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$(0,1)$ 区間上の作用素単調関数と Kwong 行列

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We review results on operator monotone functions on $(0,1)$ and Kwong matrices. For details, we refer [10].

1 Operator monotone functions on $(0,1)$

Let f be a real-valued C^1 function on an interval (a,b). For n distinct real numbers $t_1,\ldots,t_n\in(a,b)$ a *Loewner* (or *Pick*) *matrix* $L_f(t_1,\ldots,t_n)$ is defined as

$$L_f(t_1,\ldots,t_n)=\left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right].$$

In the case where $(a,b)\subsetneqq(0,\infty)$, a *Kwong* (or an *anti-Loewner*) *matrix* $K_f(t_1,\ldots,t_n)$ is defined by

$$K_f(t_1,\ldots,t_n)=\left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right].$$

In this paper we study positive operator monotone functions on $(0,1)$ to continue our preceding studies on Loewner and Kwong matrices [3, 4, 7, 8, 11]. we show that the similar results of Loewner/Kwong matrices do not hold in general. For basic facts on operator monotone functions, we refer the reader to [2, 5, 6].

The following is useful for our study.

Lemma 1.1.

$$\begin{aligned}(1)\quad K_f(t_1,\ldots,t_n)+L_f(t_1,\ldots,t_n)&=2\ \left[\frac{t_if(t_i)-t_jf(t_j)}{t_i^2-t_j^2}\right]\\&=2\ C\circ L_{tf(t)}(t_1,\ldots,t_n)\\&=2\ L_{\sqrt{t}f(\sqrt{t})}(s_1,\ldots,s_n),\end{aligned}$$

where C is given as $C = \left[\dfrac{1}{t_i + t_j}\right]$, $\circ$ stands for the Schur product and $s_i = t_i^2$.

$$\text{(2)}\quad K_f(t_1,\ldots,t_n) - L_f(t_1,\ldots,t_n) = 2\left[\frac{t_i f(t_j) - t_j f(t_i)}{t_i^2 - t_j^2}\right]$$
$$= 2\,D\left[\frac{t_i/f(t_i) - t_j/f(t_j)}{t_i^2 - t_j^2}\right] D$$
$$= 2\,C \circ \Big(DL_{t/f(t)}(t_1,\ldots,t_n)D\Big)$$
$$= 2\,DL_{\sqrt{t}/f(\sqrt{t})}(s_1,\ldots,s_n)D,$$

where C and s_i are the same as in (1) and D is given as $D = \operatorname{diag}\,(f(t_1),\ldots,f(t_n))$.

For our study we prepare the representation of positive operator monotone functions on $(0,1)$.

Theorem 1.2 A positive operator monotone function $f(s)$ on $(0,1)$ is of the form

$$f(s) = \int_{[0,1]} \frac{s}{s+\zeta-2s\zeta}\, dm(\zeta),$$

where m is a positive measure on $[0,1]$.

For $0 \leqq \zeta \leqq 1$, put

$$f_\zeta(s) := \frac{s}{(1-2\zeta)s+\zeta} = \frac{s}{s+\zeta-2s\zeta}. \tag{1.1}$$

Theorem 1.3 Let $f_\zeta(s)$ be the function in (1.1). Then $s/f_\zeta(s)$ is operator monotone if and only if $\zeta \leqq 1/2$.

Corollary 1.4 Let $f(s)$ be a positive operator monotone function on $(0,1)$ which is of the form

$$f(s) = \int_{[0,1/2]} f_\zeta(s)\, dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1-2\zeta)s+\zeta}\, dm(\zeta), \tag{1.2}$$

where m is a positive measure on $[0,1/2]$. Then $s/f(s)$ is operator monotone on $(0,1)$.

The following corresponds to Kwong [9].

Theorem 1.5 If $f(s)$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

Theorem 1.6 Let $f_\zeta(s)$ be the function in (1.1). Then all Kwong matrices associated with f_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.

The following is a counterpart to Audenaert [1].

Theorem 1.7 Let $f(s)$ be a positive function on $(0,1)$. If $\sqrt{s}f(\sqrt{s})$ or $\sqrt{s}/f(\sqrt{s})$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

For $0 \leqq \zeta \leqq 1$, let us consider the function on $(0,1)$

$$g_\zeta(s) := \frac{f_\zeta(s^2)}{s} = \frac{s}{(1-2\zeta)s^2+\zeta}. \tag{1.3}$$

We note the following:

Theorem 1.8 Let $g_\zeta(s)$ be the function in (1.3). Then $g_\zeta(s)$ is operator monotone if and only if $1/2 \leqq \zeta$, and all Kwong matrices associated with g_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.

Proposition 1.9 Let $f(s)$ be the operator monotone function in (1.2). Then for any positive integer m,

$$\left[\frac{f(s_i)^m - f(s_j)^m}{s_i^m - s_j^m}\right]$$

are positive semidefinite for all n and $s_1, \ldots, s_n$ in $(0,1)$.

参考文献

[1] K. M. R. Audenaert, *A characterisation of anti-Löwner functions,* Proc. Amer. Math. Soc., 139 (2011), 4217-4223.

[2] R. Bhatia, *Matrix Analysis,* Springer (1996).

[3] R. Bhatia and T. Sano, *Loewner matrices and operator convexity,* Math. Ann., 344 (2009), 703-716.

[4] R. Bhatia and T. Sano, *Positivity and conditional positivity of Loewner matrices,* Positivity, 14 (2010), 421-430.

[5] W. F. Donoghue, *Monotone Matrix Functions and Analytic Continuation,* Springer (1974).

[6] F. Hansen and G. K. Pedersen, *Jensen's inequality for operators and Löwner's theory,* Math. Ann., 258 (1982), 229-241.

[7] F. Hiai and T. Sano, *Loewner matrices of matrix convex and monotone functions,* J. Math. Soc. Japan, 64 (2012), 343-364.

[8] C. Hidaka and T. Sano, *Conditional negativity of anti-Loewner matrices,* Linear and Multilinear Algebra, 60 (2012), 1265-1270.

[9] M. K. Kwong, *Some results on matrix monotone functions,* Linear Algebra Appl., 118 (1989), 129-153.

[10] J. Morishita, T. Sano and S. Tachibana. *Kwong matrices and operator monotone functions on* $(0, \infty)$, Ann. Funct. Anal., 5 (2014), 121-127.

[11] T. Sano and S. Tachibana, *On Loewner and Kwong matrices,* Scientiae Mathematicae Japonicae, 75 (2012), 335-338.