数理解析研究所講究録
第 1895 巻 2014 年 79-102

Matrices of isomorphic models and morass-like structures

Tadatoshi MIYAMOTO

December 26, 2013

Abstract

We force matrices of isomorphic models of set theory by countable conditions. We show that the matrices entail semimorasses and morasses depending on the forcing method. We also present a construction of a Souslin tree that makes use of a matrix.

Introduction

The idea of matrices of isomorphic models of set theory was conceived by Todorcevic ([T1] and [T2]). Recently, Aspero and Mota make use of a matrix forced by finite conditions to force a new forcing axiom. Their iterated forcing involves new apparatus so called markers ([A-M]). We may view that the new forcing axiom is forced over the intermediate stage formed by the matrix.

We investigate the matrices of isomorphic models of set theory forced by side conditions. We consider a list of basic and additional properties of the matrices. There are two ways to force matrices. One way is by finite conditions like [A-M] and the other by countable conditions like [B-S], [V] and [Ko]. The basic properties are assured to hold by forcing a la Aspero and Mota ([A-M]). But we force by countable conditions so that the basics as well as additional properties are satisfied. Roughly speaking, the matrices forced by finite conditions entail Kurepa trees, quagmires ([Ka]), and $\square_{\omega_1}$ ([M]). The matrices forced by countable conditions entail semi-morasses ([Ko]) and morasses ([D]).

We also present a construction of a Souslin tree that is carried out along a matrix. While this approach is rather a direct one but is weaker than [V] and [I], since we assume the matrix has a type of associated diamond.

In general, we would like to view forcing by side conditions as means to provide intermediate stages in relevant constructions. For example, the matrix forced by side conditions forms an intermediate stage to make sure that the rest of the construction goes fine. Namely, the quotient satisfies, say, some type of properness, even is c.c.c. The entire construction would be done by, say, proper forcing due to the side conditions. Though the exceptions to this view would include forcing closed and cofinal subsets by finite conditions as in [F], [Mit], and [Kr].

In this note, we concentrate on the matrices forced by countable conditions ([B-S], [V] and [Ko]). The matrices forced by finite conditions are discussed in a separate note [M].

§1. A matrix

We develop a theory of structures called matrices. A matrix is a complex next to the ordinals and entails morass-like structures. To understand the following premise, it would be helpful to tell what a typical situation is. In a typical situation, a matrix is gotten by cofinality preserving proper forcing. Hence H below is H_κ in the ground model and H_κ below is H_κ in the generic extension. Hence $\kappa \subset H \subsetneq H_\kappa$ below.

1.1 Premise. Let κ be a regular cardinal with $\omega_2 \leq \kappa$. We have a fixed transitive set H such that

(1) $\kappa \subset H \subsetneq H_\kappa$.

(2) The $\in$-structure $(H, \in)$ is a transitive model of set theory without the power set axiom.

(3) The cofinatilies and cardinalites below κ are absolute between the universe and $(H, \in)$.

We would further assume other things, if needed. For example, $\{N \in H \mid N \text{ is countable (in } H)\}$ is stationary in $[H]^\omega$. Since we see no use of this yet, we drop this requirement.

1.2 Proposition. (1) The least uncountable cardinal ω_1 is definable in $(H, \in)$ with no parameters.

(2) If $\omega_2 < \kappa$, then ω_2 is definable in $(H, \in)$ with no parameters.

(3) If X is countable in H, $X \in M$, and M is an elementary substructure of $(H, \in)$, then $X \subset M$ (proper inclusion).

(4) Let M, N be elementary substructures of $(H, \in)$. M and N may or may not be in H. Let $\phi : (M, \in) \longrightarrow (N, \in)$ be an isomorphism. Let $X \in M$ and X be countable in H. Then $\phi(X) = \phi``X$.

(5) If $N, N', M \in H$ are elementary substructures of $(H, \in)$ such that $(N, \in)$ and $(N', \in)$ are isomorphic in H and $N, N' \in M$, then the isomorphism exists in M.

1.3 Definition. $\mathcal{N}$ is a *matrix* (of isomorphic countable elementary substructures of H), if

(1) For all $N \in \mathcal{N}$, N are countable in H and the $\in$-structures $(N, \in)$ are elementary substructures of $(H, \in)$.

(2) For all $N, N' \in \mathcal{N}$, if $N \cap \omega_1 = N' \cap \omega_1$, then two structures $(N, \in)$ and $(N', \in)$ are isomorphic in H and the (necessarily) unique isomorphism ϕ is the identity on the intersection $N \cap N'$. Futhermore, we demand $\phi``(\mathcal{N} \cap N) \subseteq \mathcal{N} \cap N'$. Hence, $\phi : (N, \in, \mathcal{N} \cap N) \longrightarrow (N', \in, \mathcal{N} \cap N')$ is an isomorphism.

(3) For all $\underline{N}, N' \in \mathcal{N}$, if $\underline{N} \cap \omega_1 < N' \cap \omega_1$, then there exists $N \in \mathcal{N}$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$.

(4) For all $N, N' \in \mathcal{N}$, there exists $N'' \in \mathcal{N}$ such that $N, N' \in N''$. ($\in$-directed)

(5) $H = \bigcup \mathcal{N}$. (cofinal)

1.4 Definition. A matrix $\mathcal{N}$ satisfies *LD(2)* (locally directedness with binary spliting), if

(6) For all $N \in \mathcal{N}$, (exclusively) either (0) || (limit) || (suc) holds, where

(0) $\mathcal{N} \cap N = \emptyset$.

(limit) $N = \bigcup(\mathcal{N} \cap N)$.

(suc) There exist N_1 and N_2 such that

$$N_1 \neq N_2, N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2).$$

A matrix $\mathcal{N}$ satisfies LD(2) + Δ, if in the item (6), we further have

(Δ) $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system.

Namely, $\Delta = (N_1 \cap \omega_2) \cap (N_2 \cap \omega_2)$ is a common proper initial segment of $N_1 \cap \omega_2$ and $N_2 \cap \omega_2$ and the non-empty tail $(N_2 \cap \omega_2) \setminus \Delta$ comes after the non-empty tail $(N_1 \cap \omega_2) \setminus \Delta$, or vice versa.

1.5 Definition. A matrix $\mathcal{N}$ satisfies *LD($\leq$ 2)* (locally directedness with at most binary spliting), if

(6) For all $N \in \mathcal{N}$, (exclusively) either (0) || (limit) || $(\text{suc})_1$ || $(\text{suc})_2$ holds, where

(0) $\mathcal{N} \cap N = \emptyset$.

(limit) $N = \bigcup(\mathcal{N} \cap N)$.

