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CANJAR FILTERS II: PROOFS OF $\mathfrak{b} < \mathfrak{s}$ AND $\mathfrak{b} < \mathfrak{a}$ REVISITED

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ABSTRACT. It is a result of Shelah that both $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ are consistent. Using ideas of Brendle and Raghavan, we give alternative proofs of these results.

1. INTRODUCTION

For any filter $\mathcal{F}$ on the natural numbers, we can define two forcing notions that diagonalize it (i.e. adds a pseudointersection to it) the *Laver forcing* relative to $\mathcal{F}$, denoted by $\mathbb{L}(\mathcal{F})$, which consists of all trees of height ω that have a stem and above it the set of successors of every node is a member of $\mathcal{F}$, and there is also the *Mathias forcing* relative to $\mathcal{F}$, which is defined as $\mathbb{M}(\mathcal{F}) = \{(s, A) \mid s \in [\omega]^{<\omega} \wedge A \in \mathcal{F}\}$, the order is given by $(s, A) \leq (z, B)$ whenever z is an initial segment of s, $s - z \subseteq B$ and $A \subseteq B$. These two partial orders have many properties in common; however, in general these partial orders are not equivalent as forcing notions. For every filter $\mathcal{F}$, the Laver forcing associated with it adds a dominating real, but this may not be case for its Mathias forcing. A trivial example is when $\mathcal{F}$ is the filters of all cofinite subsets of ω, in this case $\mathbb{M}(\mathcal{F})$ is forcing equivalent to Cohen forcing, so it does not add a dominating real. A more interesting example was provided by Canjar in [8] (see also [9]) where under $\mathfrak{d} = \mathfrak{c}$, he constructed an ultrafilter which Mathias forcing does not add a dominating real. For this reason, we call such type of filters *Canjar filters*. We say that an ideal $\mathcal{I}$ is a *Canjar ideal* if its dual filter $\mathcal{I}^* = \{\omega - X \mid X \in \mathcal{I}\}$ is a Canjar filter. Canjar filters have been previously investigated in [10], [4] and [9] this paper can be seen as a continuation of that line of research (in fact this article was the last chapter of [9], but the referee of the paper suggested to publish this last chapter independently). No previous knowledge of the previous articles is needed here.

It is a result of Shelah that the unboundedness number $\mathfrak{b}$ can be smaller than the splitting number $\mathfrak{s}$. He achieved this result by using a countable support iteration of a creature forcing (see [1], [7] or [13]). Using a modification of the previous forcing, he also constructed a model where the unboundedness number is smaller than the almost disjointness number $\mathfrak{a}$. Brendle and Raghavan in [7] showed that the partial

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orders of Shelah can be decomposed as an iteration of two simpler forcings. In this note, we will show how to use this decomposition to give alternative proofs of Shelah's results. The consistency of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ may also be achieved using finite support iteration, as was proved by Brendle [5], and Brendle and Fischer [6].

If $\mathcal{I}$ is an ideal we will denote by $\mathcal{I}^+$ the set of subsets of ω that are not in $\mathcal{I}$ and are called the *positive sets with respect to* $\mathcal{I}$ or $\mathcal{I}$-positive sets. Whenever a, b are two sets, $a - b$ will denote the set theoretic difference of a and b (and never the arithmetic difference, even if $a, b \in \omega$). If $\mathcal{A}$ is an almost disjoint family, we denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. If W is a countable set, we denote by $fin(W)$ the set of all non empty finite subsets of W. If $\mathcal{I}$ is an ideal on W, we define the ideal $\mathcal{I}^{<\omega}$ as the set of all $A \subseteq fin(W)$ such that there is $Y \in \mathcal{I}$ with the property that $a \cap Y \neq \emptyset$ for all $a \in A$. We will write fin instead of $fin(W)$ when it is clear from the context. Recall that $\mathcal{I}$ is a *P^+-ideal* if every decreasing sequence of positive sets has a positive pseudointersection. If $f, g \in \omega^\omega$ and $n \in \omega$ then $f <_n g$ means that $f(m) < g(m)$ for every $m \geq n$. If A is a set, we denote by $\wp(A)$ the collection of all subsets of A. We may identify $\wp(\omega)$ with 2^ω, which is homeomorphic to the Cantor set if we give it the product topology. In this way, we can talk about the topological properties (like being compact, F_σ or Borel) of families of subsets of ω. The rest of our notation is mostly standard and follows [3], where the definitions of $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{s}$ can be consulted as well as their basic properties.

