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The behavior of the interfaces in the fast reaction limits of some reaction-diffusion systems with unbalanced interactions

Masato Iida* (University of Miyazaki)
Harunori Monobe (Meiji University)
Hideki Murakawa (Kyushu University)
Hirokazu Ninomiya (Meiji University)

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^{N}$ with smooth boundary $\partial\Omega$. Hilhorst-Hout-Peletier [2, 3] investigated a simple reaction-diffusion system with a huge positive parameter k

$$\begin{cases} u_{t}=\Delta u-k\,uw & \text{in } \Omega, \\ w_{t}=-k\,uw & \text{in } \Omega \end{cases} \tag{1}$$

which describes a "fast reaction" between a diffusive reactant u and a non-diffusive one w. Assuming that the initial values of u and w are non-negative and fixing a positive number T, they derived the singular limit as $k\rightarrow\infty$ of an initial-boundary value problem in $\Omega\times(0, T)$ for a class of reaction-diffusion systems with a parameter k such as (1). Their results are summarized as follows: the solution (u_{k}, w_{k}) of their initial-boundary value problem posseses its singular limit (u_{*}, w_{*}) as $k\rightarrow\infty$ such that $u_{*}w_{*}\equiv 0$; therefore, when we use the notation

$$\begin{aligned} &\Omega^{u}(t)=\{x\in\Omega|\, u_{*}(x, t)>0\}, \quad \Omega^{w}(t)=\mathrm{Int}\overline{\{x\in\Omega|\, w_{*}(x, t)>0\}}, \\ &\Gamma(t)=\Omega\setminus(\Omega^{u}(t)\cup\Omega^{w}(t)), \end{aligned} \tag{2}$$

the region $\Omega^{u}(t)$ and the region $\Omega^{w}(t)$ are divided by an "interface" $\Gamma(t)$; moreover u_{*} satisfies the one-phase Stefan problem

$$\begin{cases} u_{*,t}=\Delta u_{*} & \text{in } \Omega^{u}(t), \\ w_{*}\big|_{\Gamma(t)+0\mathbf{n}}V_{\mathbf{n}}=-\dfrac{\partial u_{*}}{\partial\mathbf{n}}\bigg|_{\Gamma(t)-0\mathbf{n}}, \quad u_{*}\big|_{\Gamma(t)}=0 & \end{cases} \tag{3}$$

in a weak sense. Here $\mathbf{n}$ is the unit normal vector to $\Gamma(t)$ oriented from $\Omega^{u}(t)$ to $\Omega^{w}(t)$, and $V_{\mathbf{n}}$ is the velocity of $\Gamma(t)$ in the direction of $\mathbf{n}$.

In this article we consider generalized "fast reactions" between u and w:

$$\begin{cases} u_{t}=\Delta u-k\,u^{m_{1}}w^{m_{3}} & \text{in } \Omega, \\ w_{t}=-k\,u^{m_{2}}w^{m_{4}} & \text{in } \Omega, \end{cases} \tag{4}$$

where $m_{j}\geq 1$ $(j=1,2,3,4)$. We are particularly interested in the situations where $(m_{1}, m_{3})\neq(m_{2}, m_{4})$, while Hilhorst-Hout-Peletier [2, 3] investigated situations where $(m_{1}, m_{3})=(m_{2}, m_{4})$. Even in the situations where $(m_{1}, m_{3})\neq(m_{2}, m_{4})$ the corresponding singular limit (u_*, w_*) of (u_k, w_k) as $k \to \infty$, if it exists, must formally satisfies $u_* w_* \equiv 0$. However, the rapid dynamics of (4) in such situations are very different from that in the situations where $(m_1, m_3) = (m_2, m_4)$. The rapid dynamics of (4) is essentially determined by the two-dimensional dynamical system

$$\begin{cases} u_t = -u^{m_1} w^{m_3}, \\ w_t = -u^{m_2} w^{m_4}. \end{cases} \tag{5}$$

Note that all the trajectories of (5) are straight and that the trajectories toward the axis $u = 0$ intersect it slantwise if $(m_1, m_3) = (m_2, m_4)$. If $(m_1, m_3) \neq (m_2, m_4)$, then the trajectories toward the axis $u = 0$ intersect it vertically in some situations; those trajectories touch the axis $u = 0$ tangentially in other situations; in some situations among the other ones no trajectories possess intersections with the axis $u = 0$. When $(m_1, m_3) \neq (m_2, m_4)$, these various structures of the trajectories in (5) may cause any different behavior of the interface $\Gamma(t)$ in the singular limit of (4). Related problems were investigated in [6] from the aspect of numerical simulation (see also [4]).

