0, \tag{73}$$

which leads to

$$\left\|(h-H)^{+}\right\|_{L^2}^{2}\leq\left\|(h_0-H)^{+}\right\|_{L^2}^{2}=0, \tag{74}$$

showing that $h<H$ in $\Omega\times(0,T)$. Similarly, taking $w=-(h-H)^{-}$ in (16) yields $h>-H$ in $\Omega\times(0,T)$ and thus the statement is proven. An important consequence of the maximum principle is that the solution with a non-negative initial condition $h_0\in X_0(\Omega)$ remains non-negative for arbitrary $t>0$.

5. Numerical analysis on the BTE model

5.1 Conforming Petrov-Galerkin finite element method

Numerical analysis on the BTE model is carried out to further investigate behaviour of its solutions. Dhawan *et al.* (2012) reviewed numerical methods for BTE models. Although they extensively surveyed the numerical methods, the models on connected graphs were not focused on. Some authors developed practical numerical methods to solve PDEs on connected graphs; however, their methods do necessarily not guarantee regularity of the solutions at junctions (Islam and Chaudhry, 1998; Basha and Malaeb, 2007; Tumanova, N., and Čiegis, 2012). The authors developed a conforming Petrov-Galerkin finite element method (CPGFEM) that solves the BTE model (12) using weight and interpolation functions in $X(\Omega)$ and $X_0(\Omega)$ (Yoshioka *et al.*, 2013).

5.2 Test problems

Test problems are firstly considered to show that the condition (14) is essential for the maximum principle. Here the parameter M is set as 2. A locally 1-D open channel network Ω as shown in **Figure 2** is set as the computational domain. The key nodes defining the boundaries of the reaches are labeled from A through E, which are the upstream-end (A), downstream-ends (C and D) and a junction (B). Length of each reach equals to 1. The reaches A-B, B-C and B-D are labeled as R_1, R_2 and R_3, respectively. D is set as 0.001 in the entire Ω. Here the following two cases of V_i are considered.

(a) $V_1=3.0$, $V_2=2.0$ and $V_3=1.0$ (The condition (14) is satisfied)

(b) $V_1=3.0$, $V_2=1.0$ and $V_3=0.5$ (The condition (14) is not satisfied: $\Delta V<0$)

(c) $V_1=3.0$, $V_2=4.0$ and $V_3=2.0$ (The condition (14) is not satisfied: $\Delta V>0$)

The initial condition is $h=1$ in the entire Ω. The homogenous Dirichlet boundary condition $h=0$ is specified at A, C and D. The time increment Δt is set as 0.001, which is sufficiently to ensure that errors in the temporal integration procedure are negligible small.

Figures 2(a)-(c) plot the computational results of h for each case at $t = 500i\Delta t$ with i the integer, clearly showing that the maximum principle is violated in the case (b) (h exceeds 1 in Ω). In the case (c), the maximum principle is not violated but the solution has an abrupt change at B, which is not observed in the case (a). In all the cases a rarefaction wave propagates from R_1 to R_2 and to R_3, and shocks resulting from the homogenous Dirichlet boundary condition are created near C and D.

5.2 Real problem

The BTE model as a governing equation of the water depth fluctuation is applied to simulate water wave propagations in an agricultural drainage system in Japan. The computational domain is same with that of in Yoshioka *et al.* (2014). **Figures 3** shows a sketch of the domain Ω, which is identified with a connected graph having five reaches and two junctions. An underlying water flow to determine the coefficients of the BTE model is computed on the basis of a uniform depth formula at the boundaries A, B and C, respectively. The boundary conditions are the Dirichlet one $h = 0.1$ (m) at A, B and C and a free-outflow one at D. Here, V is set as

$$V = \frac{2}{3}\sqrt{g}\left[\left(h + h_0\right)^{\frac{3}{2}} - h_0^{\frac{3}{2}}\right] \tag{75}$$

where g is the gravitational acceleration and h_0 is the water depth of the underlying equilibrium flow field. The coefficient V in (75) is determined so that the celerity of the inviscid counterpart of the BTE model reduces to that of the non-dispersive gravitational wave $\sqrt{g\left(h + h_0\right)}$. The coefficient D is set as 0.1 (m^2/s) in the entire Ω. Δt is 0.004 (s). **Figure 4** plots water wave propagations in the domain at $t = 500i\Delta t$.