2. Preliminaries

Let $\mathcal{B}$ be an unbounded $\leq^*$ well-ordered family of increasing functions. We call a filter $\mathcal{F}$ a *$\mathcal{B}$-Canjar filter* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$. We will give a combinatorial characterization of this property. Given a decreasing sequence $\overline{X} = \{X_n \mid n \in \omega\} \subseteq fin$ and $f \in \mathcal{B}$, we define the set $\overline{X}_f = \bigcup\limits_{n \in \omega} (X_n \cap \wp(f(n)))$. Observe that $\overline{X}_f$ is a pseudointersection of $\overline{X}$. We say $\overline{X}$ has a *pseudointersection according to* $\mathcal{B}$ if there is $f \in \mathcal{B}$ such that $\overline{X}_f$ is positive. We call $\mathcal{F}^{<\omega}$ a *P^+-filter according to* $\mathcal{B}$ if every decreasing sequence $\overline{X}$ of positive sets has a pseudointersection according to $\mathcal{B}$. The following is a variant of the characterization of Canjar filters by Hrušák and Minami (see [10]).

Proposition 1. *A filter $\mathcal{F}$ is a $\mathcal{B}$-Canjar filter if and only if $\mathcal{F}^{<\omega}$ is a P^+-filter according to $\mathcal{B}$.*

Proof. Assume that $\mathcal{F}$ is not $\mathcal{B}$-Canjar, in other words, there is a name $\dot{g}$ for an increasing function such that $1_{\mathbb{M}(\mathcal{F})} \Vdash$ "$\dot{g}$ is an upper bound for $\mathcal{B}$". For every function $f \in \mathcal{B}$ let $s_f \in [\omega]^{<\omega}, n_f \in \omega$ and $F_f \in \mathcal{F}$ such that $(s_f, F_f) \Vdash$ "$f <_{n_f} \dot{g}$". Since $\mathcal{B}$ is an unbounded increasing chain there are $s \in [\omega]^{<\omega}, n \in \omega$ and a cofinal family $\mathcal{B}' \subseteq \mathcal{B}$ such that $s_f = s$ and $n_f = n$ for every $f \in \mathcal{B}'$.

For every $m \in \omega$ let X_m be the set of all $t \in [\omega - \bigcup s]^{<\omega}$ such that there is $F \in \mathcal{F}$ with the property that $(s \cup t, F)$ decides $\langle \dot{g}(0), \ldots, \dot{g}(m) \rangle$ and $(s \cup t, F) \Vdash$ "$\dot{g}(m) < max(t)$". It is easy to see that every $\overline{X} = \{X_m \mid m \in \omega\}$ is a decreasing sequence of positive sets. We will see that it has no pseudointersection according to $\mathcal{B}$. Since $\mathcal{B}'$ is cofinal in $\mathcal{B}$, it is enough to show that $\overline{X}$ has no pseudointersection according to $\mathcal{B}'$.

Aiming for a contradiction, assume that there is $f \in \mathcal{B}'$ such that $\overline{X}_f$ is positive. Since $\overline{X}_f \cap [F_f]^{<\omega}$ is infinite, pick $t \in \overline{X}_f \cap [F_f]^{<\omega}$ such that $t \in X_k \cap \wp(f(k))$ with $k > n$. Since $t \in X_k$ there is $F \in \mathcal{F}$ such that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq max(t)$" and note that $(s \cup t, F) \Vdash$ "$\dot{g}(k) \leq f(k)$". In this way, $(s \cup t, F_h \cap F)$ forces both $f(k) < \dot{g}(k)$ and $\dot{g}(k) \leq f(k)$, which is a contradiction.

