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Forcing proofs of Ramsey's Hindman's Theorems

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Abstract

We give proofs of Ramsey's and Hindman's theorems in which the corresponding homogeneous sets are found with a forcing argument.

1 Introduction

The basic pigeon-hole principle states that for every partition of the set of all natural numbers in finitely-many classes there is an infinite set of natural numbers that is included in one class.

Ramsey's Theorem [6], which can be seen as a generalization of this simple result, is about partitions of the set $[\mathbb{N}]^k$ of all k-element sets of natural numbers. It states that for every $k \geq 1$ and every partition of $[\mathbb{N}]^k$ into finitely-many classes, there is an infinite subset M of $\mathbb{N}$ such that all k-element subsets of M belong to the same class. Such a set is said to be *homogeneous* for the partition.

In [3], Neil Hindman proved a Ramsey-like result that was conjectured by Graham and Rothschild in [2]. Hindman's Theorem asserts that if the set

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of all natural numbers is divided into two classes, one of the classes contains an infinite set such that all finite sums of distinct members of the set remain in the same class.

We are interested in giving proofs of Ramsey's and Hindman's Theorems based on forcing arguments.

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2 Ramsey's Theorem

We are interested in functions $h : [A]^n \longrightarrow \lambda$, where n is a natural number and λ is a cardinal. We often refer to any such function h as a partition of $[A]^n$ (into $\leqslant \lambda$ classes), or a coloring of $[A]^n$ into $\leqslant \lambda$ colors.

If $h : [A]^n \longrightarrow \lambda$, a subset H of A is called *homogeneous* for h, or $h-$*homogeneous*, if and only if h is constant on $[H]^n$, i.e., if and only if $h(X) = h(Y)$ for all $X, Y \in [H]^n$. The notation $(\kappa) \longrightarrow (\alpha)^n_\lambda$, for κ and λ (finite or infinite) cardinals, α an ordinal, and n a natural number, means that for every partition h of $[\kappa]^n$ into $\leqslant \lambda$ classes there is an $h-$homogeneous set of order-type α.

Notice that if κ is an infinite cardinal, A is a subset of κ of cardinality κ and $\kappa \longrightarrow (\alpha)^n_\lambda$ holds, then every partition $f : [A]^n \longrightarrow \lambda$ has an $f-$homogenous set of order-type α.

We will give a proof of Ramsey's Theorem using forcing arguments.

Theorem 2.1 (Ramsey [6]). *For every $n, m > 0$, $\omega \longrightarrow (\omega)^n_m$.*

We will take care only of the case $m = 2$, the general case can easily be proved by induction on m.

Proof. We proceed by induction on $n \geqslant 1$. By the Pigeonhole principle we have it for $n = 1$. Given $n \geqslant 1$ and given $g : [\omega]^{n+1} \longrightarrow 2$, we must conclude that there is an infinite $g-$homogeneous set.

Assume that there is no $H \subseteq \omega$ infinite such that H is $g-$homogeneous of color 0. We will produce an infinite $g-$homogeneous set of color 1.

We define a partial order $\mathbb{P}_{g,1} = \langle P, \leqslant^* \rangle$ as follows: the elements of P are of the form (a, A) where $a \in [\omega]^{<\omega}$, $A \in [\omega]^{\omega}$, $a < A$, which means $\max(a) < \min(A)$, $g(x) = 1$ for all $x \in [a]^{n+1}$, and for every $j \in \{1, \ldots, n\}$, every $y \in [a]^j$, and every $x \in [A]^{n+1-j}$, $g(y \cup x) = 1$.

Given two elements of $(a, A), (b, B) \in P$, we define $(b, B) \leqslant^* (a, A)$, as in Mathias forcing, if and only if a is an initial segment of b, $B \subseteq A$ and for all $x \in (b \setminus a)$ $(x \in A)$.

We have that P is not empty because $(\emptyset, \omega) \in P$, and $\leqslant^*$ is clearly reflexive and transitive.