$(\text{suc})_1$ There exists N_1 such that

$$\mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1).$$

$(\text{suc})_2$ There exist N_1 and N_2 such that

$$N_1 \neq N_2, N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2).$$

We may express $(\text{suc})_1$ and $(\text{suc})_2$ combined as follows.

$(\text{suc})_{\leq 2}$ There exist N_1 and N_2 such that

$$N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2).$$

1.6 Definition. A matrix $\mathcal{N}$ is *complete*, if for any sequence $\langle e_i \mid i < \omega_2 \rangle$ of elements of H, there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that $N_1 \neq N_2, N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$, and two structures $(N_1, \in, e_i)$ and $(N_2, \in, e_j)$ are isomorphic.

1.7 Definition. A matrix $\mathcal{N}$ is Δ-*complete*, if for any sequence $\langle e_i \mid i < \omega_2 \rangle$ of elements of H, there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that $N_1 \neq N_2, N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-ystem, and two structures $(N_1, \in, e_i)$ and $(N_2, \in, e_j)$ are isomorphic.

§2. Basics on a matrix

A matrix $\mathcal{N}$ contains many elements ([A-M] and [Ko]).

2.1 Proposition. Let $\mathcal{N}$ be a matrix and $N, N' \in \mathcal{N}$.

(1) If $N' \cap \omega_1 < N \cap \omega_1$, then there exists $\underline{N} \in \mathcal{N}$ such that $\underline{N} \in N$ and $\underline{N} \cap \omega_1 = N' \cap \omega_1$.

(2) If $N \in N'$ and there exists $N'' \in \mathcal{N}$ with $N \cap \omega_1 < N'' \cap \omega_1 < N' \cap \omega_1$, then there exists $\underline{N} \in \mathcal{N}$ such that $N \in \underline{N} \in N'$ and $\underline{N} \cap \omega_1 = N'' \cap \omega_1$.

Proof. (1): Take $M \in \mathcal{N}$ such that $N' \in M$ and $M \cap \omega_1 = N \cap \omega_1$. Let $\phi : M \longrightarrow N$ be the isomorphism and set $\underline{N} = \phi(N')$. Since $N' \in M$ and N' is countable in H, we have $N' \subset M$. Then $N' \cap \omega_1 \subset M \cap N$ and so $N' \cap \omega_1 = \phi``(N' \cap \omega_1) = \phi(N' \cap \omega_1) = \phi(N') \cap \phi(\omega_1) = \phi(N') \cap \omega_1 = \underline{N} \cap \omega_1$. Hence $\underline{N} \cap \omega_1 = N' \cap \omega_1$ and $\underline{N} = \phi(N') \in N$.

(2): Take $M \in \mathcal{N}$ such that $N \in M$ and $M \cap \omega_1 = N'' \cap \omega_1$. Then take $M' \in \mathcal{N}$ such that $M \in M'$ and $M' \cap \omega_1 = N' \cap \omega_1$. Let $\phi : M' \longrightarrow N'$ be the isomorphism. Let $\underline{N} = \phi(M)$. Then $\underline{N} \in \mathcal{N} \cap N'$. Notice that $N \in N' \cap M'$ holds and so $\phi(N) = N$. Now it is routine to show this $\underline{N}$ works.

□

LD(2) and LD(≤ 2) hold level-wise. In the case of unary plitting, there exists a unique predecessor. In the case of binary plitting, there exists a unique pair.

2.2 Proposition. Let $\mathcal{N}$ be a matrix.

(1) If $N, M \in \mathcal{N}$ such that $N \cap \omega_1 = M \cap \omega_1$ and $\mathcal{N} \cap N = \emptyset$, then $\mathcal{N} \cap M = \emptyset$.

(2) If $N, M \in \mathcal{N}$ such that $N \cap \omega_1 = M \cap \omega_1$ and $N = \bigcup(\mathcal{N} \cap N)$, then $M = \bigcup(\mathcal{N} \cap M)$.

(3) Let $N, M \in \mathcal{N}$ such that $N \cap \omega_1 = M \cap \omega_1$ and $\mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1)$. Then there exists M_1 such that $M_1 \cap \omega_1 = N_1 \cap \omega_1$ and $\mathcal{N} \cap M = \{M_1\} \cup (\mathcal{N} \cap M_1)$.

(4) Let $N, M \in \mathcal{N}$ such that $N \cap \omega_1 = M \cap \omega_1$, $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Then there exist $M_1 \neq M_2$, $M_1 \cap \omega_1 = M_2 \cap \omega_1 (= N_1 \cap \omega_1)$ such that $\mathcal{N} \cap M = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$.

(5) Let $N, N_1, M_1 \in \mathcal{N}$ such that $\mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1) = \{M_1\} \cup (\mathcal{N} \cap M_1)$, then $M_1 = N_1$.

(6) Let $N, N_1, N_2 \in \mathcal{N}$ such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. If $M_1 \neq M_2$, $M_1 \cap \omega_1 = M_2 \cap \omega_1$, and $\mathcal{N} \cap N = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$, then $\{M_1, M_2\} = \{N_1, N_2\}$.

Proof. (1), (2), (3), and (4): Since $(N, \in, \mathcal{N} \cap N)$ and $(M, \in, \mathcal{N} \cap M)$ are isomorphic, we may check these items. Notice that if $N_1 \in \mathcal{N} \cap N$ and $\phi : (N, \in) \longrightarrow (M, \in)$ is an isomorphism, then $\phi \lceil N_1 : (N_1, \in) \longrightarrow (\phi(N_1), \in)$ is the isomorphism.

(5): We have $M_1 \in \mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1)$. Suppose on the contrary that $M_1 \in \mathcal{N} \cap N_1$. Then $M_1 \cap \omega_1 < N_1 \cap \omega_1$. Since $N_1 \in \mathcal{N} \cap N = \{M_1\} \cup (\mathcal{N} \cap M_1)$, we must have $N_1 \cap \omega_1 \leq M_1 \cap \omega_1$. This is a contradiction. Hence $M_1 \in \{N_1\}$. Thus $M_1 = N_1$.

(6): We have $M_1 \in \mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Suppose on the contrary that $M_1 \in (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Then $M_1 \cap \omega_1 < N_1 \cap \omega_1 = N_2 \cap \omega_1$. Since $N_1, N_2 \in \mathcal{N} \cap N = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$, we must have $N_1 \cap \omega_1 \leq M_1 \cap \omega_1 = M_2 \cap \omega_1$. This is a contradiction. Hence $M_1 \in \{N_1, N_2\}$. Similary, we conclude $M_2 \in \{N_1, N_2\}$. Thus $\{M_1, M_2\} = \{N_1, N_2\}$.