Now assume that that $\mathcal{F}$ is $\mathcal{B}$-Canjar, we will see that $\mathcal{F}^{<\omega}$ is P^+ according to $\mathcal{B}$. Let $\overline{X} = \langle X_n \mid n \in \omega \rangle$ be a decreasing sequence of positives. Let M be the Mathias real, observe that $[M]^{<\omega}$ intersect infinitely every member of $(\mathcal{F}^{<\omega})^+$. In this way, in $V[M]$ we may define an increasing function $g : \omega \longrightarrow \omega$ such that $(M - n) \cap g(n)$ contains a member of X_n. Since $\mathcal{F}$ preserves $\mathcal{B}$, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$, we will see that $\overline{X}_f$ is positive. Let $F \in \mathcal{F}$ we must prove that $\overline{X}_f \cap [F]^{<\omega}$ is not empty. Since $F \in \mathcal{F}$ then $M \subseteq^* F$ so there is $k \in \omega$ such that $g(k) < f(k)$ and $M - k \subseteq F$ and hence $\overline{X}_f \cap [F]^{<\omega} \neq \emptyset$. □

Given $A \subseteq fin$, we denote by $\mathcal{C}(A)$ the set of all $X \subseteq \omega$ such that $a \cap X \neq \emptyset$ for all $a \in A$. It is easy to see that $\mathcal{C}(A)$ is a compact set and if $A \in (\mathcal{I}^{<\omega})^+$ then $\mathcal{C}(A) \subseteq \mathcal{I}^+$ for any ideal $\mathcal{I}$. The following lemma is well known and very easy to prove.

Lemma 2. *If $\mathcal{C}$ is a compact set and $A \in [\omega]^\omega$ intersects every element of $\mathcal{C}$, then there is $s \in [A]^{<\omega}$ such that s intersects every element of $\mathcal{C}$.*

The following lemma appears in [9]. We prove it here for the convenience of the reader.

Lemma 3. *Let $\mathcal{F}$ be a filter , let $X \subseteq fin$ be such that $\mathcal{C}(X) \subseteq \mathcal{F}$ and let $\mathcal{D}$ compact with $\mathcal{D} \subseteq \mathcal{F}$. Then, for every $n \in \omega$ there is $S \in [X]^{<\omega}$ such that if $A_0, \ldots, A_n \in \mathcal{C}(S)$ and $F \in \mathcal{D}$ then $A_0 \cap \ldots \cap A_n \cap F \neq \emptyset$.*

Proof. Given $s \in X$ define $K(s)$ as the set of all $(A_0, \ldots, A_n) \in \mathcal{C}(s)^{n+1}$ with the property that there is $F \in \mathcal{D}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$. This is a compact set by the previous lemma. Note that if $(A_0, \ldots, A_n) \in \bigcap_{s \in X} K(s)$ then $A_0, \ldots, A_n \in \mathcal{C}(X) \subseteq \mathcal{F}$ and there would be $F \in \mathcal{D} \subseteq \mathcal{F}$ such that $A_0 \cap \ldots \cap A_n \cap F = \emptyset$ which is clearly a contradiction. Since the $K(s)$ are compact, then there must be $S \in [F]^{<\omega}$ such that $\bigcap_{s \in S} K(s) = \emptyset$. It is easy to see that this is the S we are looking for. □

We say $\mathcal{F}$ is *strongly Canjar* if $\mathcal{F}$ is $\mathcal{B}$-Canjar for every well-ordered and unbounded $\mathcal{B}$. We will show that all F_σ ideals are strongly Canjar.

Lemma 4. *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ a compact set and $X \in (\mathcal{F}^{<\omega})^+$. Then there is $n \in \omega$ such that if $F \in \mathcal{D}$ then $(X \cap \wp(n)) \cap [F]^{<\omega} \neq \emptyset$.*

Proof. For every $m \in \omega$ define U_m as the set of all $A \subseteq \omega$ such that $(X \cap \wp(m)) \cap [A]^{<\omega} \neq \emptyset$, clearly this is an open set. Since $\mathcal{D} \subseteq \mathcal{F}$ and $X \in (\mathcal{F}^{<\omega})^+$ we conclude that $\mathcal{D} \subseteq \bigcup_{m\in\omega} U_m$. Finally, $\mathcal{D}$ is a compact set and $\langle U_m \rangle_{m\in\omega}$ is an increasing chain of open sets, so there must be an m such that $\mathcal{D} \subseteq U_m$. □

