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A VARIANT PROOF OF Con($\mathfrak{b} < \mathfrak{a}$)

JÖRG BRENDLE AND ANDREW D. BROOKE-TAYLOR

Abstract. We present a variation of the proof in [2] of Con($\mathfrak{b} < \mathfrak{a}$), which in particular removes some of the obstacles to generalising the argument to cardinals $\kappa > \omega$.

§1. Introduction. The generalisations of cardinal characteristics of the continuum to cardinals κ greater than ω has generated significant interest recently. A particular result that has so far resisted attempts at generalisation is the statement that $\mathfrak{b} < \mathfrak{a}$ is consistent. Blass, Hyttinen and Zhang [1, Section 5] briefly survey the different approaches known for proving Con($\mathfrak{b} < \mathfrak{a}$), highlighting the difficulties each presents for a generalisation.

We present here a variation on the proof of Con($\mathfrak{b} < \mathfrak{a}$) given in [2], which we hope will be more amenable to generalisation. In particular, the proof in [2] relies on a rank argument, which of course cannot be naïvely generalised to uncountable κ. We show here that it may be replaced by a suitable formulation in terms of games, which *does* generalise to higher κ. Indeed, with this observation, the question of forcing $\mathfrak{b}_\kappa > \mathfrak{a}_\kappa$ for some suitable large cardinal κ seems to boil down to interesting questions about the existence of suitable filters on κ.

§2. Preliminaries. Let κ be an infinite cardinal. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called *almost disjoint* if $|A \cap B| < \kappa$ for any two distinct members A and B of $\mathcal{A}$. $\mathcal{A}$ is a *maximal almost disjoint family* (*mad family*, for short) if $\mathcal{A}$ is almost disjoint and maximal with this property. This means that for every $C \in [\kappa]^\kappa$ there is $A \in \mathcal{A}$ such that $|A \cap C| = \kappa$. The *almost disjointness number* $\mathfrak{a}_\kappa$ is the least

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size of a mad family on κ of size at least $cf(\kappa)$ (equivalently, of size $> cf(\kappa)$). In case $\kappa = \omega$ write $\mathfrak{a}$ for $\mathfrak{a}_\omega$.

Now assume κ is a regular cardinal. For functions $f, g \in \kappa^\kappa$, say that g *eventually dominates* f ($f \leq^* g$ in symbols) if $f(\alpha) \leq g(\alpha)$ holds for all α beyond some $\alpha_0 < \kappa$. The *unbounding number* $\mathfrak{b}_\kappa$ is the least size of an unbounded family $\mathcal{F}$ in the order $(\kappa^\kappa, \leq^*)$. That is, for all $g \in \kappa^\kappa$ there is $f \in \mathcal{F}$ with $f(\alpha) > g(\alpha)$ for cofinally many α's. Again we write $\mathfrak{b}$ instead of $\mathfrak{b}_\omega$.

Let $\mathcal{F}$ be a filter on ω. *Mathias forcing* $\mathbb{M}(\mathcal{F})$ *with* $\mathcal{F}$ consists of conditions (s, F) such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max(s) < \min(F)$. $\mathbb{M}(\mathcal{F})$ is ordered by $(t, G) \leq (s, F)$ if $s \subseteq t \subseteq s \cup F$ and $G \subseteq F$. It is well-known and easy to see that $\mathbb{M}(\mathcal{F})$ is a σ-centered forcing which introduces a pseudointersection Z of the filter $\mathcal{F}$. This means that $Z \subseteq^* F$ for all $F \in \mathcal{F}$, where $\subseteq^*$ denotes *almost inclusion:* $A \subseteq^* B$ iff $A \setminus B$ is finite.

In [2], the notion of *pseudocontinuity* is used. This notion and the corresponding basic lemma can be nicely phrased in terms of continuity with respect to an appropriate topology.

DEFINITION 1. *The* initial segment topology *on ω is the topology which has the (von Neumann) ordinals as open sets. We denote ω endowed with this topology by ω_i.*

DEFINITION 2. *A function to ω or ω^ω is* pseudocontinuous *if it is continuous as a function to ω_i or ω_i^ω respectively.*

Thus, a pseudocontinuous function $F : X \to \omega$ is one such that for every $n \in \omega$, the set of x in X with image at most n is open.

