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RELATIVE DEFINABILITY

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ABSTRACT. This is a (non-exhaustive) collection of results addressing the question "If A is such that $P(A)$, does there exists a B such that $Q(B)$ and B is definable from A?", for various properties $P(x)$, $Q(x)$, as well as closely related questions. The focus is on classical combinatorial properties at the level of $H(\omega_2)$.

1. INTRODUCTION

We all know that in ZFC one can prove the existence of such objects as Hausdorff gaps, Aronszajn trees, partitions of ω_1 into $\aleph_1$–many stationary sets, and so on. Many of these existence proofs proceed by a specific construction of the relevant type of object, where this construction is definable from any given object p satisfying a certain property P. More specifically, one establishes in ZFC the existence of some p such that $P(p)$, and then one runs the relevant construction with any fixed p such that $P(p)$ as a parameter. As a typical example, consider the notions of ladder system and of Countryman line.[1] It is a completely trivial fact that ladder systems exist under ZFC, but it is by no means obvious that the same is true for Countryman lines. However, if $\vec{C}$ is a ladder system, then there is a recursive construction of a Countryman line definable from $\vec{C}$ ([12]). In this note I analyse the net of relations of the form "if A is such that $P(A)$, then there exists a B such that $Q(B)$ and B is definable from A" for various classical properties $P(x)$, $Q(x)$ of combinatorial flavour pertaining the structure $H(\omega_2)$.

In many cases one obtains positive results by simply looking in the literature at specific constructions of objects B such that $Q(B)$ and observing that those construction are indeed definable from any A such that $P(A)$. One can also obtain positive results by more indirect considerations of the following form: Suppose A is such that $P(A)$ and, say, $Z \subseteq \omega_1$ is obtained from A in a definable way. Then argue that $L[Z]$ is sufficiently close to $\mathbf{V}$, for example in the sense that $\omega_1 = \omega_1^{L[Z]}$,

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[1]See next section for the definitions of the relevant objects.

and therefore there is some B such that $Q(B)^{L[Z]}$ and moreover such that $Q(B)$ holds in $\mathbf{V}$. Then it is enough to choose the $<_{L[Z]}$–least such B.

One can often obtain negative results by starting with a model $\mathbf{V}$ satisfying a suitable existence–pattern of the relevant type of objects at some cardinal $\kappa > \omega_1$, picking a sample of objects that do exist there, and then arguing, in some homogeneous extension in which κ becomes ω_1, that the objects we started with have been preserved and, moreover, no objects of unwanted kind are definable from them as otherwise they would be in $\mathbf{V}$ by homogeneity, which would mean that in $\mathbf{V}$ there were such objects at κ, and this was not the case. As we will see, many of the negative results on relative definability[2] in this note can be easily turned into independence results relative to ZF by going to a natural symmetric submodel of the corresponding generic extension.

One immediate conclusion emerging from this analysis is that certain innocent–looking combinatorial objects inevitably code more information than others; for example, Countryman lines, Hausdorff gaps and simplified $(\omega, 1)$–morasses always code information which, say, (ω_1, ω_1^*)–gaps in $({}^{\omega}\omega, <^*)$ or ω_1–sequences of distinct reals do not necessarily code.

The proofs in this note are quite elementary. Consequently, many of the results I am presenting here were probably known, although I could not find references for them. This note expands on part of what I explained in one section, focusing on the topic of relative definability, of the talk I gave at the workshop, with the title "Forcing locally definable well–orders of the universe without the GCH". It should be clear that the work contained in this note barely scratches the surface of this area, and that the present considerations can be extended in many directions.

Two quick words on terminology: I will say that a poset $\mathcal{P}$ is *homogeneous* in case for every two conditions p, $p' \in \mathcal{P}$ there are extensions q, q' of p and p', respectively, such that $\mathcal{P} \restriction q \cong \mathcal{P} \restriction q'$. If $\mathcal{P}$ is homogeneous in this sense, then every two conditions force the same truth value for any statement with parameters in the ground model. This notion of homogeneity will be enough for all applications in this note. Also, at times I will mention symmetric submodels of forcing extensions.[3] The basic theory for this type of construction can be found in [3].

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[2]But not all of them.

[3]Starting from any $\mathbf{V} \models \mathrm{ZF}$, any symmetric submodel of any forcing extension of $\mathbf{V}$ is a ZF model extending $\mathbf{V}$.