Claim 2.2. *Given a condition $(a, A) \in P$ we can extend it, i.e., there exists $m \in A$ and there exists $B \subseteq A$ infinite with $k > m$ for all $k \in B$ such that $g(\{m\} \cup y) = 1$ for all $y \in [B]^n$. Note that then $(b, B) \leqslant^* (a, A)$ where $b = a \cup \{m\}$.*

Proof of Claim: Assume, towards a contradiction, that for all $m \in A$ and for all $B \subseteq A$ if $g(\{m\} \cup y) = 1$ for all $y \in [B]^n$, then B is finite.

Let $m_0 = \min A$, and let $g_{m_0} : [A \setminus \{m_0\}]^n \to 2$ be defined as $g_{m_0}(y) = g(\{m_0\} \cup y)$. By inductive hypothesis there is $B_0 \subseteq A \setminus \{m_0\}$ infinite such that $g_{m_0} \restriction [B_0]^n$ is constant. By our assumption $g_{m_0} \restriction [B_0]^n$ is constant with value 0.

Assume that we have elements $m_0 < m_1 < \ldots < m_k$ in A and

$$B_k \subseteq \ldots \subseteq B_0 \subseteq A$$

are such that $m_j = \min B_{j-1}$ for all $j \in \{1, \ldots, k\}$, the function

$$g_{m_j} : [B_{j-1} \setminus \{m_j\}]^n \to 2$$

is defined as $g_{m_j}(y) = g(\{m_j\} \cup y)$, and $g_{m_j} \restriction [B_j]^n$ is constant 0 for all $j \in \{1, \ldots, k\}$.

Then let $m_{k+1} = \min B_k$, and let $g_{m_{k+1}} : [B_k \setminus \{m_{k+1}\}]^n \to 2$ be defined as

$$g_{m_{k+1}}(y) = g(\{m_{k+1}\} \cup y).$$

By the inductive hypothesis and our assumption, there exists

$$B_{k+1} \subseteq B_k \setminus \{m_{k+1}\}$$

infinite such that $g_{m_{k+1}} \restriction [B_{k+1}]^n$ is constant 0.

Inductively we have constructed $H = \{m_i : i \in \omega\}$. We claim that $g \restriction [H]^{n+1}$ is constant 0, which yields a contradiction to our initial assumption.

For $y \in [H]^{n+1}$, the least element of y is m_j, for some $j \in \omega$. Then

$$g(y) = g(\{m_j\} \cup (y \setminus \{m_j\})) = 0$$

because $y \setminus \{m_j\} \in [B_{j+1}]^n$. This proves the claim. □

For every $n \in \omega$ we define $D_n = \{(a, A) \in P : |a| \geqslant n\}$. Note that D_n is a dense set for all $n \in \omega$ and consider $\mathscr{D} = \{D_n : n \in \omega\}$. Let G be a $\mathscr{D}$-generic filter in $\mathbb{P}_{g,1}$ and

$$S := \bigcup \{a \in [\omega]^{<\omega} : \exists A \in [\omega]^{\omega} \text{ such that } (a, A) \in G\}.$$

By the claim we have that S is infinite.

We shall prove that $g \restriction [S]^{n+1}$ is constant 1. Let $y \in [S]^{n+1}$, where $y = \{y_0, \ldots, y_n\}$, then there exist $(a_0, A_0), \ldots, (a_n, A_n)$ in G such that every element $y_j \in a_j$ for all $j \leqslant n$. Since G is a filter there is $(b, B) \in G$ such that (b, B) extends (a_j, A_j) for all $j \leqslant n$. Then $y \in [b]^{n+1}$, and so $g(y) = 1$. Hence S is g-homogeneous with color 1.