Now we can prove the following

Proposition 5. *Every F_σ ideal is strongly Canjar.*

Proof. Let $\mathcal{I} = \bigcup \mathcal{C}_n$ be an F_σ ideal with $\langle \mathcal{C}_n \mid n \in \omega \rangle$ an increasing sequence of compact sets and $\mathcal{B}$ be a well-ordered family of increasing functions. Let $\overline{X} = \{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ be a decreasing sequence. By the previous lemma, we can construct $f : \omega \longrightarrow \omega$ such that if $m \in \omega$ then every element of $\mathcal{C}_m$ contains an element of $X_m \cap \wp(f(m))$. Since $\mathcal{B}$ is unbounded, there is $g \in \mathcal{B}$ that is not dominated by $\mathcal{B}$. It is easy to see that $\overline{X}_g$ is positive. □

3. A model of $\mathfrak{b} < \mathfrak{s}$

Shelah was the first to construct a model where $\mathfrak{b}$ is less than $\mathfrak{s}$ (see [1] or [13]). He achieved this by constructing a weakly ω^ω-bounding proper forcing[1] that adds a real not split by any ground model reals. Later Brendle and Raghavan in [7] showed that Shelah forcing is equivalent to a two step iteration of simpler forcings, we will work with this descomposition.

Definition 6. *Define $\mathbb{F}_\sigma$ as the set of all F_σ filters and consider it as a forcing notion ordered by inclusion.*

It is easy to see that $\mathbb{F}_\sigma$ is σ-closed and if $G \subseteq \mathbb{F}_\sigma$ is a generic filter, then $\bigcup G$ is an ultrafilter. We denote the canonical name of this ultrafilter by $\dot{\mathcal{U}}_{gen}$. In [12] Laflamme showed that this is a Canjar ultrafilter, we will reprove this below. The following lemma is easy to verify.

Lemma 7. *If $\mathcal{U}$ is an ultrafilter and $X \subseteq fin$, then $X \in (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{U}$. It follows that if $\mathcal{F}$ is an F_σ filter then $\mathcal{F} \Vdash$"$X \in \left(\mathcal{U}_{gen}^{<\omega}\right)^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{F}$".*

With the aid of the previous lemmas, we can prove the following,

Proposition 8. *Let $\mathcal{B} \in V$ be an unbounded well-ordered family. Then $\mathbb{F}_\sigma$ forces that $\dot{\mathcal{U}}_{gen}$ is $\mathcal{B}$-Canjar.*

Proof. By the previous observation and since $\mathbb{F}_\sigma$ is σ-closed, it is enough to show that if $\mathcal{F} \Vdash$ "$\overline{X} = \langle X_n \rangle_{n\in\omega} \subseteq \dot{\mathcal{U}}_{gen}^{<\omega+}$" then there is $\mathcal{G} \leq \mathcal{F}$ and $f \in \mathcal{B}$ such that $\mathcal{C}\left(\overline{X}_f\right) \subseteq \mathcal{G}$.

Let $\mathcal{F} = \bigcup \mathcal{C}_n$ where each $\mathcal{C}_n$ is compact and they form an increasing chain. By lemma 3 there is $g : \omega \longrightarrow \omega$ such that if $n \in \omega$, $F \in \mathcal{C}_n$ and $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ then $A_0 \cap \ldots \cap A_n \cap F \neq \emptyset$. Since $\mathcal{B}$ is unbounded, then there is $f \in \mathcal{B}$ such that $f \nleq^* g$. We claim that $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ generates a filter. Let $F \in \mathcal{C}_n$

[1]Recall that a forcing notion $\mathbb{P}$ is *weakly ω^ω-bounding* if $\mathbb{P}$ does not add dominating reals.

and $A_0, \ldots, A_m \in \mathcal{C}(\overline{X}_f)$. We must show that $A_0 \cap \ldots \cap A_m \cap F \neq \emptyset$. Since f is not bounded by g, we may find $r > n, m$ such that $f(r) > g(r)$. In this way, $A_0, \ldots, A_n \in \mathcal{C}(X_n \cap \wp(g(n)))$ and then $A_0 \cap \ldots \cap A_m \cap F \neq \emptyset$. Finally, we can define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}(\overline{X}_f)$. □

Unlike the ω^ω-bounding property, the weakly ω^ω-bounding property is not preserved under iteration. However, Shelah proved the following preservation result.