□

A matrix that satisfies LD(≤ 2) entails a closed and cofinal subset of ω_1.

2.3 Lemma. Let $\mathcal{N}$ be a matrix. Let $I = \{N \cap \omega_1 \mid N \in \mathcal{N}\}$.

(1) I is a cofinal subset of ω_1.

(2) If $\mathcal{N}$ satisfies LD(≤ 2), then I is closed and cofinal in ω_1.

(3) If $\mathcal{N}$ satisfies LD(2), then I is closed and cofinal in ω_1.

Proof. (cofinal) Let $k < \omega_1$. Since $k \in H = \bigcup \mathcal{N}$, we have $N \in \mathcal{N}$ such that $k \in N$ and so $k < N \cap \omega_1$.

(closed) Let $i < \omega_1$ be a limit ordinal such that $I \cap i$ is cofinal in i. Let $N \in \mathcal{N}$ be such that $i \leq N \cap \omega_1$ and $N \cap \omega_1$ is the least among those $N \cap \omega_1$. Then $i = N \cap \omega_1$ must hold. We argue in four cases by LD(≤ 2). Assume $i < N \cap \omega_1$.

Case 1. $\mathcal{N} \cap N = \emptyset$. Since $I \cap i$ is cofinal, pick any $N' \in \mathcal{N}$ with $N' \cap \omega_1 < i$. Then we may calculate a copy $\underline{N} \in \mathcal{N} \cap N$ of N'. This is a contradiction. Hence this case does not occur.

Case 2. $N = \bigcup(\mathcal{N} \cap N)$. Then there are many $N' \in \mathcal{N} \cap N$ with $i < N' \cap \omega_1 < N \cap \omega_1$. This would contradict the leastness of $N \cap \omega_1$.

Case 3. There exists N_1 such that $\mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1)$. Then $N_1 \cap \omega_1 < N \cap \omega_1$ and so $N_1 \cap \omega_1 < i < N \cap \omega_1$. Then there exists $N' \in \mathcal{N}$ such that $N_1 \in N' \in N$. This is a contradiction.

Case 4. There are $N_1 \neq N_2$ such that $N_1 \cap \omega_1 = N_2 \cap \omega_1$ and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Then $N_1 \cap \omega_1 < N \cap \omega_1$ and so $N_1 \cap \omega_1 < i < N \cap \omega_1$. Then there exists $N' \in \mathcal{N}$ such that $N_1 \in N' \in N$. This is a contradiction.

§3. A tree and copies of cardinals λ via a matrix

Let us fix a matrix $\mathcal{N}$ once for all in this section. We extract a tree out of $\mathcal{N}$ as in [M] that is based on [D].

3.1 Definition. Let $I = \{N \cap \omega_1 \mid N \in \mathcal{N}\}$. For $i \in I$, let us fix $N_i \in \mathcal{N}$ with $N_i \cap \omega_1 = i$. Transitive collase N_i onto $\overline{N_i}$. Let $F_{i\omega_1} = \{(c_N)^{-1} \mid N \in \mathcal{N}$ and $N \cap \omega_1 = i\}$. For $i, j \in I$ with $i < j$, let $F_{ij} = \{c_M \circ (c_N)^{-1} \mid N, M \in \mathcal{N},\ N \in M,\ N \cap \omega_1 = i$ and $M \cap \omega_1 = j\}$. Here $c_N \in H$ and $c_M \in H$ are the transitive collapses of N and M, respectively.

The following is a represetation of $\mathcal{N}$. Write $\overline{N_{\omega_1}} = H$.

3.2 Lemma. (1) For all $i < j$ in $I \cup \{\omega_1\}$ and all $f \in F_{ij}$, we have $f \in H$ and $f : (\overline{N_i}, \in) \longrightarrow (\overline{N_j}, \in)$ are elementary embeddings.

(2) For all $i < j$ in I, F_{ij} is a countable set.

(3) For all $i < j < k$ in $I \cup \{\omega_1\}$, we have $F_{ik} = F_{jk} \circ F_{ij}$. (pairwise compositions)

(4) For all $i_1, i_2 \in I$ and all $f_1 \in F_{i_1\omega_1}$, $f_2 \in F_{i_2\omega_1}$, there exist (g_1, g_2, h, k) such that $i_1, i_2 < k < \omega_1$ in I, $g_1 \in F_{i_1k}$, $g_2 \in F_{i_2k}, h \in F_{k\omega_1}$, $f_1 = h \circ g_1$, and $f_2 = h \circ g_2$.

(5) $\overline{N_{\omega_1}} = \bigcup\{f``\overline{N_i} \mid i \in I, f \in F_{i\omega_1}\}$.

(6) For all $i < j$ in $I \cup \{\omega_1\}$, all $f_1, f_2 \in F_{ij}$, all $\overline{e_1}, \overline{e_2} \in \overline{N_i}$, if $f_1(\overline{e_1}) = f_2(\overline{e_2})$, then $\overline{e_1} = \overline{e_2}$. (tree order)

Proof. (1): Some account for the case $j < \omega_1$. Let $f \in F_{ij}$ and let $f = c_M \circ (c_N)^{-1}$. Since $N \in M$, we have $N \prec M$. Since $c_N : N \longrightarrow \overline{N_i}$ and $c_M : M \longrightarrow \overline{N_j}$, we have $f = c_M \circ (c_N)^{-1} : \overline{N_i} \longrightarrow \overline{N_j}$.

(2): $F_{ij} = \{c_{N_j} \circ (c_N)^{-1} \mid N \in \mathcal{N} \cap N_j, N \cap \omega_1 = i\}$ holds and so F_{ij} is countable. Some details follows. Let $f \in F_{ij}$. Take $N', M \in \mathcal{N}$ such that $N' \in M$ and $f = c_M \circ (c_{N'})^{-1}$. Since $N_j \cap \omega_1 = j = M \cap \omega_1$, there exists an isomorphism $\phi : M \longrightarrow N_j$. Let $N = \phi(N')$. Then $N \in \mathcal{N} \cap N_j$, $N \cap \omega_1 = N' \cap \omega_1 = i$, $c_M = c_{N_j} \circ \phi$ and $c_{N'} = c_N \circ (\phi[N')$. Hence $f = c_{N_j} \circ (c_N)^{-1}$ holds.