□

Given a partition $g : [\omega]^2 \to 2$, by Ramsey's Theorem there exist $X \in [\omega]^{\omega}$ and some $i \in \{0, 1\}$ such that X is g-homogeneous with color i. We define

$$\mathbb{M}_X = \{(s, A) : s \in [X]^{<\omega}, A \in [X]^{\omega} \text{ and } \max s < \min A\}$$

and we define the ordering relation between elements in $\mathbb{M}_X$ as in Mathias forcing. Then $\mathbb{M}_X$ order is isomorphic to Mathias forcing.

If $\mathbb{P}$ is a partial order and $p \in \mathbb{P}$, $\mathbb{P} \restriction p$ is the suborder of $\mathbb{P}$ whose elements are in $\mathbb{P}$ below p.

Theorem 2.3. *The following statements are equivalent:*

1. *$\mathbb{P}_{g,i}$ is non trivial, i.e., every condition can be extended in the finite part.*

2. *Every infinite set of natural numbers has an infinite subset X such that $\mathbb{P}_{g,i} \restriction (\emptyset, X) = \mathbb{M}_X$.*

3. *Every infinite set of natural numbers has an infinite subset X such that X is a $g-$homogeneous set of color i.*

Proof. 1$\rightarrow$2. Assume that $\mathbb{P}_{g,i}$ is non trivial. Let Y be an infinite set of natural numbers. Consider the condition $(\emptyset, Y) \in \mathbb{P}_{g,i}$ and let G be a $\mathbb{P}_{g,i}$-generic filter such that $(\emptyset, Y) \in \mathbb{P}_{g,i}$. Then there exists $X \in [Y]^\omega$ such that $g \restriction [X]^2$ is constant with value i.

Consider the partial order $\mathbb{P}_{g,i} \restriction (\emptyset, X)$, i.e., all elements in $\mathbb{P}_{g,i}$ below the condition $(\emptyset, X)$. Then the *id* function is an isomorphism between the partial orders $(\mathbb{P}_{g,i} \restriction (\emptyset, X), \leqslant^*)$ and $(\mathbb{M}_X, \leqslant)$.

2$\rightarrow$3. Let Y be an infinite set of natural numbers, by assumption there exists $X \in [Y]^\omega$ such that $\mathbb{P}_{g,i} \restriction (\emptyset, X) = \mathbb{M}_X$. The *id* function is an isomorphism between the partial orders $(\mathbb{P}_{g,i} \restriction (\emptyset, X), \leqslant^*)$ and $(\mathbb{M}_X, \leqslant)$. Let $\{n, m\} \in [X]^2$. Assume that $n < m$ and let

$$A := \{x \in X : x > n\} \text{ and } B := \{x \in X : x > m\}.$$

Then the conditions $(\{n\}, A)$, $(\{n, m\}, B)$ belong to $\mathbb{M}(X)$ and $(\{n, m\}, B) \leqslant (\{n\}, A)$. By the assumption $(\{n, m\}, B) \leqslant^* (\{n\}, A)$, in particular $g(\{n, m\}) = i$. Hence $g \restriction [X]^2$ is constant i.

3$\rightarrow$1. It is trivial. $\square$

3 Hindman's Theorem

In this chapter we will define a partial order associated to Hindman's theorem ([3]) and we will give a proof of Hindman's theorem using forcing arguments relative to this partial ordering. Our proof uses some lemmas from Baumgartner's proof of the theorem, in [1].

Definition 3.1. Let $H \subseteq \omega$. $FS(H) = \{\sum_{n \in a} n : a \in [H]^{<\omega} \textit{ and } a \neq \emptyset\}$, $FS(H)$ is called the *sum-set* of H. For example:

$$FS(\{2, 3, 7\}) = \{2, 3, 5, 7, 9, 10, 12\}.$$

Theorem 3.2 (Hindman [3]). *If ω is finitely colored, then there exists H an infinite subset of ω, such that $FS(H)$ is monochromatic.*

Call $\mathscr{D}$ a *disjoint collection* if $\mathscr{D}$ is an infinite set of pairwise disjoint finite subsets of natural numbers.