As the first attempt to solve the behavior of the interface $\Gamma(t)$ in the situations where $(m_1, m_3) \neq (m_2, m_4)$, we will investigate typical four cases of such "unbalanced interactions" between u and w: $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$, $(1, 1, m, 1)$, $(1, m, 1, 1)$ and $(m, 1, 1, 1)$, where m is a constant larger than 1. In each case we would like to reveal the interfacial dynamics in the fast reaction limit of (4) as $k \to \infty$. Hereafter we denote $\Omega \times (0, T)$ by Q_T and consider (4) under the initial condition

$$u|_{t=0} = u_0, \quad w|_{t=0} = w_0 \quad \text{in } \Omega \tag{6}$$

and a boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{7}$$

where ν denotes the unit outer normal vector of $\partial\Omega$.

2 Singular limits in Case $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ or $(1, 1, m, 1)$: moving interfaces

In these cases we can respectively reduce (4) into a reaction-diffusion system with a "balanced interaction"; namely into a system with $(m_1, m_3) = (m_2, m_4)$ by some transformations of variables. When $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ with $1 \leq m < 2$, we put $W_k = w_k^{2-m}$ for any solution (u_k, w_k) to (4). Then (u_k, W_k) becomes a solution to

$$\begin{cases} u_t = \Delta u - kuW^{1/(2-m)} & \text{in } \Omega, \\ W_t = -(2-m)kuW^{1/(2-m)} & \text{in } \Omega. \end{cases} \tag{8}$$

The singular limits of (8) with appropriate initial-boundary conditions were studied by Hilhorst, Hout and Peletier [2, 3]. They showed that u_* of the singular limit $(u_*, W_*) =$

$\lim_{k\to\infty}(u_k, W_k)$ satisfies a one-phase Stefan problem with a finite normal velocity of the interface. In the same manner as the proofs in [2, 3], we can derive the singurar limit of (8) with an initial condition

$$u|_{t=0} = u_0, \quad W|_{t=0} = {w_0}^{2-m} \quad \text{in } \Omega \tag{9}$$

and a boundary condition (7).

Throughout this section, we impose the following assumption on the initial datum (u_0, w_0):

(H1) $(u_0, w_0) \in C(\overline{\Omega}) \times L^\infty(\Omega)$, w_0 is continuous in $\operatorname{supp} w_0$ and there exist positive constants M and m_w such that

$$\begin{aligned} u_0 w_0 = 0, \quad 0 \le u_0,\, w_0 \le M \quad \text{in } \Omega, \\ m_w \le w_0 \quad \text{in } \operatorname{supp} w_0. \end{aligned}$$

Under the assumption (H1), there exists a unique solution (u_k, W_k) of the initial-boundary value problem (8),(9) and (7) satisfying

$$\begin{aligned} &u_k \in C([0,T]; C(\overline{\Omega})) \cap C^1((0,T]; C(\overline{\Omega})) \cap C((0,T]; W^{2,p}(\Omega)) \quad (\forall p > 1), \\ &w_k \in C^1([0,T]; L^\infty(\Omega)) \end{aligned} \tag{10}$$

(see [1]). We obtain the following theorem in the same manner as the proofs in [2, 3].