2. Relative definability over $H(\omega_2)$

I will mostly focus on the following objects on ω_1: Ladder systems, (special) Aronszajn trees, Countryman lines, (indestructible) (ω_1, ω_1^*)–gaps in $({}^\omega\omega, <^*)$, ω_1–sequences of distinct reals, simplified $(\omega, 1)$–morasses, and partitions of ω_1 into stationary sets. Ladder systems will be understood to be of the form $\langle C_\delta : \delta \in \mathrm{Lim}(\omega_1)\rangle$ and will be such that $\mathrm{ot}(C_\delta) = \omega$ and C_δ is cofinal in δ for all δ. They are also known as C–sequences. A Countryman line is a linear order $(\omega_1, \leq_C)$ such that $\omega_1 \times \omega_1$ with the product order can be decomposed as a countable union of linearly ordered sets (see [10] or [12]). An (ω_1, ω_1^*)–gap $(\vec{f}, \vec{g})$ in $({}^\omega\omega, <^*)$ is indestructible if and only if it remains a gap in any outer model with the same ω_1 as $\mathbf{V}$ (equivalently, in any forcing extension of $\mathbf{V}$ by a c.c.c. poset). $(\vec{f}, \vec{g})$, for $\vec{f} = (f_\alpha)_{\alpha<\omega_1}$ and $\vec{g} = (g_\alpha)_{\alpha<\omega_1}$, is indestructible if and only if there is $m < \omega$ and strictly increasing sequences $(i_\nu)_{\nu<\omega_1}$, $(j_\nu)_{\nu<\omega_1}$ in ω_1 such that $(\langle f_{i_\nu} + m : \nu < \omega_1\rangle, \langle g_{j_\nu} : \nu < \omega_1\rangle)$ is a special gap (see e.g. [9] or [15] for this and for the notion of special gap). Also, Hausdorff gaps are indestructible (see [9]). All other notions mentioned in this paragraph should be either well–known or easy to look up. For example see [13] and [14] for information on simplified $(\omega, 1)$–morasses.

The mention of witnesses in the statement of Theorem 2.1 has the natural meaning. For example, a special Aronszajn tree with a witness is a pair (T, f), where T is an Aronszajn tree and $f : T \longrightarrow \omega$ is a specializing function, and an indestructible gap with a witness is a gap $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ in $({}^\omega\omega, <^*)$ together with an $m < \omega$ and two strictly increasing sequences $(i_\nu)_{\nu<\omega_1}$, $(j_\nu)_{\nu<\omega_1}$ in ω_1 such that $(\langle f_{i_\nu} + m : \nu < \omega_1\rangle, \langle g_{j_\nu} : \nu < \omega_1\rangle)$ is a special gap. The presence of witnesses is crucial for the present type of considerations; for example, every Aronszajn tree (resp., every (ω_1, ω_1^*)–gap) can be made special (resp., indestructible) by going to a c.c.c. extension.[4] On the other hand, as we will see, the notion of special Aronszajn tree with a witness is strictly stronger, in terms of definability power, than the notion of Aronszajn tree (and similarly for indestructible gaps with a witness vs. (ω_1, ω_1^*)–gaps).

Given two properties $P(x)$, $Q(x)$, I will say that $P(x)$ has definability strength at least that of $Q(x)$ over $\langle H(\omega_2), \in\rangle$ if there is a formula

[4] J. Moore's result on the existence of a 5 membered basis for the uncountable linear orders under PFA ([7]) can be seen as a "specialisation" result under this forcing axiom for linear orders not embedding ω_1, its converse ω_1^*, or an uncountable set of reals; indeed, the bulk of that proof consists in showing that every such linear order contains a Countryman line.

$\varphi(x,y)$ such that $Q(\{b \in H(\omega_2) : H(\omega_2) \models \varphi(A,b)\})$ for every $A \in H(\omega_2)$ such that $P(A)$.[5]

Theorem 2.1. *(ZF) The following properties have the same definability strength over $\langle H(\omega_2), \in\rangle$.*

(1) *x is a ladder system*
(2) *x is a simplified $(\omega,1)$–morass*
(3) *x is an special Aronszajn tree with a witness*
(4) *x is a Countryman line with a witness*
(5) *x is an indestructible gap with a witness*

Proof. Each of the notions in (2)–(5) can be exemplified from a ladder system. In fact, one can easily find in the literature recursive constructions of these objects explicitly definable from a given C–sequence. For example, for (2), one can check that the construction in the proof of Theorem 1.4 in [14] of a simplified $(\omega,1)$–morass is indeed definable from any fixed ladder system.

Conversely, if x satisfies any one of conditions (2)–(5), then a ladder system is definable from x. Suppose, for example, that x is as in (3), let $\theta = \omega_2$, and let $A \subseteq \omega_1$ be defined from x in $H(\omega_2)$ and such that $L_\theta(x) = L_\theta(A) = L_\theta[A]$. If $\kappa = \omega_1$, then in $L_\theta[A]$, $x = ((\kappa, \leq_C), (X_n)_{n\in\omega})$, where $\leq_C$ is a linear order on κ and $(X_n)_{n\in\omega}$ is a decomposition of $\kappa \times \kappa$ into chains. But then necessarily $\kappa = \omega_1^{L[A]}$. The reason is that $L_\theta[A]$ can see that $(\kappa, \leq_C)$ embeds neither what it thinks is ω_1, nor its converse, nor any uncountable set of reals (which is not difficult to verify and was first observed by Galvin), and therefore it believes (correctly) that $|\kappa| = \aleph_1$ (see [10]). Here is an argument: $L_\theta[A]$ sees that any partition tree (see [11]) for $(\kappa, \leq_C)$ has to be an Aronszajn tree on its ω_1, and therefore it sees also $|\kappa| = \aleph_1$ ([11]). But then $\kappa = \omega_1^{L_\theta[A]}$. Now we can pick the $<_{L_\theta[A]}$–first ladder system on ω_1 in $L_\theta[A]$. The proof in the other cases is similar. □