We denote by FIN the set of all finite non empty subsets of ω. For s and t in FIN, we write $s < t$ if $\max(s) < \min(t)$.

If X is a subset of FIN, then we write $FU(X)$ for the set of all finite unions of members of X, excluding the empty union.

Theorem 3.3 (Baumgartner [1]). *Let $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_k$. Then there exist $0 \leqslant i \leqslant k$ and a disjoint collection $\mathscr{D}$ with $FU(\mathscr{D}) \subseteq \mathscr{C}_i$.*

Lemma 3.4. *Theorem 3.3 implies Hindman's Theorem.*

Proof. Let $k \geqslant 1$ be a natural number and let $h : \omega \to k$ be a coloring of ω with k colors. Consider the canonical bijection $g : [\omega]^{<\omega} \to \omega$, that assigns each $s \in [\omega]^{<\omega}$ to $n_s = \sum_{i \in s} 2^i$. Then $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_{k-1}$ where

$$\mathscr{C}_i = \{s \in [\omega]^{<\omega} : h(g(s)) = i\}$$

for $i \in \{0, \ldots, k-1\}$. By Theorem 3.3 there exists $0 \leqslant i < k$ and a disjoint collection $\mathscr{D}$ with $FU(\mathscr{D}) \subseteq \mathscr{C}_i$.

Let $H := \{g(d) : d \in \mathscr{D}\}$. It is clear that $H \subseteq \omega$ is infinite. Let $s \in [H]^{<\omega}$ with $s \neq \emptyset$. Then $s = \{a_0, \ldots, a_m\}$, where $a_0 = g(d_{j_0}), \ldots, a_m = g(d_{j_m})$.

We have:

$$h(a_0 + \ldots + a_m) = h(g(d_{j_0}) + \ldots + g(d_{j_m})) = h(\sum_{l \in d_{j_0}} 2^l + \ldots + \sum_{l \in d_{j_m}} 2^l) =$$

$$h(\sum_{l' \in \bigcup_{l=0}^{l=m} d_{j_l}} 2^{l'}) = h(g(\bigcup_{l=0}^{l=m} d_{j_l})) = i$$

Hence $h \restriction FS(H)$ is monochromatic. $\square$

On the class of disjoint collections of finite subsets of natural numbers, we define a partial order $\sqsubseteq$ by $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ if and only if $\mathscr{D}_1 \subseteq FU(\mathscr{D})$.

Definition 3.5. Given a collection of finite subsets of natural numbers $\mathscr{C}$, we say *$\mathscr{C}$ is large for $\mathscr{D}$* if $\mathscr{C} \cap FU(\mathscr{D}_1) \neq \emptyset$ for all $\mathscr{D}_1 \sqsubseteq \mathscr{D}$.

Lemma 3.6 (Decomposition Lemma, Baumgartner [1]). *Assume that $\mathscr{C}$ is large for $\mathscr{D}$ and $\mathscr{C} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_k$. Then there exists $0 \leqslant i \leqslant k$ and $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ such that $\mathscr{C}_i$ is large for $\mathscr{D}_1$.*

Proof. By induction on k.

Let $k = 1$. If $\mathscr{C} = \mathscr{C}_0 \dot\cup \mathscr{C}_1$ and $\mathscr{C}_0$ is not large for $\mathscr{D}$, then $\mathscr{C}_0 \cap FU(\mathscr{D}_1) = \emptyset$ for some $\mathscr{D}_1 \sqsubseteq \mathscr{D}$. Let $\mathscr{D}_2 \sqsubseteq \mathscr{D}_1$. Since $\mathscr{C}$ is large for $\mathscr{D}$, $\mathscr{C} \cap FU(\mathscr{D}_2) \neq \emptyset$, so $\mathscr{C}_1 \cap FU(\mathscr{D}_2) \neq \emptyset$ (because $FU(\mathscr{D}_2) \subseteq FU(\mathscr{D}_1)$ and $\mathscr{C}_0 \cap FU(\mathscr{D}_1) = \emptyset$). Hence $\mathscr{C}_1$ is large for $\mathscr{D}_1$.

