数理解析研究所講究録
第 1893 巻 2014 年 67-71



# ON SOME INEQUALITIES WITH MATRIX MEANS

DINH TRUNG HOA, DU THI HOA BINH, AND HO MINH TOAN

ABSTRACT. Let $0 < m \leq A, B \leq M$ and $\sigma, \tau$ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map $\Phi$,

$$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$$
$$\Phi(A\sigma B) \leq K(h)\left(\Phi(A)\tau\Phi(B)\right),$$
$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B),$$

and

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ with $h = \dfrac{M}{m}$ is the Kantorovich constant.

## 1. INTRODUCTION

The axiomatic theory for connections and means for pairs of positive matrices have been studied by Kybo and Ando [4]. A binary operation $\sigma$ define on the set of positive definite matrices is called a connection if

(i) $A \leq C, B \leq D$ implies $A\sigma B \leq B\sigma D$;
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then $\sigma$ is called a mean.

Many authors study matrix inequalities containing means and linear unital positive maps on matrix algebras. Such inequalities are interesting by themselves and have many applications in quantum information theory.

In [2], Lin proved the following Theorem.

**Theorem 1.1** ([2]). *Let $0 < m \leq A, B \leq M$. Then for every positive unital linear map $\Phi$,*

$$\Phi^2(A\nabla B) \leq K^2(h)\Phi^2(A\sharp B), \tag{1}$$

*and*

$$\Phi^2(A\nabla B) \leq K^2(h)(\Phi(A)\sharp\Phi(B))^2, \tag{2}$$

*where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.*

---

2000 *Mathematics Subject Classification.* 26D15.
*Key words and phrases.* matrix means, inequalities.

It is well-known that the arithmetic mean $\nabla$ is the biggest among symmetric means (see [4]). A natural question is that *is the theorem above still true if we replace the biggest means by a smaller one*? In this note, we consider such inequalities for two different means with Kantorovich constant. In applications, we give an analogous result of Uchiyama and Yamazaki in [7].

This note is based on preprint [1].

## 2. Main results

**Lemma 2.1.** *Let $0 < m \le A, B \le M$ and $\sigma, \tau$ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map $\Phi$,*

$$\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \le M + m, \tag{3}$$

*and*

$$\Phi(A)\sigma\Phi(B) + Mm\Phi^{-1}(A\tau B) \le M + m. \tag{4}$$

*Proof.* It is easy to see that

$$(M - A)(m - A)A^{-1} \le 0,$$

or

$$mMA^{-1} + A \le M + m.$$

Consequently,

$$\Phi(A) + mM\Phi(A^{-1}) \le M + m.$$

Similarly,

$$\Phi(B) + mM\Phi(B^{-1}) \le M + m.$$

Summing up two above inequalities, we get

$$\Phi(A \triangledown B) + mM\Phi((A!B)^{-1}) \le M + m.$$

Besides, from the general theory of matrix means we know that $\triangledown \ge \sigma$ and $\tau \ge !$. Hence,

$$\begin{aligned}\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B) &\le \Phi(A\sigma B) + mM\Phi((A\tau B)^{-1}) \\ &\le \Phi(A \triangledown B) + mM\Phi((A!B)^{-1}) \\ &\le M + m.\end{aligned}$$

By a similar argument, we can get inequality (4) with using the fact that

$$\Phi(A)\sigma\Phi(B) \le \Phi(A) \triangledown \Phi(B) = \Phi(A \triangledown B).$$

□

The following Proposition is a generalization of Lin's result mentioned in Introduction.

**Proposition 2.1.** *Let* $0 < m \leq A, B \leq M$ *and* $\sigma, \tau$ *two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map* $\Phi$,

$$\Phi^2(A\sigma B) \leq K^2(h)\Phi^2(A\tau B), \tag{5}$$

$$\Phi^2(A\sigma B) \leq K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2, \tag{6}$$

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)\Phi^2(A\tau B), \tag{7}$$

*and*

$$(\Phi(A)\sigma\Phi(B))^2 \leq K^2(h)(\Phi(A)\tau\Phi(B))^2, \tag{8}$$

*where* $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$ *is the Kantorovich constant.*

*Proof.* We prove (2.1). The inequality (2.1) is equivalent to the following

$$\Phi^{-1}(A\tau B)\Phi^2(A\sigma B)\Phi^{-1}(A\tau B) \leq K^2(h),$$

or

$$||\Phi(A\sigma B)\Phi^{-1}(A\tau B)|| \leq K(h).$$

On the other hand, it is well known that [5, Theorem 1] for $A, B \geq 0$,

$$||AB|| \leq \frac{1}{4}||A+B||^2.$$

So, it is necessary to prove that

$$\frac{1}{4mM}||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)||^2 \leq \frac{(M+m)^2}{4Mm},$$

or,

$$||\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)|| \leq M + m.$$

The last inequality follows from Lemma 2.1.