3. Some negative results

Given a pre-gap $(\vec{h},\vec{k}) = (\langle h_\alpha : \alpha < \lambda\rangle, \langle k_\alpha : \alpha < \lambda\rangle)$ in $({}^\omega\omega, <^*)$ (for λ a limit ordinal), let $\mathbb{P}_{\vec{h},\vec{k}}$ denote the forcing for adding a real splitting $(\vec{h},\vec{k})$ defined as follows: conditions in $\mathbb{P}_{\vec{h},\vec{k}}$ are triples (A,B,s) with A, $B \in [\lambda]^{<\omega}$, $s \in {}^{<\omega}\omega$, $h_\alpha(m) + 1 < k_\beta(m)$ for all $\alpha \in A$, $\beta \in B$ and

[5]Then, of course, $P(x)$ and $Q(x)$ have the same definability strength over $\langle H(\omega_2), \in\rangle$ if $P(x)$ has definability strength at least that of $Q(x)$ and the same exchanging $P(X)$ and $Q(x)$ (over $\langle H(\omega_2), \in\rangle$). And one can certainly define obvious variations of this notion of strength, for example by looking at definability over other structures or over the universe.

$m \geq \mathrm{dom}(s)$, and (A_1, B_1, s_1) extends (A_0, B_0, s_0) if $s_0 \subseteq s_1$, $A_0 \subseteq A_1$, $B_0 \subseteq B_1$ and $h_\alpha(m) < s(m) < k_\beta(m)$ for all $\alpha \in A_0$, $\beta \in B_0$ and $s \in \mathrm{dom}(s_1) \setminus \mathrm{dom}(s_0)$.

Let $\kappa \geq \omega_1$ be a regular cardinal. There is a natural finite support c.c.c. iteration, that I will denote by $\langle \mathcal{P}^\kappa_\alpha, \dot{\mathcal{Q}}_\beta : \beta < \kappa, \alpha \leq \kappa \rangle$, for adding a (κ, κ^*)–gap $(\vec{f}, \vec{g}) = (\bigcup_{\alpha<\kappa} \vec{f}_\alpha, \bigcup_{\alpha<\kappa} \vec{g}_\alpha)$ in $({}^\omega\omega, <^*)$, where each $(\vec{f}_\alpha, \vec{g}_\alpha)$ is a pre-gap added by $\mathcal{P}^\kappa_\alpha$ and $\vec{f}_\alpha$ and $\vec{g}_\alpha$ are initial segments of $\vec{f}_{\alpha'}$ and $\vec{g}_{\alpha'}$, respectively, for $\alpha < \alpha' < \kappa$, and where in fact $(\vec{f}_\alpha, \vec{g}_\alpha) = (\bigcup_{\beta<\alpha} \vec{f}_\beta, \bigcup_{\beta<\alpha} \vec{g}_\beta)$ for limit ordinals α. Given $\beta < \kappa$, $\dot{\mathcal{Q}}_\beta$ is forced to be $\mathbb{P}_{\vec{h},\vec{k}}$, where $(\vec{h}, \vec{k})$ is an (ω, ω^*)–pre-gap equivalent to $(\vec{f}_\beta, \vec{g}_\beta)$ if $\mathrm{cf}(\beta) = \omega$, and otherwise $(\vec{h}, \vec{k}) = (\vec{f}_\beta, \vec{g}_\beta)$. Then $(\vec{f}_{\beta+1}, \vec{g}_{\beta+1})$ is a $(\mathrm{dom}(\vec{f}_\beta) + \omega, (\mathrm{dom}(\vec{g}_\beta) + \omega)^*)$–pre-gap defined naturally from $(\vec{f}, \vec{g})$ together with the split of $(\vec{f}_\beta, \vec{g}_\beta)$ added by $\dot{\mathcal{Q}}_\beta$. (Cf. [1], where a very similar iteration appears; the corresponding proof from [1] can be easily adapted to yield that $\Vdash_{\mathcal{P}_\beta}$ "$\dot{\mathcal{Q}}_\beta$ is c.c.c." for every $\beta < \kappa$). It is also easy to see that any real r in any extension by $\mathcal{P}^\kappa_\kappa$ appears already at some initial segment α and therefore, by a standard density argument, it cannot split the generic pre–gap as in fact $\dot{f} \not<^* r$ for any $\dot{f}$ added to the $\vec{f}$ part of the pre-gap after stage α.