Assume now that the statement is true for k.

Let $\mathscr{C} = \mathscr{C}_0 \dot\cup \ldots \dot\cup \mathscr{C}_{k+1}$. Assume that $\mathscr{C}_0$ is not large for $\mathscr{D}$. Then there is $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ that is large for $\mathscr{C}_1 \dot\cup \ldots \dot\cup \mathscr{C}_{k+1}$. By the inductive hypothesis, there is $\mathscr{D}_2 \sqsubseteq \mathscr{D}_1$ and $i \in \{1, \ldots, k+1\}$ such that $\mathscr{D}_2$ is large for $\mathscr{C}_i$. □

Define $\mathscr{C} - s := \{c \in \mathscr{C} : c \cap s = \emptyset\}$.

Lemma 3.7 (Baumgartner). *If $\mathscr{C}$ is large for $\mathscr{D}$ and s is a finite subset of ω, then $\mathscr{C} - s$ is large for $\mathscr{D}$.*

Proof. Suppose that there is $\mathscr{D}_1 \sqsubseteq \mathscr{D}$ such that $(\mathscr{C} - s) \cap FU(\mathscr{D}_1) = \emptyset$. Let $\mathscr{D}_2 = \{d \in \mathscr{D}_1 : d \cap s = \emptyset\}$. Note that $\mathscr{D}_2$ is infinite since s is finite. Then $\mathscr{C} \cap FU(\mathscr{D}_2) = \emptyset$, but $\mathscr{D}_2 \sqsubseteq \mathscr{D}$, and we reach a contradiction.

□

Lemma 3.8 (Baumgartner). *If $\mathscr{C}$ is large for $\mathscr{D}$, there exist $s \in FU(\mathscr{D})$ and $\mathscr{D}_1 \sqsubseteq \mathscr{D} - s$ such that $\mathscr{C}_1 = \{t \in \mathscr{C} - s : t \cup s \in \mathscr{C}\}$ is large for $\mathscr{D}_1$.*

Proof. Let us first prove the following.

Claim 3.9. *There exist n and $d_1, \ldots, d_n \in \mathscr{D}$ such that, for every $d_{n+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_n$, there exists non-empty $I \subseteq \{1, \ldots, n\}$ such that $d_{n+1} \cup \bigcup_{i \in I} d_i \in \mathscr{C}$.*

Proof of Claim: Suppose, otherwise. If $I \subseteq \{1, \ldots, n\}$, let us write d_I for $\bigcup_{i \in I} d_i$. Thus, for all $n \in \omega$ and for all $d_1, \ldots, d_n \in \mathscr{D}$ there is $d_{n+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_n$ such that $d_{n+1} \cup d_I \notin \mathscr{C}$ for all $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$.

Suppose that we have $d_1, \ldots, d_k$, elements of $FU(\mathscr{D})$, that are pairwise disjoint and such that every finite union of them does not belong to $\mathscr{C}$. By assumption, there is $d_{k+1} \in FU(\mathscr{D})$ disjoint from $d_1 \cup \ldots \cup d_k$ such that $d_{k+1} \cup d_I \notin \mathscr{C}$ for all $I \subseteq \{1, \ldots, k\}$, $I \neq \emptyset$. In this way, we construct $\mathscr{D}' = \{d_1, d_2, \ldots\} \sqsubseteq \mathscr{D}$ such that $\mathscr{C} \cap FU(\mathscr{D}') = \emptyset$, which is a contradiction.