Theorem 2.1 (Hilhorst, Hout and Peletier [2, 3]) *Let (u_k, W_k) be the solution of (8) under the initial and boundary conditions (9) and (7), where $1 \le m < 2$. Then there exist subsequences $\{u_{k_n}\}$, $\{W_{k_n}\}$ and functions $(u_*, W_*) \in L^2(0,T; H^1(\Omega)) \times L^2(Q_T)$ such that*

$$\begin{aligned} &u_{k_n} \to u_* \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T; H^1(\Omega)), \\ &W_{k_n} \to W_* \quad \text{strongly in } L^2(Q_T), \end{aligned}$$

as k_n tends to infinity, where

$$u_* W_* = 0, \quad u_* \ge 0, \quad W_* \ge 0 \qquad a.e.\ in\ Q_T.$$

Moreover, u_* and W_* satisfy

$$\iint_{Q_T} \{-(u_* - \lambda W_*)\zeta_t + \nabla u_* \cdot \nabla \zeta\}\, dx dt = \int_\Omega \left(u_0 - \lambda {w_0}^{2-m}\right)\zeta(\cdot, 0)\, dx \tag{11}$$

for all functions $\zeta \in C^\infty(\overline{Q_T})$ such that $\zeta(x,T) = 0$, where $\lambda = 1/(2-m)$.

Since $u_* W_* \equiv 0$, we can rewrite (11) as a classical one-phase Stefan problem with a finite propagation speed. Here we use $\Omega^u(t)$, $\Omega^w(t)$ and $\Gamma(t)$ defined by (2) where $w_* = {W_*}^{1/(2-m)}$ with $1 \le m < 2$. Also we use the following notation:

$$Q_T^u = \bigcup_{0<t<T} \Omega^u(t) \times \{t\}, \quad Q_T^w = \bigcup_{0<t<T} \Omega^w(t) \times \{t\}, \quad \Gamma = \bigcup_{0<t<T} \Gamma(t) \times \{t\}. \tag{12}$$

Theorem 2.2 *Set* $(m_1, m_2, m_3, m_4) = (1, 1, 1, m)$ *where* $1 \leq m < 2$. *Let* (u_k, w_k) *be the solution of* (4) *under the initial-boundary conditions* (6)-(7) *and set* $W_k = {w_k}^{2-m}$. *Namely* (u_k, W_k) *is the solution of* (8) *satisfying* (9) *and* (7). *Let* (u_*, W_*) *be the limit given in Theorem 2.1 and set* $w_* = {W_*}^{1/(2-m)}$. *Suppose that* $\Gamma(t)$ *is a smooth, closed and orientable hypersurface satisfying* $\Gamma(t) \cap \partial\Omega = \emptyset$ *for all* $t \in [0, T]$. *Also assume that* $\Gamma(t)$ *smoothly moves with a normal velocity* V_n *from* $\Omega^u(t)$ *to* $\Omega^w(t)$, *and* u_* *is continuous in* Q_T *and smooth on* $\overline{Q_T^u}$, *and* w_* *is smooth on* $\overline{Q_T^w}$. *Then the following relations hold.*

$$w_*(t) = w_0, \qquad \text{in } Q_T^w;$$

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T^u, \\ u_* = 0, \quad \dfrac{{w_0}^{2-m}}{2-m} V_n = -\dfrac{\partial u_*}{\partial n} & \text{on } \Gamma, \\ \dfrac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}. \end{cases}$$

When $(m_1, m_2, m_3, m_4) = (1, 1, m, 1)$ with $m \geq 1$, we put $W_k = w_k^m$ for any solution (u_k, w_k) to (4). Then (u_k, W_k) becomes a solution to

$$\begin{cases} u_t = \Delta u - kuW & \text{in } \Omega, \\ W_t = -mkuW & \text{in } \Omega. \end{cases} \tag{13}$$

Taking the fast reaction limit of (13) under the boundary condition (7) and an initial condition

$$u|_{t=0} = u_0, \quad W|_{t=0} = {w_0}^m \quad \text{in } \Omega, \tag{14}$$

we can similarly derive the same conclusions as those of Theorem 2.1 where $\lambda = 1/m$. Thus we obtain the following theorem. Here we use the notation $\Omega^u(t)$, $\Omega^w(t)$, $\Gamma(t)$, Q_T^u, Q_T^w and Γ defined by (2) and (12) where $w_* = {W_*}^{1/m}$ with $m \geq 1$.