Part (1) of the following result shows how to separate our strongest level of definability strength from the join of all members from a (natural) second level. Part (2) (together with Theorem 2.1) answers a question in [8], asking whether the existence of an (ω_1, ω_1^*)–gap implies in ZF the existence of a Hausdorff gap. The use of an inaccessible in the even–numbered parts is necessary.[6]

Theorem 3.1. (1) *It is consistent that there is an Aronszajn tree T, an (ω_1, ω_1)–gap $(\vec{f}, \vec{g})$ in $({}^\omega\omega, <^*)$ and a partition $\vec{S}$ of ω_1 into $\aleph_1$–many stationary sets such that no ladder system is definable from $(T, (\vec{f}, \vec{g}), \vec{S})$.*

(2) *If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of $\mathbf{V}$: There is an Aronszajn tree, an (ω_1, ω_1)–gap in $({}^\omega\omega, <^*)$ and a partition of ω_1 into $\aleph_1$–many stationary sets but there is no ladder system on ω_1.*

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[6]For part (2), if $\kappa = \omega_1$ is regular but there is no ladder system, then ω_1 is inaccessible in L: If $\kappa = (\lambda^+)^L$ and $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa) \rangle \in L$ is a club–sequence with $\mathrm{cf}(C_\alpha) = \mathrm{cf}(\alpha)^L$ for all α, then is is easy to define a ladder system from $\vec{C}$ together with a bijection $f : \omega \longrightarrow \lambda$. This was already observed by Blass (cf. [8]). A similar argument works for parts (4) and (6).

(3) *It is consistent that there is an Aronszajn tree T, an ω_1-sequence $\vec{r}$ of distinct reals and a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets such that no (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$ is definable from $(T, \vec{r}, \vec{S})$.*

(4) *If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of $\mathbf{V}$: There is an Aronszajn tree, an ω_1-sequence of distinct reals and a partition of ω_1 into $\aleph_1$-many stationary sets but there is no (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$.*[7]

(5) *It is consistent, relative to* ZFC, *that there is an Aronszajn tree T and a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets such that no ω_1-sequence of distinct reals is definable from $(T, \vec{S})$.*

(6) *If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of $\mathbf{V}$: There is an Aronszajn tree and a partition of ω_1 into $\aleph_1$-many stationary sets but there is no ω_1-sequence of distinct reals.*

Proof. For part (1) we start with a model with an $\aleph_2$–Aronszajn tree T. Let $\vec{S} = (S_\nu)_{\nu<\omega_2}$ be any partition of ω_2 into stationary sets. Let G be $\mathcal{P}_{\omega_2}^{\omega_2}$–generic and let $(\vec{f}, \vec{g}) = (\langle f_\alpha \mid \alpha < \omega_2\rangle, \langle g_\beta \mid \beta < \omega_2\rangle)$ be the generic gap added by G.

Claim 3.2. *Every c.c.c. forcing $\mathcal{Q}$ preserves the Aronszajnness of T. In particular, T is Aronszajn in $\mathbf{V}[G]$.*

Proof. Otherwise there is a $\mathcal{Q}$–name $\dot{b}$ for a cofinal branch through T and a subtree $T' \subseteq T$ of height ω_2 with countable levels such that $\Vdash_{\mathcal{Q}} \dot{b} \restriction \alpha \in T'_\alpha$ for every $\alpha < \omega_2$. But for every regular κ and $\lambda < \kappa$, every tree of height κ with levels of size less than λ has a κ–branch (cf. [4], Prop. 7.9), and so T', and therefore also T, has an ω_2–branch, which is a contradiction. □

In $\mathbf{V}[G]^{\mathrm{Coll}(\omega,\omega_1)}$, every S_ν remains a stationary subset of $\omega_2^{\mathbf{V}[G]} = \omega_2^{\mathbf{V}}$, $\omega_1 = \omega_2^{\mathbf{V}}$, and $(\vec{f}, \vec{g})$ is still a gap: Suppose G' is $\mathrm{Coll}(\omega, \omega_1)$–generic over $\mathbf{V}[G]$ and r is a real in $\mathbf{V}[G][G']$. Then $r \in \mathbf{V}[G \restriction \alpha][G']$ for some $\alpha < \omega_2$. But then r cannot split $(\vec{f}, \vec{g})$ by essentially the same argument that we have seen at the beginning of this section. Finally,

[7] A related observation of Todorčević from the 1980's is that although it is easy to see that ZF + "there is an (ω_1, ω_1)–gap" implies the existence of a $\subseteq$-increasing ω_1–sequence of G_δ sets covering $\mathbb{R}$ (see e.g. [3], Exercise 29.8), this conclusion does not follow from ZF+DC + "There is an ω_1–enumeration of distinct reals." (He also proved that this last theory proves the existence of a partition of $\mathbb{R}$ into $\aleph_1$–many $F_\sigma \cap G_\delta$–sets.)

in $\mathbf{V}[G]^{\mathrm{Coll}(\omega,\omega_1)}$ there cannot be any ladder system on $\omega_1 = \omega_2^{\mathbf{V}[G]}$ definable from $(T, (\vec{f}, \vec{g}), \vec{S})$. Otherwise, by homogeneity of the collapse this ladder system would be in $\mathbf{V}[G]$, which is impossible.