6 Conclusions

This paper analytically and numerically studied the BTE model. The homogenous Dirichlet boundary condition was assumed in this paper for the simplicity, but linear non-homogenous conditions can also be implemented without any technical difficulties. The mathematical analysis revealed that the BTE model is well-posed if the coefficient V satisfies the balance law (14). The constraint was essential in order to obtain the energy estimate and the maximum principle for the model. Another theoretical analysis focusing on a steady BTE model with a source terms revealed that its solution is uniquely determined if the source is sufficiently regular. Numerical simulation carried out with the CPGFEM showed that the solutions to the BTE model have singular behaviour around the unction J if (14) is not satisfied.

The analyses carried out in this paper revealed a part of the basic properties of the BTE model. This paper a priori assumed the constraint (14) as a sufficient condition in order to obtain the energy estimate and the Maximum principle. However, it is not sure at the present whether it also serves as a necessary condition or not. In addition, this paper does not cover the models with nonlinear source terms as discussed in the researches (Tersenov, 2010; 2012). Furthermore, there exists a BTE model having a degenerate diffusion term (Mizumura, 2010) whose solutions are expected to behave more irregularly than the non-degenerate counterparts, which also serves as an effective reduced mathematical model of the 1-D SWEs. Future research will focus on investigations of the well-posedness and mathematical properties of the extended BTE models on connected graphs, such as the ones with a degenerate diffusion term and/or a non-linear source term.

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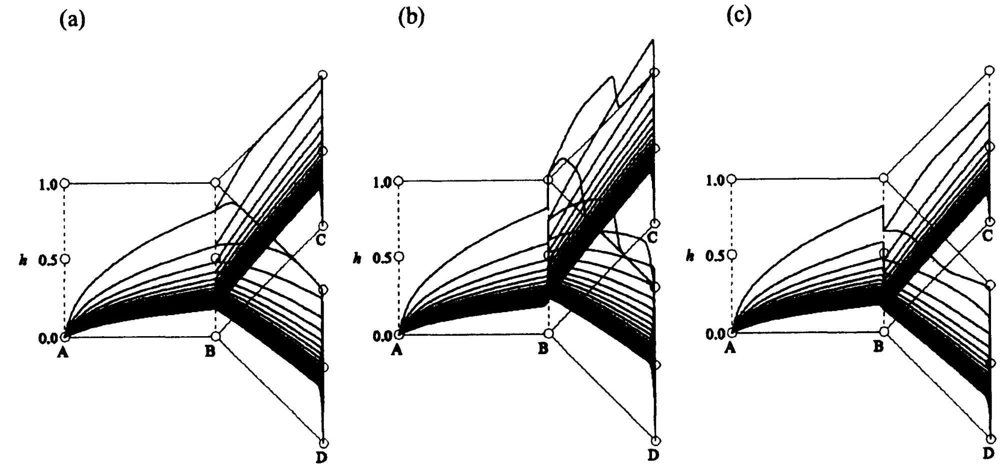

Figure 2. Computational results of the test problems.

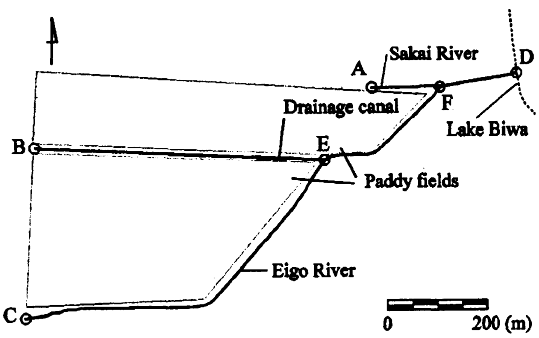

Figure 4. A sketch of Ω for the real problem.

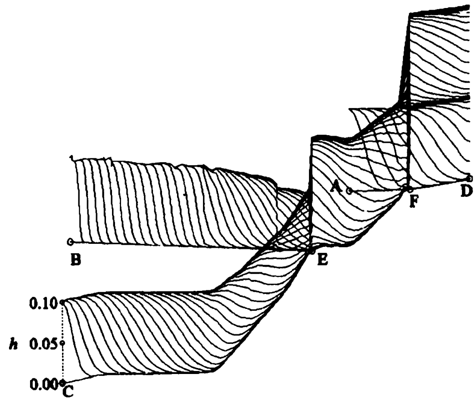

Figure 5. Computational results of the real problem.

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