Proposition 9 (Shelah, see [1]). *If $\gamma \leq \omega_2$ is limit and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \gamma \rangle$ is a countable support iteration of proper forcings and each $\mathbb{P}_\alpha$ is weakly ω^ω-bounding (over V) then $\mathbb{P}_\gamma$ is weakly ω^ω-bounding.*

Note that $\mathbb{P}$ is weakly ω^ω-bounding if and only if it preserves the unboundedness of every dominating family. By applying the result of Shelah we can easily conclude the following result.

Corollary 10. *If V satisfies* CH *(it is enough to assume that V has a well ordered dominating family) and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ is a countable support iteration of proper forcings such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ preserves the unboundedness of all well-ordered unbounded families, then $\mathbb{P}_{\omega_2}$ is weakly ω^ω-bounding.*

We are now in position to build a model where the unboundedness number is smaller than the splitting number.

Theorem 11 (Shelah). *There is a model where $\mathfrak{b} < \mathfrak{s}$.*

Proof. Assume that V satisfies CH and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ be the countable support iteration, where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$". By the previous results, it follows that $\mathbb{P}_{\omega_2}$ is weakly ω^ω-bounding and then $\mathfrak{b} = \omega_1$ in the final model. On the other hand, since $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ adds an ultrafilter and then diagonalice it, it follows that it destroys all splitting families of the ground model. Therefore $\mathfrak{s} = \omega_2$ in the extension. □

Before constructing the model of $\mathfrak{b} < \mathfrak{a}$ we would like to make some remarks. Recall the definition of almost ω^ω-bounding forcings,

Definition 12. *We say that a forcing notion $\mathbb{P}$ is* almost ω^ω-bounding *if for every name $\dot{f}$ for a real and $p \in \mathbb{P}$, there is an increasing $g : \omega \longrightarrow \omega$ such that for all $A \in [\omega]^\omega$ there is $p_A \leq p$ with the property that $p_A \Vdash$ "$g \restriction A \not\leq^* \dot{f} \restriction A$."*

The following is well known.

Lemma 13. *If $\mathbb{P}$ is almost ω^ω-bounding then $\mathbb{P}$ preserves all unbounded families of the ground model.*

Proof. Let $\mathcal{B}$ be unbounded, let $\dot{f}$ a name for a real and let $p \in \mathbb{P}$. Find $g : \omega \longrightarrow \omega$ as above. Since $\mathcal{B}$ is unbounded, there is $h \in \mathcal{B}$ and $A \in [\omega]^\omega$ such that $g \restriction A \leq h \restriction A$. It then clearly follows that p_A forces that $\dot{f}$ does not dominate $\mathcal{B}$. □

Given $A \in [\omega]^\omega$ denote by $e_A : \omega \longrightarrow A$ its enumerating function. It is a well known result of Talagrand (see [2]) that a filter $\mathcal{F}$ is non-meager if and only if $\{e_A \mid A \in \mathcal{F}\}$ is unbounded. It follows that no almost ω^ω-bounding forcing can diagonalize a non-meager filter. Since ultrafilters are non-meager, we conclude the following.

Corollary 14. *If $\mathcal{U}$ is an ultrafilter, then $\mathbb{M}(\mathcal{U})$ is not almost ω^ω-bounding.*

It follows by the theorems of Shelah, Brendle and Raghavan (see [1] and [7]) that $\mathbb{F}_\sigma * \mathbb{M}(\dot{\mathcal{U}}_{gen})$ is almost ω^ω-bounding, in spite the fact that $\mathbb{M}(\dot{\mathcal{U}}_{gen})$ is not.

4. A model of $\mathfrak{b} < \mathfrak{a}$

The first model where $\mathfrak{b} < \mathfrak{a}$ was constructed by Shelah using countable support iteration of proper forcings. Later, Brendle in [5] constructed a model of this result using finite support iteration. Although we will also use countable support iteration, the following proof was inspired by the work of Brendle.[2]