For part (2), let κ be an inaccessible cardinal such that there is a κ–Aronszajn tree T, let $(S_\nu)_{\nu<\kappa}$ be a partition of κ into stationary sets, and let G be generic for $\mathcal{P}_\kappa^\kappa$. Our model $\mathbf{W}$ will be the symmetric submodel of the extension of $\mathbf{V}[G]$ by $\mathrm{Coll}(\omega, <\kappa)$ generated by the names fixed by an automorphism of $\mathrm{Coll}(\omega, <\kappa)$ fixing $\mathrm{Coll}(\omega, <\alpha)$ for some $\alpha < \kappa$. In $\mathbf{W}$, every α is collapsed to ω and so $\omega_1 = \kappa$, each S_ν is clearly stationary, T remains Aronszajn (a cofinal branch through T in $\mathbf{W}$ would have to be in $\mathbf{V}[G][H]$ for a $\mathrm{Coll}(\omega, <\alpha)$–generic H for some $\alpha < \kappa$), and $(\vec{f}, \vec{g})$ remains unsplit (by the same proof as in the first part). Also, in $\mathbf{W}$ there is no ladder system on ω_1 as such an object would be in $\mathbf{V}[G][H]$ for some H as above, which is impossible.

I will not say anything here about the proofs of parts (3)–(6), except that parts (3) and (4) use a c.c.c. forcing due to Kunen which, starting from a regular $\kappa > \omega_1$ such that $\diamondsuit(\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\})$ holds, produces a model with $2^{\aleph_0} \geq \kappa$ in which there are no (κ, κ^*)–gaps (see [9]), and that for parts (5) and (6) we start with a regular $\kappa > 2^{\aleph_0}$ such that there is a κ–Aronszajn tree. Using this, the remainder of the proofs is as in parts (1) and (2). □

Theorem 3.3. *Suppose there is a weakly compact cardinal. Then there is a partial order $\mathcal{P}$ such that*

(1) *$\mathcal{P}$ forces that there is a partition $\vec{S}$ of ω_1 into $\aleph_1$–many stationary sets and an (ω_1, ω_1^*)–gap $(\vec{f}, \vec{g})$ in $({}^\omega\omega, <^*)$ such that no Aronszajn tree is definable from $(\vec{S}, (\vec{f}, \vec{g}))$, and*

(2) *there is a symmetric submodel of a forcing extension by $\mathcal{P}$ satisfying that there is a partition of ω_1 into $\aleph_1$–many stationary sets, an (ω_1, ω_1^*)–gap in $({}^\omega\omega, <^*)$, but no Aronszajn tree.*

Proof. Let κ be weakly compact, let $\vec{S}$ be a partition of κ into κ–many stationary sets, let G be $\mathcal{P}_\kappa^\kappa$–generic and $(\vec{f}, \vec{g})$ the corresponding generic gap, let M be a transitive model of enough of ZFC containing everything relevant and such that $V_\kappa \subseteq M$ and let $j : M \longrightarrow N$ be an elementary embedding, N transitive, with critical point κ. Then j can be extended to an elementary embedding $j : M[G] \longrightarrow N[G][H]$, where H is $(\mathcal{P}_{j(\kappa)}^{j(\kappa)})^N/G$–generic over $N[G]$ (since $(\mathcal{P}_\alpha^\kappa)_{\alpha\leq\kappa}$ is the initial segment of $(\mathcal{P}_\alpha^{j(\kappa)})^N_{\alpha<j(\kappa)}$ of length κ). Now, every κ–tree T in $M[G]$ acquires a κ–branch in $N[G][H]$ since $j(T) \restriction \kappa = T$. But $(\mathcal{P}_{j(\kappa)}^{j(\kappa)})^N/G$ has the c.c.c. in $N[G]$, and therefore T already had a κ–branch in $N[G]$

by Claim 3.2. It follows from this that there are no κ–Aronszajn trees after forcing with $\mathcal{P}^\kappa_\kappa$. Now it is easy to see, as in the proof of Theorem 3.1, that $\mathcal{P} = \mathcal{P}^\kappa_\kappa * \mathrm{Coll}(\omega, <\kappa)$ is as desired. □

The weakly compact cardinal in Theorem 3.3 is necessary for part (2) by a classical result of Silver (see [3], Thm. 28.23, and note that $\mathbf{V} \models \mathsf{AC}$ is not needed for that proof). I don't know if the weakly compact cardinal is necessary for part (1), though.

3.1. **Partitions of ω_1 into stationary sets.** The following fact is well–known.