LEMMA 3. *Compact sets in ω_i and ω_i^ω are bounded. In particular, any pseudocontinuous image in ω or ω^ω of a compact set must be bounded.*

PROOF. The Lemma is clear for ω_i. Similarly, compact $K \subset \omega_i^\omega$ are in fact bounded in the strict (not just $\leq^*$) sense. Otherwise, there would be some m in ω such that $f(m)$ is unbounded in ω for $f \in K$, and then the open sets $\mathcal{O}_{m,n} = \{f \in \omega_i^\omega \mid f(m) \leq n\}$ for $n < \omega$ would form an open cover of K with no finite subcover. ⊣

As usual we may identify $\mathcal{P}(\omega)$ with 2^ω by way of the map taking sets to their characteristic functions, $\chi : X \mapsto \chi_X$. We give $\mathcal{P}(\omega)$ the corresponding topology, making χ a homeomorphism from $\mathcal{P}(\omega)$ to the Cantor space 2^ω.

DEFINITION 4. *For any cardinal* λ, *we call a filter* $\mathcal{G} \subseteq \mathcal{P}(\omega)$ *a* K_λ-filter *if it is generated by the union of fewer than* λ *many compact subsets of* $\mathcal{P}(\omega)$. *We write* K_σ *for* $K_{\aleph_1}$.

LEMMA 5. *If* $K_0, \ldots, K_{n-1}$ *are (finitely many) compact subsets of* $\mathcal{P}(\omega)$, *then the pointwise intersection*

$$\bigwedge_{i<n} K_i = \Big\{ \bigcap_{i<n} G_i \;\Big|\; (G_0, \ldots, G_{n-1}) \in \prod_{i<n} K_i \Big\}$$

and the pointwise union

$$\bigvee_{i<n} K_i = \Big\{ \bigcup_{i<n} G_i \;\Big|\; (G_0, \ldots, G_{n-1}) \in \prod_{i<n} K_i \Big\}$$

are compact. Furthermore, for any compact set $K \subseteq \mathcal{P}(\omega)$, *the upward closure*

$$\bar{K} = \{A \in \mathcal{P}(\kappa) \mid \exists B \in K(A \supseteq B)\}$$

is also compact.

PROOF. The product $\prod_{i<n} K_i$ is compact by the Tychonoff theorem, and the functions $\mathcal{P}(\omega)^n \to \mathcal{P}(\omega)$ given by $(G_0, \ldots, G_{n-1}) \mapsto \bigcap_{i<n} G_i$ and $(G_0, \ldots, G_{n-1}) \mapsto \bigcup_{i<n} G_i$ are clearly continuous, so $\bigwedge_{i<n} K_i$ and $\bigvee_{i<n} K_i$ are compact. Finally, for compact $K \subseteq \mathcal{P}(\omega)$, $\bar{K}$ is just $K \vee \mathcal{P}(\omega)$. ⊣

§3. The proof. We work in a model V of ZFC in which $\lambda = \mathfrak{c}^V$ is a regular cardinal satisfying $2^\lambda = \lambda^+$, and there is an unbounded, $<^*$-well-ordered sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of strictly increasing functions from ω to ω. For example, any model of GCH will suffice as a ground model, and these properties will be preserved in intermediate stages of our forcing iteration.

Let $\mathcal{A}$ be an infinite maximal almost disjoint family in V of subsets of ω.

THEOREM 6. *There is a ccc forcing* $\mathbb{P}(\mathcal{A})$ *such that*

$$\Vdash_{\mathbb{P}(\mathcal{A})} \mathcal{A} \textit{ is not mad and } \langle f_\alpha : \alpha < \lambda \rangle \textit{ is still unbounded.}$$

PROOF. Let $\mathcal{F} = \mathcal{F}(\mathcal{A})$ be the dual filter of $\mathcal{A}$, that is, the filter generated by the sets whose complements are finite or in $\mathcal{A}$. Note that this filter is proper: if for some $k < \omega$ there were $\{A_i \mid i < k\} \subset \mathcal{A}$ such that $|\bigcap_{i<k} \omega \setminus A_i| < \omega$, any other element of $\mathcal{A}$ would have infinite intersection with one of the A_i, violating almost disjointness. Note that the generic subset of ω introduced by Mathias forcing with

$\mathcal{F}$, or any filter extending $\mathcal{F}$, will end the madness of $\mathcal{A}$, as it will be almost contained in $\omega \setminus A$ for every $A \in \mathcal{A}$.