□

Continuing with the proof of the Lemma, fix $d_1, \ldots, d_n \in \mathscr{D}$ and write d^* for $d_1 \cup \ldots \cup d_n$. For $\emptyset \neq I \subseteq \{1, \ldots, n\}$ we let

$$\mathscr{C}_I = \{c \in \mathscr{C} : c \cap d^* = \emptyset, c \cup d_I \in \mathscr{C}\}.$$

We claim that $\mathscr{C}_{I_1} \dot{\cup} \ldots \dot{\cup} \mathscr{C}_{I_k}$ is large for $\mathscr{D}$, where $\{I_1, \ldots, I_k\}$ is a list of all nonempty subsets of $\{i, \ldots, n\}$. For if $\mathscr{D}' \sqsubseteq \mathscr{D}$, then define

$$\mathscr{D}^* := \{d \in \mathscr{D}' : d > d^*\}.$$

So, $\mathscr{D}^* \sqsubseteq \mathscr{D}$ and since $\mathscr{C}$ is large for $\mathscr{D}$ there exists $d \in FU(\mathscr{D}^*) \cap \mathscr{C}$. In particular, $d \in FU(\mathscr{D}') \cap FU(\mathscr{D})$ and d is disjoint from d^* so $d \cup d_{I_i} \in \mathscr{C}$ for some $i \in \{1, \ldots, k\}$. Thus,

$$d \in \{c \in \mathscr{C} : c \cap d^* = \emptyset \text{ and } c \cup d_{I_i} \in \mathscr{C}\} = \mathscr{C}_{I_i}.$$

Since $\mathscr{D} - d^* \sqsubseteq \mathscr{D}$,

$$\mathscr{C}_{I_1} \dot{\cup} \ldots \dot{\cup} \mathscr{C}_{I_k}$$

is also large for $\mathscr{D} - d^*$, so by Lemma 3.6 there is $\mathscr{C}_{I_i}$ large for some $\mathscr{D}' \sqsubseteq \mathscr{D} - d^*$. And this proves the lemma with $s = d_{I_i}$, because $\mathscr{C}_{I_i} \subseteq \mathscr{C}_1$. □

Lemma 3.10 (Baumgartner). *If $\mathscr{C}$ is large for $\mathscr{D}$, then there exist $s' \in FU(\mathscr{D}) \cap \mathscr{C}$ and $\mathscr{D}' \sqsubseteq \mathscr{D} - s'$, such that*

$$\mathscr{C}' = \{t \in \mathscr{C} : t \cap s' = \emptyset \text{ and } t \cup s' \in \mathscr{C}\}$$

is large for $\mathscr{D}'$.

Proof. Notice that only the requirement $s' \in \mathscr{C}$ distinguishes Lemma 3.10 from Lemma 3.8. We apply Lemma 3.8 repeatedly. Beginning with $\mathscr{C}_0 = \mathscr{C}$, $\mathscr{D}_0 = \mathscr{D}$, we find, for $i \geqslant 1$, s_i, $\mathscr{C}_i$, $\mathscr{D}_i$ with $s_{i+1} \in FU(\mathscr{D}_i)$ so that

$$\mathscr{C}_{i+1} = \{T \in \mathscr{C}_i : T \cap s_{i+1} = \emptyset, T \cup s_{i+1} \in \mathscr{C}_i\}$$

is large for $\mathscr{D}_{i+1} \sqsubseteq \mathscr{D}_i$ and $D \cap \bigcup_{j=1}^{i+1} s_j = \emptyset$, for all $D \in FU(\mathscr{D}_{i+1})$.

Note that $\mathscr{C}_{i+1} \subseteq \mathscr{C}_i$ for all $i \in \omega$, and if $T \in \mathscr{C}_{i+1}$ then $T \cup s \in \mathscr{C}$ and $T \cap s = \emptyset$ for all partial unions s of the $s_1, \ldots, s_{i+1}$.