(3): Let $i < j < k < \omega_1$ in I. The case $k = \omega_1$ is similar. Let $f = c_M \circ (c_N)^{-1} \in F_{ik}$ with $N \in M$. Take $N' \in \mathcal{N}$ such that $N \in N' \in M$ and $N' \cap \omega_1 = j$. Then $c_{N'} \circ (c_N)^{-1} \in F_{ij}$ and $c_M \circ (c_{N'})^{-1} \in F_{jk}$. It is clear that $f = (c_M \circ (c_{N'})^{-1}) \circ (c_{N'} \circ (c_N)^{-1}) \in F_{jk} \circ F_{ij}$. Conversely, let $f \in F_{ij}$ and $g \in F_{jk}$. Then $g = c_{N_k} \circ (c_M)^{-1}$. Since M and N_j are isomorphic, we may assume $f = c_M \circ (c_N)^{-1}$ for some $N \in M \in N_k$. Hence $g \circ f = (c_{N_k} \circ (c_M)^{-1}) \circ (c_M \circ (c_N)^{-1}) = c_{N_k} \circ (c_N)^{-1} \in F_{ik}$.

(4): Let $f_1 = (c_{N_1})^{-1}$ and $f_2 = (c_{N_2})^{-1}$. Since $\mathcal{N}$ is $\in$-directed, there exists $N \in \mathcal{N}$ such that $N_1, N_2 \in N$. Let $k = N \cap \omega_1$, $h = (c_N)^{-1}$, $g_1 = c_N \circ (c_{N_1})^{-1}$ and $g_2 = c_N \circ (c_{N_2})^{-1}$. Then $h \in F_{k\omega_1}$, $g_1 \in F_{i_1 k}$, $g_2 \in F_{i_2 k}$ and $f_1 = h \circ g_1$, $f_2 = h \circ g_2$ hold.

(5): Let $e \in H = \bigcup \mathcal{N}$. Then there exists $N \in \mathcal{N}$ with $e \in N$. Let $i = N \cap \omega_1$. Then e is in the range of $(c_N)^{-1} \in F_{i\omega_1}$.

(6): First with $j = \omega_1$. Let $f_1 = (c_{N_1})^{-1}$ and $f_2 = (c_{N_2})^{-1}$ with $N_1 \cap \omega_1 = N_2 \cap \omega_1 = i$. Let $e = f_1(\overline{e_1}) = f_2(\overline{e_2})$. Then $e \in N_1 \cap N_2$. Since two structures $(N_1, \in)$ and $(N_2, \in)$ are isomorphic and the isomorphim $\phi : N_1 \longrightarrow N_2$ is the identity on $N_1 \cap N_2$, we have $\overline{e_1} = c_{N_1}(e) = (c_{N_2} \circ \phi)(e) = c_{N_2}(e) = \overline{e_2}$.

Next $i < j < \omega_1$ in I. Let $f_1(\overline{e_1}) = f_2(\overline{e_2})$. Take any $h \in F_{j\omega_1}$. Then $(h \circ f_1)(\overline{e_1}) = (h \circ f_2)(\overline{e_2})$. Hence we have seen that $\overline{e_1} = \overline{e_2}$.

□

Following [D], we considered a tree order in [M].

3.3 Definition. ([M]) Let $T = \{(i, \overline{e}) \mid i \in I \cup \{\omega_1\}, \overline{e} \in \overline{N_i}\}$. For $t_1 = (i_1, \overline{e_1}), t_2 = (i_2, \overline{e_2})$, we set $t_1 <_T t_2$, if $i_1 < i_2$ and there exists $f \in F_{i_1 i_2}$ with $f(\overline{e_1}) = \overline{e_2}$.

3.4 Lemma. ([M]) (1) $(T, <_T)$ is a tree.

(2) For $e \in \overline{N_{\omega_1}}$, let $i_e \in I$ be the least $i \in I$ such that $e \in N$ for some $N \in \mathcal{N}$ with $N \cap \omega_1 = i$. Then for all $i \in I$ with $i \geq i_e$, there exists a unique $\pi_i(e) \in \overline{N_i}$ such that there exists $h \in F_{i\omega_1}$ with $h(\pi_i(e)) = e$. The set $\{(i, \pi_i(e)) \mid i_e \leq i \in I\} \cup \{(\omega_1, e)\}$ forms a chain in $(T, <_T)$.

(3) For different $e_1, e_2 \in \overline{H_{\omega_1}}$, $\{(i, \pi_i(e_1)) \mid i \geq i_{e_1}$ in $I\}$ and $\{(i, \pi_i(e_2)) \mid i \geq i_{e_2}$ in $I\}$ split at some point.

Proof. (1): (irreflexive) $(i, \overline{e}) <_T (i, \overline{e})$ does not hold, as $i < i$ does not hold.

(transitive) Let $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2}) <_T (i_3, \overline{e_3})$. Then $i_1 < i_2 < i_3$, $f(\overline{e_1}) = \overline{e_2}$, $g(\overline{e_2}) = \overline{e_3}$. Hence $i_1 < i_3$ and $(g \circ f)(\overline{e_1}) = \overline{e_3}$.

(comparison below a node) Let $(i_1, \overline{e_1}), (i_2, \overline{e_2}) <_T (i, \overline{e})$. We have $f_1(\overline{e_1}) = \overline{e} = f_2(\overline{e_2})$. Let $i_1 = i_2$. Then we know $\overline{e_1} = \overline{e_2}$. Two nodes are identical in this case. Let $i_1 < i_2$. Then $f_1 = h \circ g$ with $g \in F_{i_1 i_2}$ and $h \in F_{i_2 i}$. Then $h(g(\overline{e_1})) = f_2(\overline{e_2})$. Hence $g(\overline{e_1}) = \overline{e_2}$. Therefore $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2})$. The remaining case is similar.

(linear order below any node is well-ordered) Since $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2})$ entails $i_1 < i_2$, the linear order below any node is well-ordered.

(2): Let $c_N(e) = \pi_{i_e}(e)$. Then for any $i > i_e$ in I, we have $f_i \in F_{i_e i}$ and $h_i \in F_{i\omega_1}$ such that $(c_N)^{-1} = h_i \circ f_i$. Hence let $\pi_i(e) = f_i(\pi_{i_e}(e))$. Then $h_i(\pi_i(e)) = e$ and so $(i, \pi_i(e)) <_T (\omega_1, e)$. Hence if $i_e \leq i_1 < i_2$ in I, we have $(i_1, \pi_{i_1}(e)) <_T (i_2, \pi_{i_2}(e))$.

(3): Take $N \in \mathcal{N}$ with $e_1, e_2 \in N$. Let $i_{e_1 e_2} = N \cap \omega_1$. Then for any $i \in I$ with $i \geq i_{e_1 e_2}$, we see that $\pi_i(e_1)$ and $\pi_i(e_2)$ are different.