Theorem 2.3 *Set* $(m_1, m_2, m_3, m_4) = (1, 1, m, 1)$ *where* $m \geq 1$. *Let* (u_k, w_k) *be the solution of* (4) *under the initial-boundary conditions* (6)-(7) *and set* $W_k = {w_k}^m$. *Namely* (u_k, W_k) *is the solution of* (13) *satisfying* (14) *and* (7). *Set* $w_* = {W_*}^{1/m}$ *for the limit* (u_*, W_*) *given in Theorem 2.1 where* (8), (9) *and* (11) *are replaced by* (13), (14) *and*

$$\iint_{Q_T} \{-(u_* - \lambda W_*)\zeta_t + \nabla u_* \cdot \nabla\zeta\} \, dxdt = \int_\Omega (u_0 - \lambda {w_0}^m)\, \zeta(\cdot, 0)\, dx \tag{15}$$

with $\lambda = 1/m$, *respectively. Suppose that* $\Gamma(t)$ *is a smooth, closed and orientable hypersurface satisfying* $\Gamma(t) \cap \partial\Omega = \emptyset$ *for all* $t \in [0, T]$. *Also assume that* $\Gamma(t)$ *smoothly moves with a normal velocity* V_n *from* $\Omega^u(t)$ *to* $\Omega^w(t)$, *and* u_* *is continuous in* Q_T *and smooth* *on $\overline{Q_T^u}$, and w_* is smooth on $\overline{Q_T^w}$. Then the following relations hold.*

$$w_*(t) = w_0, \qquad in\ Q_T^w;$$

$$\begin{cases} u_{*,t} = \Delta u_* & in\ Q_T^u, \\ u_* = 0, \quad \dfrac{{w_0}^m}{m} V_n = -\dfrac{\partial u_*}{\partial n} & on\ \Gamma, \\ \dfrac{\partial u_*}{\partial \nu} = 0 & on\ \partial\Omega \times (0,T), \\ u_* = u_0 & on\ \Omega^u(0) \times \{0\}. \end{cases}$$

3 Singular limits in Case $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$: immovable interfaces

A free boundary appears in the fast reaction limit also in this case; however, this free boundary does not move.

Throughout this section, we impose (H1) on the initia datum (u_0, w_0) again, and assume $m > 1$. Under the assumption (H1), there exists a unique solution (u_k, w_k) of the initial-boundary value problem (4),(6) and (7) satisfying (10).

We give a result on the convergence of (u_k, w_k).

Theorem 3.1 *Set $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$ where $m > 1$. Let (u_k, w_k) be the solution of (4) under the initial and boundary conditions (6) and (7). Then there exist subsequences $\{u_{k_n}\}$ and $\{w_{k_n}\}$ of $\{u_k\}$ and $\{w_k\}$, respectively, and functions u_*, w_* and a distribution U_* such that*

$$u_*, {u_*}^{\frac{m}{2}} \in L^\infty(Q_T) \cap L^2(0,T; H^1(\Omega)), \ w_* \in L^\infty(Q_T), \ U_* \in H^{-1}(Q_T), \tag{16}$$

$$0 \le u_*, w_* \le M, \quad u_* w_* = 0 \quad a.e.\ in\ Q_T, \qquad U_* \ge 0 \quad in\ H^{-1}(Q_T), \tag{17}$$

$$u_{k_n} \to u_* \quad \text{strongly in } L^p(Q_T)(\forall p \ge 1),\ \text{a.e. in } Q, \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and weakly} * \text{ in } L^\infty(Q_T), \tag{18}$$

$$w_{k_n} \to w_* \quad \text{weakly in } L^p(Q_T)(\forall p \ge 1) \text{ and weakly} * \text{ in } L^\infty(Q_T), \tag{19}$$

$$\left|\nabla u_{k_n}^{\frac{m}{2}}\right|^2 \to U_* \quad \text{weakly in } H^{-1}(Q_T) \tag{20}$$

as k_n tends to infinity. Moreover u_*, w_* and U_* satisfy

$$\iint_{Q_T} \left\{ -\left(\frac{1}{m}{u_*}^m - w_*\right)\zeta_t + \frac{2}{m}{u_*}^{\frac{m}{2}} \nabla {u_*}^{\frac{m}{2}} \cdot \nabla\zeta \right\} dxdt + \frac{4(m-1)}{m^2} {}_{H^{-1}(Q_T)}\langle U_*, \zeta \rangle_{H_0^1(Q_T)} = 0 \tag{21}$$

for all $\zeta \in H_0^1(Q_T)$.