We define $\mathscr{D}^* := \{s_i : i \geqslant 1\}$. So, $\mathscr{D}^*$ is a disjoint collection and $\mathscr{D}^* \sqsubseteq \mathscr{D}$. Since $FU(\mathscr{D}^*) \cap \mathscr{C} \neq \emptyset$, we can find $i_1 < \ldots < i_k$ such that

$$s' = s_{i_1} \cup \ldots \cup s_{i_k} \in \mathscr{C}.$$

If $t \in \mathscr{C}_{i_k}$, then $t \in \mathscr{C}'$ and Lemma 3.10 holds with $\mathscr{D}' = \mathscr{D}_{i_k}$, as $\mathscr{C}_{i_k} \subseteq \mathscr{C}'$. □

Definition 3.11. Let I be a natural number or $I = \omega$. A *finite (an infinite) block sequence* is a sequence $\mathscr{D} = \langle d_i \rangle_{i \in I}$ of finite subsets of $\mathbb{N}$ such that $d_i < d_{i+1}$ for all $i \in I$. The set $(FIN)^{\omega}$is the collection all infinite block sequences of elements of FIN.

Notice that given a disjoint collection $\mathscr{D}$ we can obtain an infinite block sequence from it, in fact, some $\mathscr{D}' \subseteq \mathscr{D}$ is an infinite block sequence. By convenience when we say x is an element of a finite (an infinite) block sequence, we mean that x is equal to some element of the range of the finite (infinite) block sequence.

Theorem 3.12. *If $\mathscr{C}$ is large for $\mathscr{D}'$, then there exists $\mathscr{E} \sqsubseteq \mathscr{D}'$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}$.*

Proof. Assume that $\mathscr{C}$ is large for $\mathscr{D}'$. We are going to define a partial order and by a forcing argument we shall obtain $\mathscr{E}$ with the desired property.

We define

$$\mathbb{P}_{\mathscr{C},\mathscr{D}'} = \langle P, \leqslant_{\mathscr{C}} \rangle \tag{3.1}$$

as follows: the elements of P are of the form $(A, \mathscr{D})$, where $A = \langle x_0, \ldots, x_m \rangle$ is a finite block sequence of finite subsets of natural numbers such that $FU(A) \subseteq \mathscr{C}$, $\mathscr{D} = \langle d_{i i \in \omega} \rangle$ is an infinite block sequence such that $\mathscr{D} \sqsubseteq \mathscr{D}'$ and $A < \mathscr{D}$, i.e., $\max(x_m) < \min(d_0)$, and

$$\mathscr{C}^* = \{y \in FU(\mathscr{D}) \cap \mathscr{C} : \forall x \in FU(A)(x \cup y \in \mathscr{C})\}$$

is large for $\mathscr{D}$.

Given two elements in P, $(A, \mathscr{D})$ and $(B, \mathscr{B})$, we let $(B, \mathscr{B}) \leqslant_{\mathscr{C}} (A, \mathscr{D})$ if and only if A is an initial subsequence of B, in this context we only write $B \supseteq A$, $\mathscr{B} \sqsubseteq \mathscr{D}$ and $\forall x \in B \setminus A$ $(x \in FU(\mathscr{D}))$.

Note that $(\langle\rangle, \mathscr{D}') \in P$ and the ordering relation $\leqslant_{\mathscr{C}}$ is reflexive and transitive.

Claim 3.13. *Every condition in P can be extended in the finite part.*

Proof of Claim: Let $(A, \mathscr{D})$ be a condition in P, with $A = \langle x_0, \ldots, x_m \rangle$. We have, in particular, that $\mathscr{C}^*$ is large for $\mathscr{D}$.

By Lemma 3.10 there are $s \in \mathscr{C}^* \cap FU(\mathscr{D})$ and $\mathscr{E} \sqsubseteq \mathscr{D} - s$ such that

$$\mathscr{C}' = \{z \in \mathscr{C}^* : s \cap z = \emptyset \text{ and } z \cup s \in \mathscr{C}^*\}$$

is large for $\mathscr{E}$.