□

For all cardinals λ with $\omega_2 \leq \lambda \leq \kappa$, we find copies of them that are single-rooted in the tree.

3.5 Lemma. (1) Let λ be a cardinal such that $\omega_2 \leq \lambda < \kappa$ and for all $N \in \mathcal{N}$, $\lambda \in N$. Then there exists $(i_0, \overline{\xi_0})$ such that $i_0 \in I$ and $\overline{\xi_0} \in \overline{N_{i_0}}$, and that

$$K = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \in [\lambda]^\lambda.$$

(2) Let $\lambda = \kappa$. Then there exists $(i_0, \overline{\xi_0})$ such that $i_0 \in I$, $\overline{\xi_0} \in \overline{N_{i_0}}$, and that

$$K = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \in [\lambda]^\lambda.$$

Proof. (1): Since $\overline{N_{\omega_1}} = \{f(\overline{e}) \mid i \in I, f \in F_{i\omega_1}, \overline{e} \in \overline{N_i}\}$ and $\{(i, \overline{e}) \mid i \in I, \overline{e} \in \overline{N_i}\}$ is of size ω_1, there exists $i_0 \in I$ and $\overline{\xi_0} \in \overline{N_{i_0}}$ such that $\{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \subseteq \lambda$ is of size λ.

Some details. Since $\lambda = \{f(\overline{\xi}) \mid i \in I, \overline{\xi} < (\lambda)^{\overline{N_i}}, f \in F_{i\omega_1}\}$, there exists $i_0 \in I$ and $\overline{\xi_0} < (\lambda)^{\overline{N_{i_0}}}$ such that $\{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \in [\lambda]^\lambda$.

(2): Since $\overline{N_{\omega_1}} = \{f(\overline{e}) \mid i \in I, f \in F_{i\omega_1}, \overline{e} \in \overline{N_i}\}$ and $\{(i, \overline{e}) \mid i \in I, \overline{e} \in \overline{N_i}\}$ is of size ω_1, there exists $i_0 \in I$ and $\overline{\xi_0} \in \overline{N_{i_0}}$ such that $\{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \subseteq \lambda$ is of size λ.

Some details. For all $\overline{\xi} < \mathrm{o}(\overline{N_i}) =$ (the set of ordinals in the collapse $\overline{N_i}$) and all $f \in F_{i\omega_1}$, we have $f(\overline{\xi}) < \kappa = \lambda$ and so $\lambda = \{f(\overline{\xi}) \mid i \in I, \overline{\xi} < \mathrm{o}(\overline{N_i}), f \in F_{i\omega_1}\}$. Hence there exists $(i_0, \overline{\xi_0})$ such that $\overline{\xi_0} < \mathrm{o}(\overline{N_{i_0}})$ and $\{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \in [\lambda]^\lambda$.

□

The single-rooted copies K enjoy the following.

3.6 Lemma. Let $i_0 \in I$ and $\overline{e_0} \in \overline{N_{i_0}}$, and $K = \{f(\overline{e_0}) \mid f \in F_{i_0\omega_1}\} \subseteq H$. Then we have

(1) If $N \in \mathcal{N}$ with $N \cap \omega_1 = i_0$, then $N \cap K = \{\overline{e_0}^N\}$, where $\overline{e_0}^N = (c_N)^{-1}(\overline{e_0})$. (one-point)

(2) For all $N \in \mathcal{N}$ with $i_0 < i = N \cap \omega_1$,

$$N \cap K = \{(c_N)^{-1} \circ f(\overline{e_0}) \mid f \in F_{i_0 i}\} = \{(\overline{e_0})^{N_0} \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\}$$

$$= \bigcup\{\underline{N} \cap K \mid \underline{N} \in \mathcal{N} \cap N, i_0 \leq \underline{N} \cap \omega_1\}.$$

(3) For all $N, N' \in \mathcal{N}$, if $i_0 \leq N \cap \omega_1 = N' \cap \omega_1$, then $(N, \in, K \cap N)$ and $(N', \in, K \cap N')$ are isomorphic.

(4) For all $N, N' \in \mathcal{N}$ with $i_0 \leq N \cap \omega_1, N' \cap \omega_1$, if $N \cap K \subset N' \cap K$ (proper inclusion-ship), then $N \cap \omega_1 < N' \cap \omega_1$.

Proof. (1): Let $N \in \mathcal{N}$ with $N \cap \omega_1 = i_0$. Let $e \in N \cap K$. Then there exists $f \in F_{i_0\omega_1}$ such that $e = f(\overline{e_0})$. Let $M \in \mathcal{N}$ such that $f = (c_M)^{-1}$. Then $e = (c_M)^{-1}(\overline{e_0}) \in M \cap N$. Hence $c_N(e) = c_M(e)$ and so $e = (c_N)^{-1}(\overline{e_0})$. Conversely, let $(\overline{e_0})^N = (c_N)^{-1}(\overline{e_0})$. Then $(\overline{e_0})^N \in N \cap K$.

(2): Let $N \in \mathcal{N}$ with $i_0 < i = N \cap \omega_1$. Let $e \in N \cap K$. Then there exists $f \in F_{i_0\omega_1}$ such that $e = f(\overline{e_0})$. Snice $f \in F_{i_0\omega_1}$ and $i_0 < i$, there exists (g, h) such that $g \in F_{i_0 i}$, $h \in F_{i\omega_1}$, and $f = h \circ g$. Since $h \in F_{i\omega_1}$, there exists $M \in \mathcal{N}$ such that $h = (c_M)^{-1}$. Since $e = (c_M)^{-1} \circ g(\overline{e_0})$, e is in M and so in $M \cap N$. Hence $c_N(e) = c_M(e)$ and so $e = (c_N)^{-1} \circ g(\overline{e_0})$. Conversely, let $g \in F_{i_0 i}$. Then $(c_N)^{-1} \circ g \in F_{i_0\omega_1}$ and so $(c_N)^{-1} \circ g(\overline{e_0}) \in N \cap K$. Since $F_{i_0 i} = \{c_N \circ (c_{N_0})^{-1} \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\}$, we have

$\{(\overline{e_0})^{N_0} \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\} = \{(c_{N_0})^{-1}(\overline{e_0}) \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\} = \{(c_N)^{-1} \circ c_N \circ (c_{N_0})^{-1}(\overline{e_0}) \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\} = \{(c_N)^{-1} \circ f(\overline{e_0}) \mid f \in F_{i_0 i}\}$.