Fact 3.4. (1) *(ZF + the club–filter on ω_1 is normal) If $\vec{C}$ is a C–sequence, then there is a partition of ω_1 into $\aleph_1$–many stationary sets definable from $\vec{C}$.*

(2) *(ZF + DC) If $\vec{r} = (r_\alpha)_{\alpha<\omega_1}$ is a one–to–one ω_1–sequence of reals, then there is a partition of ω_1 into $\aleph_0$–many stationary sets definable from $\vec{r}$.*

I don't know if the normality of the club–filter is need in the first part and if DC is needed in the second part. In fact I don't even know whether ZF alone implies that if $\vec{r}$ is a one–to–one ω_1–sequence of reals, then there is a stationary and co–stationary subset of ω_1 definable from $\vec{r}$.

Theorem 3.5. *Let $\lambda \geq \omega$ be a nonzero cardinal. The following theories are equiconsistent.*

(1) ZFC + *There is a measurable cardinal.*

(2) ZFC + *There is a partition $(S_i)_{i<\lambda}$ of ω_1 into stationary sets such that no partition of ω_1 into more than λ–many stationary sets is definable from $(S_i)_{i<\lambda}$.*

Proof. Let κ be measurable. By a classical result of Kunen–Paris ([6], see also [4]) we may assume that there are distinct normal measures $\mathcal{U}_i$ on κ for $i < \lambda$. We may then find stationary subsets S_i of κ, for $i < \lambda$, such that for all $i^* < \lambda$, i^* is the unique $i < \lambda$ such that $S_i \in \mathcal{U}_i$. We may assume that each S_i consists of inaccessible cardinals. In $\mathbf{V}^{\mathrm{Coll}(\omega, <\kappa)}$, let $\dot{\mathcal{P}}$ be a homogeneous forcing preserving the stationarity of all S_i and adding a club C of $\kappa = \omega_1$, $C \subseteq \bigcup_{i<\lambda} S_i$, together with enumerations $(X^i_\alpha)_{\alpha<\kappa}$ of $\mathcal{U}_i$ for each $i < \lambda$ such that for all $\alpha \in C \cap S_i$, $\alpha \in \bigcap_{\beta<\alpha} X^i_\beta$: A condition in $\dot{\mathcal{P}}$ can be taken to be a pair of the form $(c, \langle (X^i_\alpha)_{\alpha<\bar{\alpha}} : i < \lambda\rangle)$, for some $\bar{\alpha} < \kappa$, such that c is a closed subset of $(\bar{\alpha}+1) \cap \bigcup_{i<\lambda} S_i$, $\{X^i_\alpha : \alpha < \bar{\alpha}\} \subseteq \mathcal{U}_i$ for all i, and such that for all $i < \lambda$ and all $\alpha \in c \cap S_i$, $\alpha \in \bigcap_{\beta<\alpha} X^i_\beta$. Given $p_\epsilon = (c^\epsilon, \langle (X^{i,\epsilon}_\alpha)_{\alpha<\bar{a}_\epsilon} :$

$i < \lambda\rangle) \in \dot{\mathcal{P}}$, $\epsilon = 0, 1$, p_1 extend p_0 if $\bar{a}_0 \leq \bar{a}_1$, $c^1 \cap (\bar{a}_0 + 1) = c^0$, and $(X^{i,1}_\alpha)_{\alpha<\bar{a}_0} = (X^{i,0}_\alpha)_{\alpha<\bar{a}_0}$.

It is easy to see that $\dot{\mathcal{P}}$ is homogeneous in $\mathbf{V}^{\mathrm{Coll}(\omega, <\kappa)}$: Given conditions $p_\epsilon = (c^\epsilon, \langle (X^{i,\epsilon}_\alpha)_{\alpha<\bar{a}_\epsilon} : i < \lambda\rangle)$, $\epsilon = 0, 1$, it is easy to check that $\dot{\mathcal{P}} \restriction q^0 \cong \dot{\mathcal{P}} \restriction q^1$, where $q^\epsilon = (c^\epsilon, \langle (Y^{i,\epsilon}_\alpha)_{\alpha<\bar{\alpha}_0+\bar{\alpha}_1} : i < \lambda\rangle)$ for $\epsilon = 0, 1$, and where for $i < \lambda$ and $\epsilon = 0, 1$, $(Y^{i,\epsilon}_\alpha)_{\alpha<\bar{\alpha}_\epsilon} = (X^{i,\epsilon}_\alpha)_{\alpha<\bar{\alpha}_\epsilon}$ and $\{Y^{i,\epsilon}_\alpha : \alpha < \bar{\alpha}_0 + \bar{\alpha}_1\} = \{X^{i,0}_\alpha, : \alpha < \bar{\alpha}_0\} \cup \{X^{i,1}_\alpha : \alpha < \bar{\alpha}_1\}$. To see that $\dot{\mathcal{P}}$ preserves the stationarity of each i, let $\dot{C}$ be a $\dot{\mathcal{P}}$–name in $\mathbf{V}^{\mathrm{Coll}(\omega, <\kappa)}$ for a club of κ, let $N \preccurlyeq H(\theta)$ for some large enough θ, $N \in \mathbf{V}$, such that N contains the relevant objects, $|N| < \kappa$, $N \cap \kappa \in S_i$, and such that $\delta := N \cap \kappa \in X$ for every $X \in \mathcal{U}_i \cap N$.[8] Now, if G is $\mathrm{Coll}(\omega, <\kappa)$–generic over $\mathbf{V}$ and $\mathcal{P} = (\dot{\mathcal{P}})_G$, $N[G \cap \mathrm{Coll}(\omega, <\delta)] = N[G]$ is countable in $\mathbf{V}[G]$. Let $(p_n)_{n<\omega} \in \mathbf{V}[G]$ be an $(N[G], \mathcal{P})$–generic sequence, $p_n = (c_n, \langle (X^{i,n}_\alpha)_{\alpha<\bar{\alpha}_n} : i < \lambda\rangle)$ for all n, and let $p = (\bigcup_n c_n \cup \{\delta\}, \langle \bigcup_n (X^{i,n}_\alpha)_{\alpha<\bar{\alpha}_n} : i < \lambda\rangle)$. Then p extends all p_n and forces $\delta \in \dot{C}$. Finally, a standard density argument shows that the generic club and generic enumerations of $\mathcal{U}_i$ (for $i < \lambda$) added by $\mathcal{P}$ are as desired.