First we add λ many Cohen reals. It is well-known that the unboundedness of $\langle f_\alpha : \alpha < \lambda \rangle$ is preserved in this intermediate extension. In case $\mathcal{A}$ is not mad anymore in this extension we are done. Also, if $\mathcal{F}$ is contained in a K_λ filter $\mathcal{G}$ in the intermediate extension, we may simply force with $\mathbb{M}(\mathcal{G})$ for it is well-known, and easy to see [2, 3.2], that Mathias forcing with a K_λ-filter does not destroy the unboundedness of $\langle f_\alpha : \alpha < \lambda \rangle$. So assume that $\mathcal{F}$ is not contained in any K_λ-filter.

We shall recursively construct a filter $\mathcal{G} \supseteq \mathcal{F}$ such that furthermore

$$(*) \qquad \Vdash_{\mathbb{M}(\mathcal{G})} \langle f_\alpha : \alpha < \lambda \rangle \text{ is unbounded.}$$

Along the construction we shall take care of every potential $\mathbb{M}(\mathcal{G})$-name for a function in ω^ω, either "killing it" or "sealing it off".

To be precise: let us refer to partial functions $\tau : [\omega]^{<\omega} \times \omega \dashrightarrow \omega$ as *preterms*, and let $\mathcal{T} = \{\tau_\beta : \beta < \lambda\}$ be an enumeration of the set of all preterms. Note in particular that if $\mathcal{G} \supseteq \mathcal{F}$ is a filter and $\dot{g}$ is an $\mathbb{M}(\mathcal{G})$-name for a function in ω^ω, then $\tau = \tau_{\dot{g}}$ given by

$$\tau(s,m) = n \text{ iff } \exists G \in \mathcal{G}\, ((s,G) \Vdash \dot{g}(m) = n)$$

is a preterm, the *preterm associated with* $\dot{g}$. We shall constrain attention to names $\dot{g}$ such that $\mathbb{1} \Vdash_{\mathbb{M}(\mathcal{G})} \dot{g} \in \omega^\omega$, since every function from ω to ω in the generic extension has such a name; we call such names *total names.*

We construct filters $\mathcal{G}_\beta$ for $0 \le \beta \le \lambda$, starting from $\mathcal{G}_0 = \mathcal{F}$, such that

- for each $\beta < \lambda$, $\mathcal{G}_{\beta+1}$ is generated by $\mathcal{G}_\beta$ and a K_σ filter $\mathcal{H}_\beta$,
- $\mathcal{G}_\delta = \bigcup_{\beta<\delta} \mathcal{G}_\beta$ for each limit ordinal $\delta \le \lambda$,

and either

(KILL): for all filters $\mathcal{H} \supseteq \mathcal{G}_{\beta+1}$, τ_β is not associated with any total $\mathbb{M}(\mathcal{H})$-name, or

(SEAL): there is an $\alpha < \lambda$ such that for all filters $\mathcal{H} \supseteq \mathcal{G}_{\beta+1}$ and all $\mathbb{M}(\mathcal{H})$-names $\dot{g}$, if $\tau_{\dot{g}} = \tau_\beta$ then $\Vdash_{\mathbb{M}(\mathcal{H})} \dot{g} \not\geq^* \check{f}_\alpha$.

Clearly any filter $\mathcal{G} \supseteq \mathcal{G}_\lambda$ will then satisfy $(*)$.

So suppose $\mathcal{G}_\beta$ has been defined for some $\beta < \lambda$; we wish to find an appropriate K_σ filter $\mathcal{H}_\beta$. Note that $\mathcal{G}_\beta$ is generated by $\mathcal{F}$ and a K_λ filter $\mathcal{G}'_\beta$; without loss of generality we may assume that $\mathcal{F}$ contains all cofinite subsets of ω. Let $\mathcal{K}_\beta$ be a family of fewer than λ many compact subsets of 2^ω generating $\mathcal{G}'_\beta$. By Lemma 5, we may assume that $\mathcal{K}_\beta$ is closed under finite pointwise intersections, and that for all $K \in \mathcal{K}_\beta$, K is upwards-closed under $\subseteq$, so that $\mathcal{G}'_\beta = \bigcup \mathcal{K}_\beta$.