And, trivially

$N \cap K = \{(\overline{e_0})^{N_0} \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\} = \bigcup\{N_0 \cap K \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\} = \bigcup\{\underline{N} \cap K \mid \underline{N} \in \mathcal{N} \cap N, i_0 \leq \underline{N} \cap \omega_1\}$.

(3): Let $\phi : (N, \in) \longrightarrow (N', \in)$ be the isomorphism that is the identity on $N \cap N'$. Then $c_N = c_{N'} \circ \phi$. First assume $i_0 = N \cap \omega_1 = N' \cap \omega_1$. Then $N \cap K = \{(c_N)^{-1}(\overline{e_0})\}$ and $N' \cap K = \{(c_{N'})^{-1}(\overline{e_0})\}$. Since $c_N = c_{N'} \circ \phi$, we have

$$c_{N'}(\phi((\overline{e_0})^N)) = (c_N)((\overline{e_0})^N) = \overline{e_0}.$$

And so $\phi((\overline{e_0})^N) = \phi((c_N)^{-1}(\overline{e_0})) = (c_{N'})^{-1}(\overline{e_0}) = (\overline{e_0})^{N'}$. Next, assume $i_0 < N \cap \omega_1 = N' \cap \omega_1$. Since $N \cap K = \{(c_N)^{-1} \circ g(\overline{e_0}) \mid g \in F_{i_0 i}\}$ and $N' \cap K = \{(c_{N'})^{-1} \circ g(\overline{e_0}) \mid g \in F_{i_0 i}\}$, we have

$$\phi\text{“}(N \cap K) = \{\phi \circ (c_N)^{-1} \circ g(\overline{e_0}) \mid g \in F_{i_0 i}\} = \{(c_{N'})^{-1} \circ g(\overline{e_0}) \mid g \in F_{i_0 i}\} = N' \cap K.$$

(4): Let $N \cap K \subset N' \cap K$. If $N \cap \omega_1 = N' \cap \omega_1$, then let $\phi : (N, \in) \longrightarrow (N', \in)$ be the isomorphism that is the identity on $N \cap N'$. We calculate $N \cap K \subset N' \cap K = \phi\text{“}(N \cap K) \subset N'$ and so $N \cap K \subset N \cap N'$. Hence $N' \cap K = \phi\text{“}(N \cap K) = N \cap K$. This is a contradiction. If $N' \cap \omega_1 < N \cap \omega_1$, then let $M \in \mathcal{N}$ such that $N' \in M$ and $M \cap \omega_1 = N \cap \omega_1$. Then there exists an isomorphism $\phi : (M, \in) \longrightarrow (N, \in)$ that is the identity on $N \cap M$. Then $\phi``(N' \cap K) \subseteq \phi``(M \cap K) = N \cap K \subset N' \cap K \subset M$ and so $\phi``(N' \cap K) \subseteq N \cap M$. Thus $N' \cap K = \phi``(N' \cap K) \subseteq N \cap K \subset N' \cap K$. This is a contradiction. Thus $N \cap \omega_1 < N' \cap \omega_1$.

□

§4. A matrix $\mathcal{N}_\lambda$ for λ with $\omega_2 < \lambda < \kappa$

Let $\mathcal{N}$ be a matrix that satisfies LD(2) and is complete. Let λ be a cardinal with $\omega_2 < \lambda < \kappa$. Since λ is not expected to be definable in the structure $(H, \in)$, we can not expect that for all $N \in \mathcal{N}$, $\lambda \in N$. However, we construct a subfamily $\mathcal{N}_\lambda$ of $\mathcal{N}$ such that $\mathcal{N}_\lambda$ is a matrix that satisfies LD(≤ 2) and that for all $N \in \mathcal{N}_\lambda$, $\lambda \in N$.

4.1 Lemma. Let $\mathcal{N}$ be a matrix that satisfies LD(2) and is complete. Let $\omega_2 < \lambda < \kappa$.

(1) For any $k < \omega_1$ and $e \in H$, there exist $N, N_1, N_2 \in \mathcal{N}$ such that $N_1 \neq N_2$, $e, \lambda \in N_1 \cap N_2$, $k < N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.

(2) Let $N, N_1, N_2 \in \mathcal{N}$ be such that $N_1 \neq N_2$, $\lambda \in N_1 \cap N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$ and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Then for any $M \in \mathcal{N}$ such that $\lambda \in M$ and $M \cap \omega_1 = N \cap \omega_1$, there exist $M_1 \neq M_2$ such that $M_1 \cap \omega_1 = M_2 \cap \omega_1$, $\lambda \in M_1 \cap M_2$, and $\mathcal{N} \cap M = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$.

(3) Let $J = \{i < \omega_1 \mid$ there exists $N \in \mathcal{N}$ such that $\lambda \in N$ and $N \cap \omega_1 = i$, for all $N \in \mathcal{N}$ such that $\lambda \in N$ and $N \cap \omega_1 = i$, there exists $N_1 \neq N_2$ such that $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\lambda \in N_1 \cap N_2$, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)\}$. Then $J = \{i < \omega_1 \mid$ there exists $N, N_1, N_2 \in \mathcal{N}$ such that $N \cap \omega_1 = i$, $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\lambda \in N_1 \cap N_2$, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)\}$ is cofinal in ω_1.

(4) Let $\mathcal{N}_\lambda = \{N \in \mathcal{N} \mid \lambda \in N$ and $N \cap \omega_1 \in J \cup J^*\}$, where J^* denotes the set of countable ordinals that are accumulation points of J. Then this $\mathcal{N}_\lambda$ is a matix that satisfies LD(≤ 2).

Proof. (1): Let $M_i = \{k, e, \lambda\}$ for all $i < \omega_2$. By the completeness, there exist $N, N_1, N_2 \in \mathcal{N}$ such that $N_1 \neq N_2$, $(N_1, \in, k, e, \lambda)$ and $(N_2, \in, k, e, \lambda)$ are isomorphic, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. We have $N_1 \neq N_2$, $e, \lambda \in N_1 \cap N_2$ and $k < N_1 \cap \omega_1 = N_2 \cap \omega_1$.

(2): Let N, N_1, N_2 and M be as in the assumption. Since $N \cap \omega_1 = M \cap \omega_1$, there exists an isomorphism $\phi : (N, \in) \longrightarrow (M, \in)$ that is the identity on $N \cap M$. Since $\lambda \in N \cap M$, we have $\phi(\lambda) = \lambda$. Let $M_1 = \phi(N_1)$ and $M_2 = \phi(N_2)$. Then $M_1 \cap \omega_1 = M_2 \cap \omega_1$, $\lambda \in M_1 \cap M_2$, and $\mathcal{N} \cap M = \phi``(\mathcal{N} \cap N) = \{\phi(N_1), \phi(N_2)\} \cup \phi``(\mathcal{N} \cap N_1) \cup \phi``(\mathcal{N} \cap N_2) = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$.