Now let H be $\mathrm{Coll}(\omega, <\kappa) * \dot{\mathcal{P}}$–generic over $\mathbf{V}$, let C be the generic club of κ added by $\dot{\mathcal{P}}$ over $\mathbf{V}^{\mathrm{Coll}(\omega, <\kappa)}$, and suppose, towards a contradiction, that there is a cardinal $\lambda' > \lambda$ and a partition $(A_i)_{i<\lambda'}$ of $\omega_1^{\mathbf{V}[H]} = \kappa$ into stationary sets definable from $(S_i)_{i<\lambda}$. By homogeneity of $\mathrm{Coll}(\omega, <\kappa) * \dot{\mathcal{P}}$, $(A_i)_{i<\lambda'} \in \mathbf{V}$. There must then be some $i^* < \lambda$ and two distinct $i_0, i_1 < \lambda'$ such that both $A_{i_0} \cap S_{i^*}$ and $A_{i_1} \cap S_{i^*}$ are stationary in $\mathbf{V}[H]$. There can be at most one $\epsilon \in \{0, 1\}$ such that $A_{i_\epsilon} \cap S_{i^*} \in \mathcal{U}_{i^*}$. In that case it follows that a final segment of $A_{i_\epsilon} \cap S_{i^*}$ is contained in C. But then $A_{i_{1-\epsilon}} \cap S_{i^*}$ is non-stationary, which is a contradiction. And if no $A_{i_\epsilon} \cap S_{i^*}$ is in $\mathcal{U}_{i^*}$, then of course no $A_{i_\epsilon} \cap S_{i^*}$ is stationary, which again is a contradiction.

For the other direction, suppose $(S_i)_{i<\lambda}$ is a partition of ω_1 into stationary sets such that there is no partition of ω_1 into more than λ–many stationary sets definable from $(S_i)_{i<\lambda}$ (equivalently, definable from $(S_i)_{i<\lambda}$ together with any ordinal). Let A be a set of ordinals definable from, and coding $(S_i)_{i<\lambda}$. We show that $\kappa = \omega_1$ is measurable in the ZFC–model $\mathbf{HOD}(A)$.[9] This is easy if λ is finite; in fact, in

[8] To find N, let first $(N_\xi)_{\xi<\kappa}$ be a $\subseteq$–continuous chain of elementary substructures of $H(\theta)$ of size less than κ containing everything relevant. Let $(X_\nu)_{\nu<\kappa}$ enumerate $\mathcal{U}_i \cap \bigcup_{\xi<\kappa} N_\xi$, and let $X = S_i \cap \Delta_{\nu<\kappa} X_\nu \in \mathcal{U}_i$. Now we may fix some $\delta \in X$ such that $N_\delta \cap \kappa = \delta$ and N_δ is as desired.

[9] Where $\mathbf{HOD}(A)$ is the class of all x such that $\mathrm{TC}(\{x\}) \subseteq \mathbf{OD}(A)$ and $\mathbf{OD}(A)$ is the class of all sets definable in $\mathbf{V}$ from A together with some ordinal.