Everything that has come so far can actually be considered to have occurred in a partial extension model, between the original model and the full extension with λ-many Cohens. More explicitly, all (codes of) elements of $\mathcal{K}_\beta$ belong to this intermediate model.

Let $\subset_{\mathrm{ee}}$ denote the strict end-extension relation on $[\omega]^{<\omega}$: that is, $s \subset_{\mathrm{ee}} s'$ if and only if $s \subset s'$ and $\max(s) < \min(s' \setminus s)$; define $\subseteq_{\mathrm{ee}}$, $\supset_{\mathrm{ee}}$ and $\supseteq_{\mathrm{ee}}$ accordingly.

In [2], a rank function was used. For our generalisation, we take a different approach using games, but use these games to much the same end as the rank function is used in [2]. It should be noted that our games are very closely related to the games independently introduced by Guzmán, Hrušák, and Martínez [3], also in the context of a proof of Con($\mathfrak{b} < \mathfrak{a}$).

Let $\tau = \tau_\beta$.

DEFINITION 7. *Given $\tau \in \mathcal{T}$, the τ* nominalisation exercise *is the following game. There are two players, Sensei and Student. On turn 0, Sensei chooses an $m \in \omega$ and $t_0 \in [\omega]^{<\omega}$. At odd stages $2d + 1$, Student plays a filter set $F(d) \in \mathcal{F}$ and a compact set $K(d) \in \mathcal{K}_\beta$. At even stages $2d + 2$, Sensei plays an element t_{d+1} of $[\omega]^{<\omega}$ such that*

- *t_{d+1} end-extends t_d*
- *$t_{d+1} \setminus t_d \subseteq F(d)$*
- *$t_{d+1} \setminus t_d$ meets every member of $K(d)$.*

If there is $s \subseteq t_{d+1}$ end extending t_0 such that $(s, m) \in \mathrm{dom}(\tau)$, Sensei declares Student to have passed and the game ends. If the game continues for infinitely many stages, then (clearly) Student has failed.

Note that, since $\mathcal{G}_\beta$ is a filter, and by compactness of $K(d)$, a t_{d+1} satisfying the requirements always exists. Also notice that if Student wins, he wins after finitely many steps. Hence the game is open and, by the classical Gale-Stewart Theorem, determined.

As in [2], we now distinguish two cases (in [2] they are *Subcases*), corresponding to options (KILL) and (SEAL) above.

3.1. Case a. There are $m \in \omega$ and $t_0 \in [\omega]^{<\omega}$ such that Sensei has a winning strategy in the τ nominalisation exercise with 0th move (m, t_0): play will continue for infinitely many steps. In this case we shall choose $\mathcal{H}_\beta$ in such a way that (KILL) holds: τ will not correspond to a name for a function $\omega \to \omega$ in the generic extension. The reader may wish to remember which case is which by the mnemonic "the τ that can be named is not the eternal τ."

We shall actually work in the extension of such the intermediate model by one further Cohen function $c : \omega \to \omega$.

Consider the tree T of all possible sequences of plays $(t_0, t_1, t_2, \dots)$ for Sensei according to his strategy, corresponding to all possible plays of Student. Note that T is infinitely branching since $\mathcal{F}$ extends the Frechet filter. Use the Cohen function c to choose a branch through T, and denote the union of the t_i of this branch by G. There is no (s, m) with m from Sensei's first move and $t_0 \subseteq_{\mathsf{ee}} s \subseteq G$ such that $(s, m) \in \mathrm{dom}(\tau_\beta)$. Indeed otherwise, the τ_β nominalisation exercise would have ended once Sensei played t_d sufficiently long to cover s. Thus, for any filter $\mathcal{H} \ni G$, $\tau \neq \tau_{\dot{g}}$ for any total $\mathbb{M}(\mathcal{H})$ name $\dot{g}$. We may therefore simply take $\mathcal{H}_\beta = \{G\}$ in order to satisfy (KILL). To check that $\{G\} \cup \mathcal{G}_\beta$ generates a filter, consider any $F \in \mathcal{F}$ and $G' \in \mathcal{G}'_\beta$, say G' is in the compact set $K \in \mathcal{K}_\beta$. For every $t_d \in T$, there is a successor node t_{d+1} in the tree T that is Sensei's response, according to his strategy, to Student playing F and K, and so in particular this t_{d+1} meets the intersection of F and every member of K. Thus, by Cohen genericity we have that $|G \cap F \cap G'| = \omega$, completing Case a. (Note that G' may not belong to the intermediate model; this, however, is irrelevant for it is sufficient that K does. By genericity the Cohen real c will produce infinitely many d such that $t_{d+1} \setminus t_d$ is contained in F and meets every $G'' \in K$, and this is clearly absolute and thus also holds for G'.)