(3): This follows from (1) and (2).

(4): We check that $\mathcal{N}_\lambda$ is a matrix that satisfies LD(≤ 2).

(1) For all $N \in \mathcal{N}_\lambda$, N are countable in H, the $(N, \in)$ are elementary substructures of the structure $(H, \in)$.

Proof. Since $\mathcal{N}_\lambda \subseteq \mathcal{N}$, this follows.

□

(2) For all $N, N' \in \mathcal{N}_\lambda$, if $N \cap \omega_1 = N' \cap \omega_1$, then two structures $(N, \in, \mathcal{N}_\lambda \cap N)$ and $(N', \in, \mathcal{N}_\lambda \cap N')$ are isomorphic and the unique isomorphism ϕ is the identity on the intersection $N \cap N'$.

Proof. Since $\mathcal{N}_\lambda \subseteq \mathcal{N}$ and $\lambda \in N \cap N'$, the isomorphism $\phi : (N, \in) \longrightarrow (N', \in)$ fixes λ. Hence $\phi``(\mathcal{N}_\lambda \cap N) = \{\phi(M) \mid M \in \mathcal{N} \cap N, \lambda \in M, M \cap \omega_1 \in J \cup J^*\} = \{M' \mid M' \in \mathcal{N} \cap N', \lambda \in M', M' \cap \omega_1 \in J \cup J^*\} = \mathcal{N}_\lambda \cap N'$.

□

(3) For all $\underline{N}, N' \in \mathcal{N}_\lambda$, if $\underline{N} \cap \omega_1 < N' \cap \omega_1$, then there exists $N \in \mathcal{N}_\lambda$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$.

Proof. Since $\mathcal{N}_\lambda \subseteq \mathcal{N}$, there exists $N \in \mathcal{N}$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$. Then $N \in \mathcal{N}_\lambda$ holds.

□

(4) For all $N, N' \in \mathcal{N}_\lambda$, there exists $N'' \in \mathcal{N}_\lambda$ such that $N, N' \in N''$. ($\in$-directed)

Proof. Since $\mathcal{N}_\lambda \subseteq \mathcal{N}$, there exists $M \in \mathcal{N}$ such that $N, N' \in M$. By the completeness of $\mathcal{N}$, there exists $N'' \in \mathcal{N}$ such that $N'' \cap \omega_1 \in J$ and $M \in N''$. Then we have $N, N' \in N'' \in \mathcal{N}_\lambda$.

□

(5) $H = \bigcup \mathcal{N}_\lambda$. (cofinal)

Proof. Let $e \in H$. Then there exists $N \in \mathcal{N}$ with $e, \lambda \in N$. By the completeness of $\mathcal{N}$, there exists M such that $N \in M$ and $M \cap \omega_1 \in J$. Then $e \in M \in \mathcal{N}_\lambda$.

□

(6) For all $N \in \mathcal{N}_\lambda$, either (0) || (limit) || $(\mathrm{suc})_{\leq 2}$ holds, where

(0) $\mathcal{N}_\lambda \cap N = \emptyset$.

(limit) $N = \bigcup(\mathcal{N}_\lambda \cap N)$.

$(\mathrm{suc})_{\leq 2}$ There exist N_1 and N_2 such that

$$N_1 \cap \omega_1 = N_2 \cap \omega_1, \mathcal{N}_\lambda \cap N = \{N_1, N_2\} \cup (\mathcal{N}_\lambda \cap N_1) \cup (\mathcal{N}_\lambda \cap N_2).$$

Proof. Let $N \in \mathcal{N}_\lambda$. We have three cases.

Case 1. $\mathcal{N} \cap N = \emptyset$. Then $\mathcal{N}_\lambda \cap N = \emptyset$ holds.

Case 2. $N = \bigcup(\mathcal{N} \cap N)$. We observe $N = \bigcup(\mathcal{N}_\lambda \cap N)$ as follows. We first show that $N \cap \omega_1 \in J^*$. If $N \cap \omega_1 \in J$, then, since $\lambda \in N$, there exists N_1, N_2 such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\lambda \in N_1 \cap N_2$, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. But $N = \bigcup(\mathcal{N} \cap N)$. This entails a contradiction. Hence $N \cap \omega_1 \in J^*$. Now let $e \in N$. Pick $N' \in \mathcal{N} \cap N$ such that $e, \lambda \in N'$. Since $N \cap \omega_1 \in J^*$, there exists $M \in \mathcal{N} \cap N$ such that $N' \in M$ and $M \cap \omega_1 \in J$. Then we have $e \in M \in \mathcal{N}_\lambda$.

Case 3. $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Then $N \cap \omega_1 \in J$. Since $N \cap \omega_1 \in J$, there exists $M_1 \neq M_2$, $M_1 \cap \omega_1 = M_2 \cap \omega_1$, $\lambda \in M_1 \cap M_2$, and $\mathcal{N} \cap N = \{M_1, M_2\} \cup (\mathcal{N} \cap M_1) \cup (\mathcal{N} \cap M_2)$. Since $\{M_1, M_2\} = \{N_1, N_2\}$, we may assume $N_1 = M_1$ and $N_2 = M_2$.

Subcase 1. $\mathcal{N}_\lambda \cap N = \emptyset$. Done.

Subcase 2. $\mathcal{N}_\lambda \cap N \neq \emptyset$. Let $i = \max(N \cap \omega_1 \cap (J \cup J^*))$. Since $J \cup J^*$ is closed, this is well-defined and we have $i \in J \cup J^*$.

Subsubcase 1. $N_1 \cap \omega_1 = i$. Since $\lambda \in N_1 \cap N_2$ and $N_1 \cap \omega_1 = N_2 \cap \omega_1 = i \in J \cup J^*$, we have $N_1, N_2 \in \mathcal{N}_\lambda$ and so $\mathcal{N}_\lambda \cap N = \{N_1, N_2\} \cup (\mathcal{N}_\lambda \cap N_1) \cup (\mathcal{N}_\lambda \cap N_2)$.

Subsubcase 2. $i < N_1 \cap \omega_1$. Then there exist no elements $j \in J \cup J^*$ with $i < j < N \cap \omega_1 \in J$.