We can prove $U_* = \left|\nabla {u_*}^{\frac{m}{2}}\right|^2 \in L^1(Q_T)$ under additional conditions. Here we use the notation $\Omega^u(t)$, $\Omega^w(t)$, $\Gamma(t)$, Q_T^u, Q_T^w and Γ defined by (2) and (12). Then we can give an explicit equation of motion for the free boundary.

Theorem 3.2 *Let u_*, w_*, U_* be the functions satisfying* (16)-(20). *Suppose that $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0,T]$. Also assume that $\Gamma(t)$ smoothly moves with a normal velocity V_n from $\Omega^u(t)$ to $\Omega^w(t)$, and u_* is continuous in Q_T and smooth on $\overline{Q^u_T}$, and w_* is smooth on $\overline{Q^w_T}$. Then the following relations hold.*

$$V_n = 0 \text{ on } \Gamma, \quad \text{that is,} \quad \Omega^u(t) \equiv \Omega^u(0), \ \Omega^w(t) \equiv \Omega^w(0), \ \Gamma(t) \equiv \Gamma(0);$$

$$w_*(t) = w_0, \quad U_* = \left|\nabla u_*^{\frac{m}{2}}\right|^2 \qquad \text{in } Q_T;$$

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } Q^u_T = \Omega^u(0) \times (0,T), \\ u_* = 0 & \text{on } \Gamma = \Gamma(0) \times (0,T), \\ \dfrac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}. \end{cases}$$

See [5] for the proofs of Theorems 3.1 and 3.2.

4 Singular limits in Case $(m_1, m_2, m_3, m_4) = (m, 1, 1, 1)$: vanishing interfaces

In this case the non-diffusive reactant w consumes much faster than diffusive one u in the limit as $k \to \infty$. This fact makes the propagation speed of $\Gamma(t)$ too rapid. At least if $m > 2$, then $\Omega^u(t)$ spread too rapidly for us to follow its boundary $\Gamma(t)$: actually we cannot observe any free boundary.

Throughout this section, we impose the following assumptions on the initial data:

(H2) $(u_0, w_0) \in C^2(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ satisfy

$$u_0(x)w_0(x) = 0, \quad 0 \le u_0(x) \le M_u, \quad 0 \le w_0(x) \le M_w$$

for any $x \in \Omega$, where $\alpha \in (0,1)$ represents a Hölder exponent and

$$M_u := \max_{x\in\overline{\Omega}} |u_0|, \quad M_w := \max_{x\in\overline{\Omega}} |w_0|.$$

(H3) u_0 holds the homogeneous Neumann boundary condition:

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega.$$

We can derive the following result on the singular limit of (4) (see [5]).

Theorem 4.1 *Set $(m_1, m_2, m_3, m_4) = (m, 1, 1, 1)$ where $m > 1$. Let (u_k, w_k) be the solution of* (4) *under the initial and boundary conditions* (6) *and* (7). *Then*

$$\begin{array}{lll} u_k \to u_* & \text{in } C^0(\overline{Q}_T) & \text{as } k \to \infty, \\ w_k \to 0 & \text{in } C^0(\overline{\Omega} \times [\varepsilon, T]) & \text{as } k \to \infty \quad \text{for any } \varepsilon \in (0,T), \end{array}$$

where $u_*(x,t)$ *belongs to* $C^{2,1}(\overline{Q_T})$ *and satisfies the heat equation in the whole domain as follows :*

$$\begin{cases} u_{*,t} = \Delta u_* & \text{in } Q_T, \\ \dfrac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_* = u_0 & \text{on } \Omega \times \{0\}. \end{cases}$$

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Masato Iida
Center for Science and Engineering Education
Faculty of Engineering
University of Miyazaki
Miyazaki 889-2192
Japan
E-mail address: iida@cc.miyazaki-u.ac.jp