Since $s \in \mathscr{C}^*$, $s \in FU(\mathscr{D}) \cap \mathscr{C}$, and for all $x \in FU(A)$ we have $x \cup s \in \mathscr{C}$, so $FU(A^\frown\langle s\rangle) \subseteq \mathscr{C}$.

Let $\mathscr{D}^* = \{d \in \mathscr{E} : d > s\}$.

We shall prove that the set

$$\mathscr{F} := \{y \in FU(\mathscr{D}^*) \cap \mathscr{C} : \forall x \in FU(A^\frown\langle s\rangle)(x \cup y \in \mathscr{C})\}$$

is large for $\mathscr{D}^*$. Let $\mathscr{D}'' \sqsubseteq \mathscr{D}^*$. Since $\mathscr{C}'$ is large for $\mathscr{E}$, $FU(\mathscr{D}'') \cap \mathscr{C}' \neq \emptyset$. So there exists $z \in FU(\mathscr{D}'')$ such that $z \in \mathscr{C}^*$, $z > s$, and $z \cup s \in \mathscr{C}^*$. Thus, $z \in \mathscr{F}$.

Hence $(A^\frown\langle s\rangle, \mathscr{D}^*) \in P$ and $(A^\frown\langle s\rangle, \mathscr{D}^*) \leqslant_{\mathscr{C}} (A, \mathscr{D})$. We have proved the Claim. $\square$

For every $n \in \omega$ we define $D_n := \{(A, \mathscr{D}) \in P : |A| \geqslant n\}$. Note that D_n is a dense set for all $n \in \omega$ and consider $D = \{D_n : n \in \omega\}$. Let G be a D-generic filter in $\mathbb{P}_{\mathscr{C},\mathscr{D}'}$ and let

$$\mathscr{E} := \bigcup\{A : \exists \mathscr{D} \text{ such that } (A, \mathscr{D}) \in G\}.$$

It is clear, by standard density arguments, that $\mathscr{E}$ is infinite. Moreover, $FU(\mathscr{E}) \subseteq \mathscr{C}$, because if $x \in FU(\mathscr{E})$, then there exist $x_0, \ldots, x_m \in \mathscr{E}$ such that $x = \bigcup_{j=0}^{j=m} x_j$, hence there are $(A_j, \mathscr{D}_j) \in G$ such that x_j is an element of the finite block sequence A_j. Since G is a filter there exists $(B, \mathscr{D}) \in G$ such that $(B, \mathscr{D}) \leqslant_{\mathscr{C}} (A_j, \mathscr{D}_j)$ for all $j \in \{0, \ldots, m\}$. By definition of the partial order, $FU(B) \subseteq \mathscr{C}$, and so $x \in \mathscr{C}$.

$\square$

Corollary 3.14. *If $[\omega]^{<\omega} = \mathscr{C}_0 \dot\cup \mathscr{C}_1$, then there exists an infinite block sequence $\mathscr{E}$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}_i$ for some $i \in \{0, 1\}$.*

Proof. Since $[\omega]^{<\omega}$ is large for $\langle\{i\}\rangle_{i\in\omega}$, by the Decomposition Lemma 3.6, there is $\mathscr{D}' \sqsubseteq \langle\{i\}\rangle_{i\in\omega}$ such that $\mathscr{C}_i$ is large for $\mathscr{D}'$ for some $i \in \{0, 1\}$. By Theorem 3.12, there exists $\mathscr{E} \sqsubseteq \mathscr{D}'$ such that $FU(\mathscr{E}) \subseteq \mathscr{C}_i$. $\square$

Remark 3.15. The Corollary above remains true if, instead of partitioning $[\omega]^{<\omega}$ one partitions $FU(\mathscr{D})$ where $\mathscr{D}$ is a block sequence. Then the homogeneous set $\mathscr{E}$ given by the theorem is such that $\mathscr{E} \sqsubseteq \mathscr{D}$.

Note that Corollary 3.14 implies Theorem 3.3, by Remark 3.15.

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