3.2. Case b. The negation of Case a: for every 0th move (m, t_0) by Sensei, Student has a winning strategy in the τ_β nominalisation exercise. In this case we wish to choose $\mathcal{H}_\beta$ in such a way that (SEAL) holds.

Since Sensei chooses his moves from a countable set, there are clearly only countable many filter sets $F_\ell \in \mathcal{F}$, $\ell \in \omega$, which appear as $F(d)$ in some $2d + 1$st move of Student playing according to his strategy.

Suppose that for all but less than λ many members A of $\mathcal{A}$, there is $G \in \mathcal{G}'_\beta$ such that $A \cap G$ is finite. Then, adding less than λ many sets of the form $\omega \setminus A$, $A \in \mathcal{A}$, to $\mathcal{G}'_\beta$ results in a K_λ filter containing $\mathcal{F}$. This contradicts our initial assumption. Hence, for λ many $A \in \mathcal{A}$, $A \cap G$ is infinite for all $G \in \mathcal{G}'_\beta$. Let A_j, $j \in \omega$, be countably many such A's such that for each j and ℓ, A_j is almost contained in F_ℓ: this is possible because $\mathcal{F}$ is the dual filter of the mad family $\mathcal{A}$.

For each $G' \in \mathcal{G}'_\beta$, $k \in \omega$, $j \in \omega$, and finite subset T of $[\omega]^{<\omega}$, we define a function $f_{G',k,j,T} : \omega \to \omega$ as follows.

$$f_{G',k,j,T}(m) = \min\{n \mid \text{for any partition } A_j = \bigcup_{i<k} B_i \text{ there is } i < k \text{ s.t.}$$
$$\forall t \in T \exists s \supset_{\text{ee}} t (s \setminus t \subseteq B_i \cap G' \wedge \tau_\beta(s,m) \leq n)\}.$$

LEMMA 8. *For every* $G' \in \mathcal{G}'_\beta$, $k, j \in \omega$, *and* $T \in [[\omega]^{<\omega}]^{<\omega}$, $f_{G',k,j,T}$ *is well-defined.*

PROOF. Fix $m \in \omega$. Given a partition $\{B_i \mid i < k\}$ of A_j, let "n suffices for $\{B_i \mid i < k\}$" mean the natural thing in the context of the definition of $f_{G',k,j,T}$, namely, that there is $i < k$ such that for every $t \in T$ there is $s \supset_{\text{ee}} t$ with $s \setminus t \subseteq B_i \cap G'$ and $\tau_\beta(s,m) \leq n$. So now fix a partition $\{B_i \mid i < k\}$ of A_j; we shall show that there is a $n \in \omega$ that suffices for it. Let $i < k$ be such that $|B_i \cap G' \cap G| = \omega$ for every $G \in \mathcal{G}'_\beta$: such an i must exist, since A_j has infinite intersection with every member of the filter $\mathcal{G}'_\beta$. Finally, fix $t \in T$.

Consider a play of the τ_β naming exercise in which Student follows his strategy, Sensei's 0th move is (m, t_0) with $t_0 = t$, and his later moves always satisfy the additional requirement $t_{d+1} \setminus t_d \subseteq B_i \cap G'$. Since B_i is almost contained in all $F(d)$ played by Student according to his strategy and since B_i has infinite intersection with all $G \in \mathcal{G}'_\beta$, Sensei always has a valid such move.

So we have that eventually Sensei plays a t_d such that

$$\exists n_t \in \omega \exists s \subseteq t_d (s \supset_{\text{ee}} t \wedge \tau_\beta(s,m) = n_t).$$

Of course, by the construction of the game, $s \setminus t_0 \subseteq B_i \cap G'$. Taking such an n_t for each $t \in T$ and setting $n = \max_{t \in T}(n_t)$, we have that n suffices for $\{B_i \mid i < k\}$.