Given an AD family $\mathcal{A}$ define $\mathbb{F}_\sigma(\mathcal{A}) = \{\mathcal{F} \in \mathbb{F}_\sigma \mid \mathcal{I}(\mathcal{A}) \cap \mathcal{F} = \emptyset\}$ and order it by inclusion. As before, it is easy to see that $\mathbb{F}_\sigma(\mathcal{A})$ is a σ-closed filter and it adds an ultrafilter, which we will denote by $\dot{\mathcal{U}}_{\mathcal{A}}$. The *Brendle game* $\mathcal{BR}(\mathcal{A})$ is defined as follows,

I		Y_0		Y_1		Y_2		$\cdots$
II	$\mathcal{F}, X$		s_0		s_1		s_2	$\cdots$

Where

(1) $\mathcal{F} \in \mathbb{F}_\sigma(\mathcal{A})$, $\mathcal{F} = \bigcup \mathcal{C}_n$, where the $\mathcal{C}_n$ are compact and increasing, $X \subseteq fin$ and $\mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$,

(2) $Y_m \in \mathcal{I}(\mathcal{A})^*$, $s_m \in [Y_m]^{<\omega}$ intersects all the elements of $\mathcal{C}_m$ and $\max(s_m) < \min(s_{m+1})$.

The player I wins the game if $\bigcup\limits_{n\in\omega} s_n$ contains an element of X.

Note that this is an open game for I, i.e., if she wins, then she wins already in a finite number of steps. In the following, $V[C_{\omega_1}]$ denotes an extension of V by adding ω_1 Cohen reals.

Lemma 15. *If $\mathcal{A}$ is an AD family in V, then in $V[C_{\omega_1}]$ the player I has a winning strategy for $\mathcal{BR}(\mathcal{A})$.*

Proof. Assume this is not the case. Since $\mathcal{BR}(\mathcal{A})$ is an open game it follows from the Gale-Stewart theorem (see [11]) that II has a winning strategy, call it π. Let $\mathcal{F}, X$, $\mathcal{F} = \bigcup\limits_{n\in\omega} \mathcal{C}_n \in \mathbb{F}_\sigma(\mathcal{A})$ and $X \subseteq fin$ be the first play of II according to

[2] A similar but different approach has also been found recently by Andrew Brooke-Taylor and Joerg Brendle

π (so $\mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$). By standard Cohen forcing arguments, we may as well assume that $\mathcal{F}$, X and π are ground model sets. Call $\mathbb{P}$ the set of all $p = \langle s_0, \ldots s_n \rangle$ such that there are $Y_0, \ldots Y_n \in \mathcal{I}(\mathcal{A})^*$ with the property that $(\mathcal{F}, X, Y_0, s_0, \ldots, Y_n, s_n)$ is a partial play and the s_n are chosen using π. We order $\mathbb{P}$ by extension, note that $\mathbb{P}$ is countable, therefore it is isomorphic to Cohen forcing and if $p = \langle s_0, \ldots s_n \rangle \in \mathbb{P}$ then $\bigcup\limits_{i<n} s_i$ does not contain an element of X.

Given $Y \in \mathcal{I}(\mathcal{A})^*$ and $m \in \omega$ the set D_{Ym} of all conditions p such that p contains a response to Y and $|p| > m$ is open dense. Let $G \in V[C_{\omega_1}]$ be a $(\mathbb{P}, V)$ generic filter. By the above observation, we conclude that $D = \bigcup G$ is a legal play of the game, and it is a winning run for II, so D does not contain any element of X. By genericity $D \in \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle^+$ however, $\omega - D \in \mathcal{C}(X) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$ which is obviously a contradiction. □

We will need the following important definition.

Definition 16. *We say a MAD family $\mathcal{A}$ is a* Laflamme family *if $\mathcal{I}(\mathcal{A})$ can not be extended to an F_σ ideal.*

Given $X \subseteq fin$ and $A \in [\omega]^\omega$ let $Catch(X, A) = \{s \in X \mid s \subseteq A\}$. With the previous lemma we can prove the following dichotomy.