this case, for every $i < \lambda$, the club filter on ω_1 restricted to S_i is, in $\mathbf{HOD}(A)$, a κ–complete ultrafilter on κ. If $\lambda = \omega$, fix any $i < \lambda$ and assume towards a contradiction that there is no stationary $S \subseteq S_i$ in $\mathbf{HOD}(A)$ such that the club filter on κ is an ultrafilter in $\mathbf{HOD}(A)$. Then we can define from A a $\subseteq$–maximal assignment $(S_t : t \in T)$ of stationary subsets of S_i, for some tree $T \subseteq {}^{<\kappa}2$, such that $S_{t'} \subseteq S_t$ for all $t \subseteq t'$ in T, and with the property that for every $t \in T$, if t is not a maximal node in T, then $\{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle\} \subseteq T$ and $\{S_{t^\frown\langle 0\rangle}, S_{t^\frown\langle 1\rangle}\}$ is a partition of S_t into stationary sets. By $\subseteq$–maximality of $(S_t : t \in T)$ and the countable completeness of the nonstationary ideal it follows then that there is $X \subseteq T$ of size $\aleph_1$ definable from A such that $\{S_t : t \in X\}$ is a set of pairwise disjoint stationary sets, which contradicts our choice of $(S_i)_{i<\omega}$ and A. □

It would be interesting to explore the possibilities for (other) large cardinal axioms to be equiconsistent with "ZFC+$P(x)$ has definability strength strictly greater than $Q(x)$" for other natural pairs of properties $P(x)$, $Q(x)$.

Recall that, for a nonzero $n \in \omega$, $\boldsymbol{\delta}^1_n$ denotes the supremum of the lengths of all $\mathbf{\Delta}^1_n$–pre-wellorderings of the reals. It is not clear how to convert the proof of Theorem 3.5 into a corresponding consistency result over ZF, but one can easily prove such results starting with a model of ZF + AD. For example, a classical well–known result of Solovay (cf. [4]) is that, under AD, the club filter on $\boldsymbol{\delta}^1_1 = \omega_1$ is an ultrafilter and therefore ω_1 cannot be partitioned into 2 stationary sets. The following theorem generalises this.

Theorem 3.6. *(*ZF + AD*) For every $n < \omega$, $\boldsymbol{\delta}^1_{2n+1}$ is a successor cardinal and regular and, letting κ be such that $\kappa^+ = \boldsymbol{\delta}^1_{2n+1}$, $\mathrm{Coll}(\omega, \kappa)$ forces that there is a partition of $(\boldsymbol{\delta}^1_{2n+1})^{\mathbf{V}} = \omega_1$ into $2^{n+1} - 1$ many stationary sets but no partition of ω_1 into more than $2^{n+1} - 1$ many stationary sets.*

Proof. Let $\lambda = \boldsymbol{\delta}^1_{2n+1}$. By a result of Kechris, λ is a successor cardinal. By a result of Jackson ([2]) there are exactly $2^{n+1} - 1$ many infinite regular cardinals κ below λ and λ has the strong partition property (i.e., $\lambda \longrightarrow (\lambda)^\lambda_\alpha$ for all $\alpha < \lambda$), which entails the regularity of λ and is more than enough (cf. [5]) to guarantee that λ is measurable and that, moreover, letting $\mathcal{C}^\lambda_\mu$ be, for every infinite regular cardinal $\mu < \lambda$, the filter generated by the μ–closed and unbounded subsets of λ, $\{\mathcal{C}^\lambda_\mu : \mu < \lambda, \mu \text{ an infinite regular cardinal}\}$ is the set of normal measures on λ. Let now κ be such that $\kappa^+ = \lambda$ and note that $\mathrm{Coll}(\omega, \kappa)$ can be well–ordered in length κ since a condition in $\mathrm{Coll}(\omega, \kappa)$ can be

canonically coded by a finite tuple in κ and thus by an ordinal in κ. It follows from this that $\mathrm{Coll}(\omega, \kappa)$ forces $\omega_1 = \lambda$. It follows also that every club of λ in any extension by $\mathrm{Coll}(\omega, \kappa)$ contains a club in $\mathbf{V}$, and hence $\mathrm{Coll}(\omega, \kappa)$ preserves the stationarity of $\lambda \cap \mathrm{cf}^{\mathbf{V}}(\mu)$ for every infinite $\mathbf{V}$–regular $\mu < \lambda$. Finally, let $\dot{S}$ be a name for a subset of $\lambda \cap \mathrm{cf}^{\mathbf{V}}(\mu)$, for some such μ, let $\bar{p} \in \mathrm{Coll}(\omega, \kappa)$, and note that $\bar{p}$ forces $\dot{S} \subseteq \bigcup_{p \in \mathrm{Coll}(\omega,\kappa) \restriction \bar{p}} \{\alpha \in \mathrm{cf}^{\mathbf{V}}(\mu) : p \Vdash_{\mathrm{Coll}(\omega,\kappa)} \alpha \in \dot{S}\}$. Again by the fact that $\mathrm{Coll}(\omega, \kappa)$ can be well–ordered in length κ, it follows that there is a μ–club $D \in \mathbf{V}$ such that either $D \cap S = \emptyset$ or $D \subseteq \{\alpha \in \mathrm{cf}(\mu) : p \Vdash_{\mathrm{Coll}(\omega,\kappa)} \alpha \in \dot{S}\}$ for some $p \leq \bar{p}$. From this we immediately get that $\mathrm{Coll}(\omega, \kappa)$ implies that there is no partition of κ into more than $2^{n+1} - 1$ many stationary sets. □

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