Now, with k still fixed but allowing the partition $\{B_i \mid i < k\}$ to vary, let us denote by $n(\{B_i \mid i < k\})$ the least n that suffices for $\{B_i \mid i < k\}$. The space of partitions of A_j into k pieces can be identified with k^{A_j} and thus when endowed with the product topology is a compact topological space. Moreover, with this topology on the space of partitions, the function n sending $\{B_i \mid i < k\}$ to $n(\{B_i \mid i < k\})$ is clearly pseudocontinuous, since n being sufficient for $\{B_i \mid i < k\}$ is witnessed by finitely many finite tuples $s \setminus t$ from B_i, which of course define an open set in k^{A_j}. Thus by Lemma 3 the image of the function n is bounded below ω. The least such upper bound will be $f_{G',k,j,T}(m)$, and it follows that $f_{G',k,j,T}$ is well-defined. ⊣

LEMMA 9. *There exists an $\alpha < \lambda$ such that for all $G' \in \mathcal{G}'_\beta$, $k, j \in \omega$ and $T \in [[\omega]^{<\omega}]^{<\omega}$, $f_\alpha \not\leq^* f_{G',k,j,T}$.*

PROOF. We first note that, given k, j, T, and compact $K \in \mathcal{K}_\beta$, the function $f_{\cdot,k,j,T}$ sending G' to $f_{G',k,j,T}$ is pseudocontinuous from K to ω^ω, by much the same argument as in the proof of Lemma 8. Indeed, fixing m and n, $\{G' \mid f_{G',k,j,T}(m) \leq n\}$ is open in K.

We thus have from Lemma 3 that for each $K \in \mathcal{K}_\beta$, $f_{\cdot,k,j,T}$"K is bounded in ω^ω, say by h_K. Since $\mathcal{K}_\beta$ has fewer than λ many elements, there is an $\alpha < \lambda$ such that f_α is not eventually dominated by any of the h_K, and hence not by any $f_{G',k,j,T}$. ⊣

We now show that α as given by Lemma 9 will make (SEAL) hold for an appropriate choice of $\mathcal{H}_\beta$. Given $t \in [\omega]^{<\omega}$, $G \in \mathcal{P}(\omega)$, and $m \in \omega$, let

$$g^\beta_{t,G}(m) = \min\{n \mid \exists s \supseteq_{\text{ee}} t(s \setminus t \subseteq G \wedge \tau_\beta(s, m) = n)\}$$

if the set on the right hand side is non-empty, and otherwise put $g^\beta_{t,G}(m) = \omega$. Thus, $g^\beta_{t,G}$ is a function in $(\omega + 1)^\omega$. Let $\alpha < \lambda$ be such that f_α is not dominated by any $f_{G',k,j,T}$, as given by Lemma 9, and define

$$\mathcal{H}_\beta = \{H \subseteq \omega \mid \exists t \in [\omega]^{<\omega}(g^\beta_{t,\omega \setminus H} \geq^* f_\alpha)\}.$$

Note that given $t \in [\omega]^{<\omega}$ and $m_0 \in \omega$, the set

$$\{H \subseteq \omega \mid \forall m \geq m_0 (g^\beta_{t,\omega\setminus H}(m) \geq f_\alpha(m))\}$$

is closed in $\mathcal{P}(\omega)$, and hence compact. Therefore, $\mathcal{H}_\beta$ is a K_σ set.

To see that this set is an appropriate choice of $\mathcal{H}_\beta$ as called for above, we check the following.

CLAIM 10. *Any filter $\mathcal{H} \supseteq \mathcal{H}_\beta$ satisfies (SEAL).*

PROOF. Let $\mathcal{H} \supseteq \mathcal{H}_\beta$ be a filter, and assume $\tau_\beta = \tau_{\dot{g}}$ for some $\mathbb{M}(\mathcal{H})$-name $\dot{g}$ for a function in ω^ω. Suppose there were $(t, G) \in \mathbb{M}(\mathcal{H})$ and $m_0 \in \omega$ such that

$$(t, G) \Vdash_{\mathbb{M}(\mathcal{H})} \forall m \geq \check{m}_0 \left(\dot{g}(m) \geq \check{f}_\alpha(m)\right).$$