Lemma 17. *Let $\mathcal{A} \in V$ be an AD family, then in $V[C_{\omega_1}]$ one of the following holds,*

1. *$\mathcal{A}$ is not a Laflamme family or,*
2. *For every $\mathcal{F} \in \mathbb{F}(\mathcal{A})$ and $\{X_n \mid n \in \omega\} \subseteq fin$ with the property that $\mathcal{C}(X_n) \subseteq \langle \mathcal{I}(\mathcal{A})^* \cup \mathcal{F} \rangle$ for all $n \in \omega$, there is $A \in \mathcal{A} \cap \mathcal{F}^+$ such that if $B \in \wp(A) \cap \mathcal{F}^+$ then $Catch(X_n, B) \in (\mathcal{F}^{<\omega})^+$ for every $n \in \omega$.*

Proof. Assume that $\mathcal{A}$ is a Laflamme family and let $\mathcal{F}$ and X_n as above. By the previous lemma, let π be a winning strategy for player I. Consider the games where II began by playing $\mathcal{F}, X_n$ and call $\mathcal{W}$ the countable set of elements of $\mathcal{I}(\mathcal{A})^*$ that were played by I following π in any of these games. Note that if $W \in \mathcal{W}$ then W almost contains every element of $\mathcal{A}$ except for finitely many. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the countable set of all those elements of $\mathcal{A}$ that are not almost contained in every element of $\mathcal{W}$. Since $\mathcal{I}(\mathcal{A})^*$ can not be extended to an F_σ filter it is not contained in $\langle \mathcal{F} \cup \{\omega - B \mid B \in \mathcal{A}'\} \rangle$ so there is $A \in \mathcal{A}$ such that $\omega - A \notin \langle \mathcal{F} \cup \{\omega - B \mid B \in \mathcal{A}'\} \rangle$. This implies that $A \in \mathcal{F}^+$ and A is almost contain in every member of $\mathcal{W}$. Let $B \in \wp(A) \cap \mathcal{F}^+$ we will now show that $Catch(X_n, B)$ is positive for each $n \in \omega$. Let $F \in \mathcal{F}$ and consider the following play,

I		W_0		W_1		W_2		$\cdots$
II	$\mathcal{F}, X_n$		s_0		s_1		s_2	$\cdots$

where the W_n are played by I according to π and $s_i \in [B \cap F]^{<\omega}$ and intersects every element of $\mathcal{C}_i$. This is possible since $B \cap F$ is positive and is almost contained in every W_n. Since π is a winning strategy, this means that I wins the game,which entails that $\bigcup s_n \subseteq B \cap F$ contains an element of X_n. □

Given $A \in [\omega]^{\omega}$ and $l \in \omega$ define $Part_l(A)$ as the set of all sequences $\langle B_1, ..., B_l \rangle$ such that $A = \bigcup_{i \leq l} B_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. Note that $Part_l(A_i)$ is a compact space with the natural topology. Also it is clear that if $A \in \mathcal{F}^+$ and $\langle B_1, ..., B_l \rangle \in Part_l(A)$ then there is $j \leq l$ such that $B_j \in \mathcal{F}^+$.

Lemma 18. *Let $\mathcal{F}$ be a filter, let $\mathcal{C} \subseteq \mathcal{F}$ be compact and let $X \in (\mathcal{F}^{<\omega})^+$. Assume that A is such that if $B \in \wp(A) \cap \mathcal{F}^+$ then $Catch(X, B) \in (\mathcal{F}^{<\omega})^+$ and let $l \in \omega$. Then there is $n \in \omega$ with the property that for all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ there is $i \leq l$ such that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F.*

Proof. Let U_n be the set of all $\langle B_1, ..., B_l \rangle \in Part_l(A)$ such that there is $i \leq l$ with the property that if $F \in \mathcal{C}$ then $X \cap \wp(B_i \cap n)$ contains a subset of F. Note that $\{U_n \mid n \in \omega\}$ is an open set cover by lemma 4 and the result follows since $Part_l(A)$ is compact. □

It is easy to see that if $\mathcal{F} \in \mathbb{F}_{\sigma}(\mathcal{A})$ and $X \subseteq fin$, then $\mathcal{F} \Vdash$ "$X \in \dot{\mathcal{U}}_{\mathcal{A}}^{<\omega+}$" if and only if $\mathcal{C}(X) \subseteq \langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. With this we may prove the following result.