Remain inequalities in Proposition can be proved analogously. □

*Remark* 1. As we mentioned in the proof of Proposition 2.1 that for any positive matrices $A, B$, $\Phi(A\sigma B) \leq \Phi(A\nabla B)$. From that, it can rise a wrong intuition that the proof of Proposition 2.1 can be obtained easily from Theorem 1.1. Unfortunately, the last inequality could not be squared as it was shown in [2, Proposition 1.2].

**Theorem 2.1.** *Let* $0 < m \leq A, B \leq M$ *and* $\sigma, \tau$ *are two arbitrary symmetric means. Then for every positive unital linear map* $\Phi$,

$$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$$
$$\Phi(A\sigma B) \leq K(h)\left(\Phi(A)\tau\Phi(B)\right),$$
$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B),$$

*and*

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$$

*where* $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$ *is the Kantorovich constant.*

*Proof.* The proof follows from Proposition 2.1 and the fact that the function $f(t) = t^{1/2}$ is operator monotone on $[0, \infty)$. □

**Corrolary 2.1.** *Let $f, g$ be symmetric operator monotone functions on $[0, \infty)$. Then for any pair $0 < m < M$,*

$$(9) \qquad max\{\frac{f(t)}{g(t)}, \frac{f(t)}{g(t)}\} \leq K(h) = \frac{(m+M)^2}{4mM}, \quad t \in [m, M].$$

*Proof.* It is necessary to apply above Theorem for the symmetric matrix means $\sigma$ and $\delta$ corresponding to the functions $f$ and $g$, and definition of matrix means via it representation functions. □

Inequality (9) is interesting by itself, and the authors do not know an elementary proof even in the case when $f(t) = \sqrt{t}$.

As an application, now we give a similar result as in [7]. Uchiyama and Yamazaki showed that for an operator monotone function $f$ on $[0, \infty)$ if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leq I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$. By applying Theorem 2.1 we get a similar result for any symmetric means.

**Corrolary 2.2.** *Let $f$ be operator monotone function on $[0, \infty)$ and $\sigma$ an arbitrary mean between harmonic and arithmetic ones. If for a given pair of positive invertible matrices $A, B$,*

$$f(\lambda B + I)^{-1} \sigma f(\lambda A + I) \leq K$$

*for all sufficiently small $\lambda > 0$ (where $K$ is Kantorovich constant), then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$.*

## References

[1] Dinh Trung Hoa, Du Thi Hoa Binh, Ho Minh Toan, On some inequalities with matrix means. Preprint.

[2] M. Lin. Squaring a reverse AM-GM inequality. Studia Math. **215** (2013), 187-194.

[3] M. Lin, On an operator Kantorovich inequality for positive linear maps. J. Math. Anal. Appl. **402** (2013) 127-132.

[4] F. Kubo, T. Ando, Means of positive linear operators. Math. Ann. **246** (1979/80), no. 3, 205-224.

[5] R. Bhatia, F. Kittaneh, Notes on matrix arithmeticgeometric mean inequalities. Linear Algebra Appl. **308** (2000) 203-211.

[6] R. Kaura, M. Singh, J. S. Aujla, Generalized matrix version of reverse Holder inequality. Linear Algebra and its Applications. **434** (2011) 636-640.

[7] M. Uchiyama, T. Yamazaki, A converse of Loewner-Heinz inequality and applications to operator means. Journal of Mathematical Analysis and Applications. **413**(1) (2014) 422429.

Duy Tan University, K7/25 Quang Trung, Danang, Vietnam
*E-mail address*: trunghoa.math@gmail.com

Ha Tay College of Pedagogy, Ha Noi, Vietnam
*E-mail address*: hoabinhcdsp@gmail.com

Institute of Mathematics
*E-mail address*: hmtoan@math.ac.vn