By the definition of $g^\beta_{t,G}$, we must then also have $g^\beta_{t,G}(m) \geq f_\alpha(m)$ for all $m \geq m_0$. So $\omega \setminus G \in \mathcal{H}_\beta \subseteq \mathcal{H}$, contradicting the fact that $\mathcal{H}$ is a filter. ⊣

CLAIM 11. *$\mathcal{H}_\beta \cup \mathcal{G}_\beta$ generates a filter.*

PROOF. We take $F \in \mathcal{F}$, $G' \in \mathcal{G}'_\beta$, and for some $k < \omega$, $H_i \in \mathcal{H}_\beta$ for $i < k$, and argue that $F \cap G' \cap \bigcap_{i<k} H_i$ has cardinality ω. Assume for the sake of contradiction that $F \cap G' \subseteq^* \bigcup_{i<k} \omega \setminus H_i$. For each $i < k$, fix $t_i \in [\omega]^{<\omega}$ such that $g^\beta_{t_i, \omega \setminus H_i} \geq^* f_\alpha$. Also fix j such that $A_j \subseteq^* F$. Without loss of generality, we may take $a < \omega$ such that $A_j \setminus a \subseteq F$, $F \cap G' \setminus a \subseteq \bigcup_{i<k} \omega \setminus H_i$ and $\max(t_i) \geq a$ for every $i < k$ (if necessary by extending each t_i with a sufficiently large element of $\omega \setminus H_i$: this can only increase the values of $g^\beta_{t_i, \omega \setminus H_i}$). Fix $m_0 \in \omega$ such that $g^\beta_{t_i, \omega \setminus H_i}(m) \geq f_\alpha(m)$ for all $m \geq m_0$ and $i < k$. Let $T = \{t_i \mid i < k\}$ and let $\{B_i \mid i < k\}$ be a partition of A_j such that $B_i \cap G' \setminus a \subseteq \omega \setminus H_i$ for all $i < k$. By the definition of f_α, there is some $m > m_0$ such that $f_\alpha(m) > f_{G',k,j,T}(m)$; take such a m, and denote $f_{G',k,j,T}(m)$ by n. By the definition of $f_{G',k,j,T}$, there is an i such that for all $t \in T$, there is $s \sqsupset_{\mathrm{ee}} t$ such that $\tau_\beta(s, m) \leq n$ and $s \setminus t$ is a subset of the intersection of G' and B_i. In particular, $\min(s \setminus t_i) > \max(t_i) \geq a$, $s \setminus t_i \subset B_i \cap G'$, and $\tau_\beta(s, m) \leq n$. Thus $s \setminus t_i \subseteq \omega \setminus H_i$, from which we have $g^\beta_{t_i, \omega \setminus H_i}(m) \leq n < f_\alpha(m)$, contradicting the choice of m_0. ⊣

This completes the construction of $\mathcal{G}_{\beta+1}$ from $\mathcal{G}_\beta$, and hence the proof of Theorem 6. ⊣

We are now ready for the consistency of $\mathfrak{b} < \mathfrak{a}$. Recall from the beginning of this section that our ground model V satisfies $\mathfrak{c} = \lambda$ is regular, $2^\lambda = \lambda^+$, and $\langle f_\alpha : \alpha < \lambda \rangle$ is unbounded $<^*$-well-ordered.

THEOREM 12. *There is a ccc forcing $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}} \mathfrak{a} = \lambda^+ \text{ and } \langle f_\alpha : \alpha < \lambda \rangle \text{ is still unbounded.}$$

In particular, $\mathfrak{b} \leq \lambda < \lambda^+ = \mathfrak{a}$ is consistent.

PROOF. Perform a finite support iteration of orderings of type $\mathbb{P}(\mathcal{A})$ of length λ^+, going through all (names for) mad families along the way by a bookkeeping argument (this is possible by the assumption $2^\lambda = \lambda^+$). The unboundedness of $\langle f_\alpha : \alpha < \lambda \rangle$ is preserved in the successor step of the iteration by Theorem 6 and in the limit step, by standard preservation results. ⊣

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GROUP OF LOGIC, STATISTICS AND INFORMATICS,
GRADUATE SCHOOL OF SYSTEM INFORMATICS,
KOBE UNIVERSITY
ROKKO-DAI 1-1,
NADA, KOBE, 657-8501
JAPAN
E-mail: brendle@kurt.scitec.kobe-u.ac.jp
E-mail: andrewbt@kurt.scitec.kobe-u.ac.jp