Proposition 19. *Let $\mathcal{B} \in V$ be a well-ordered unbounded family and let $\mathcal{A}$ an AD family, then in $V[C_{\omega_1}]$ either $\mathcal{A}$ is not Laflamme or $\mathbb{F}_{\sigma}(\mathcal{A}) \Vdash$ "$\dot{\mathcal{U}}_{\mathcal{A}}$ is $\mathcal{B}$-Canjar".*

Proof. Assume that $\mathcal{A}$ is Laflamme after adding ω_1 Cohen reals. In $V[C_{\omega_1}]$ let $\mathcal{F} \in \mathbb{F}_{\sigma}(\mathcal{A})$ and let a sequence $\overline{X} = \langle X_n \mid n \in \omega \rangle$ be such that $\mathcal{F}$ forces that each X_n is in $\dot{\mathcal{U}}_{\mathcal{A}}^{<\omega+}$, so all the $\mathcal{C}(X_n)$ are contained $\langle \mathcal{F} \cup \mathcal{I}(\mathcal{A})^* \rangle$. We will find an extension of $\mathcal{F}$ that forces that the $\overline{X}$ has a positive pseudointersection. Applying the previous lemma ω times, we may find distinct $A_0, A_1, A_2, \ldots \in \mathcal{A}$ such that $Catch(X_m, B) \in (\mathcal{F}^{<\omega})^+$ for every $B \in \wp(A_n) \cap \mathcal{F}^+$ and $n, m \in \omega$.

Let $\mathcal{F} = \bigcup_{m \in \omega} \mathcal{C}_m$, where $\langle \mathcal{C}_m \rangle_{m \in \omega}$ is an increasing sequence of compact sets. Define an increasing function $g : \omega \longrightarrow \omega$ such that if $n \in \omega$ then for all $\langle B_1, ..., B_{2^n} \rangle \in Part_{2^n}(A_n)$ there is $j \leq 2^n$ such that if $F \in \mathcal{C}_n$ then $X_n \cap \wp(B_j \cap (g(n) - n))$ contains a subset of F. Since $\mathcal{B}$ is unbounded, we can find $f \in \mathcal{B}$ that is not dominated by g.

We will now show that $\mathcal{F} \cup \mathcal{C}(\overline{X}_f) \cup \mathcal{I}^*(\mathcal{A})$ generates a filter. Let $F \in \mathcal{F}$, $C_0, ..., C_n \in \mathcal{C}(\overline{X}_f)$ and $D_0, ..., D_n \in \mathcal{A}$, we must show $F \cap C_0 \cap ... \cap C_n \cap (\omega - D_0) \cap ... \cap (\omega - D_n) \neq \emptyset$. We first find $m \in \omega$ such that,

(1) $n \leq m$,
(2) $F \in \mathcal{C}_m$,
(3) $A_m \cap (D_0 \cup ... \cup D_n) \subseteq m$,
(4) $g(m) < f(m)$.

For every $s : m \longrightarrow 2$ define B_s as the set of all $a \in A_m$ such that $a \in C_i$ if and only if $s(i) = 1$. Clearly $\langle B_s \rangle_{s \in 2^m} \in Part_{2^m}(A_m)$ and then we conclude that there is s such that $X_m \cap \wp(B_s \cap (g(m) - m))$ contains an element of F and then so does $X_m \cap \wp(B_s \cap (f(m) - m))$. Since $C_0, ..., C_n \in \mathcal{C}(\overline{X}_f)$ we conclude that s must be the constant 1 function and this entails that $F \cap C_0 \cap ... \cap C_n \cap (\omega - D_0) \cap ... \cap (\omega - D_n) \neq \emptyset$.

Finally, if we define $\mathcal{G}$ as the filter generated by $\mathcal{F} \cup \mathcal{C}\left(\overline{X}_f\right)$ then $\mathcal{G} \in \mathbb{F}_\sigma(\mathcal{A})$ and it forces that $\overline{X}$ has a positive pseudointersection. □

We are now in position to prove the result of Shelah.

Theorem 20 (Shelah). *There is a model where* $\mathfrak{b} < \mathfrak{a}$.

Proof. Assume that V satisfies CH, define the countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \omega_2 \rangle$ such that (with a suitable bookkeeping device) we destroy every MAD $\mathcal{A}$ family either by adding Cohen reals, by forcing with the Mathias forcing of an F_σ filter or with $\mathbb{F}_\sigma(\mathcal{A}) * \mathbb{M}(\dot{\mathcal{U}}_A)$. It is clear that this construction works. □

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