Subsubsubcase 1. $i \in J$. There exists unique M_1 (M_2) such that $\lambda \in M_1 \in \mathcal{N} \cap N_1$ and $M_1 \cap \omega_1 = i$ ($\lambda \in M_2 \in \mathcal{N} \cap N_2$ and $M_2 \cap \omega_1 = i$), respectively. Then $M_1, M_2 \in \mathcal{N}_\lambda$ and $\mathcal{N}_\lambda \cap N = \{M_1, M_2\} \cup (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$. It is plausible that $M_1 = M_2$. Some details follow. Since $i \in J$, there exists $N' \in \mathcal{N}$ such that $\lambda \in N'$ and $N' \cap \omega_1 = i$. Pick $N'' \in \mathcal{N}$ such that $N' \in N''$ and $N'' \cap \omega_1 = N_1 \cap \omega_1$. Then $\lambda \in N_1 \cap N''$. By calculating the isomorphic copy $M_1 \in N_1$ of N', we see that there exists M_1 such that $\lambda \in M_1 \in \mathcal{N} \cap N_1$ and $M_1 \cap \omega_1 = i$. Similary, we have M_2 such that $\lambda \in M_2 \in \mathcal{N} \cap N_2$ and $M_2 \cap \omega_1 = i$. To show the uniqueness, say, let $M_1' \neq M_1$ be such that $\lambda \in M_1' \in \mathcal{N} \cap N_1$ and $M_1' \cap \omega_1 = i$. We derive a contradiction. Since $M_1, M_1' \in N_1$, there exist $N_1' \in \mathcal{N}$ such that $\{M_1, M_1'\} \subset N_1'$ and $N_1' \cap \omega_1$ is the least among those N_1'. Then $i < N_1' \cap \omega_1 \leq N_1 \cap \omega_1$. Hence $N_1' \cap \omega_1 \notin J$. Since $N_1' \in \mathcal{N}$ that satisfies LD(2), we must have $\mathcal{N} \cap N_1' = \{\overline{M_1}, \overline{M_1'}\} \cup (\mathcal{N} \cap \overline{M_1}) \cup (\mathcal{N} \cap \overline{M_1'})$ for some $\overline{M_1} \neq \overline{M_1'}$ such that $M_1 \in \overline{M_1} \cup \{\overline{M_1}\}$ and $M_1' \in \overline{M_1'} \cup \{\overline{M_1'}\}$. Since $\lambda \in M_1 \cap M_1'$, we have $N_1' \cap \omega_1 \in J$. This is a contradiction. Hence $M_1 = M_1'$. To see that $\mathcal{N}_\lambda \cap N = \{M_1, M_2\} \cup (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$, let $M \in \mathcal{N}_\lambda \cap N \subseteq \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.

Since $\lambda \in M$ and $M \cap \omega_1 \in J \cup J^*$, we have $M \cap \omega_1 \leq i$. If $M \cap \omega_1 = i$, then $\lambda \in M \in N_1(N_2)$ would entail $M = M_1(M = M_2)$ by the uniqueness. If $M \cap \omega_1 < i$, then $M \in (\mathcal{N}_\lambda \cap N_1) \cup (\mathcal{N}_\lambda \cap N_2)$. Then by the uniqueness, if $M \in N_1(N_2)$, then $M \in M_1 \in N_1$ $(M \in M_2 \in N_2)$. Hence, $M \in (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$. Since $M_1, M_2 \in \mathcal{N}_\lambda \cap N$, the converse holds.

Subsubsubcase 2. $i \in J^*$. Since $J^* \subset I$, there exists $N' \in \mathcal{N}$ such that $\lambda \in N'$ and $N' \cap \omega_1 = i$. Some details follow. Since $i \in J^*$, there exists $N'' \in \mathcal{N}$ such that $N'' \cap \omega_1 < i$ and $\lambda \in N''$. Pick any $N' \in \mathcal{N}$ such that $N'' \in N'$ and $N' \cap \omega_1 = i$ and so $\lambda \in N'$. Then there exists unique M_1 (M_2) such that $\lambda \in M_1 \in \mathcal{N} \cap N_1$ and $M_1 \cap \omega_1 = i$ $(\lambda \in M_2 \in \mathcal{N} \cap N_2$ and $M_2 \cap \omega_1 = i)$, respectively. Then $M_1, M_2 \in \mathcal{N}_\lambda$ and $\mathcal{N}_\lambda \cap N = \{M_1, M_2\} \cup (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$. It is plausible that $M_1 = M_2$. Details are identical to the previous case as follows. By considering copies M_1 and M_2 of N', we have M_1 and M_2 such that $M_1 \cap \omega_1 = M_2 \cap \omega_1 = N' \cap \omega_1 = i$, $\lambda \in M_1 \in \mathcal{N} \cap N_1$, and $\lambda \in M_2 \in \mathcal{N} \cap N_2$. To show the uniqueness of, say, M_1, let $M_1' \neq M_1$ such that $\lambda \in M_1' \in \mathcal{N} \cap N_1$. Let $N_1' \in \mathcal{N}$ be such that $\{M_1, M_1'\} \subset N_1'$ and $N_1' \cap \omega_1$ is the least among those $\underline{N_1'} \cap \omega_1$. Then $i < N_1' \cap \omega_1 \leq N_1 \cap \omega_1$. Since $\underline{N_1'} \in \mathcal{N}$ that satisfies LD(2), we must have $\mathcal{N} \cap N_1' = \{\overline{M_1}, \overline{M_1'}\} \cup (\mathcal{N} \cap \overline{M_1}) \cup (\mathcal{N} \cap \overline{M_1'})$ for some $\overline{M_1} \neq \overline{M_1'}$ such that $M_1 \in \overline{M_1} \cup \{\overline{M_1}\}$ and $M_1' \in \overline{M_1'} \cup \{\overline{M_1'}\}$. Since $\lambda \in M_1 \cap M_1'$, we have $N_1' \cap \omega_1 \in J$. This is a contradiction. Hence $M_1 = M_1'$. To see that $\mathcal{N}_\lambda \cap N = \{M_1, M_2\} \cup (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$, let $M \in \mathcal{N}_\lambda \cap N \subseteq \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. Since $\lambda \in M$ and $M \cap \omega_1 \in J \cup J^*$, we have $M \cap \omega_1 \leq i$. If $M \cap \omega_1 = i$, then $\lambda \in M \in N_1(N_2)$ would entail $M = M_1(M = M_2)$ by the uniqueness. If $M \cap \omega_1 < i$, then $M \in (\mathcal{N}_\lambda \cap N_1) \cup (\mathcal{N}_\lambda \cap N_2)$. Then by the uniqueness, if $M \in N_1(N_2)$, then $M \in M_1 \in N_1$ $(M \in M_2 \in N_2)$. Hence, $M \in (\mathcal{N}_\lambda \cap M_1) \cup (\mathcal{N}_\lambda \cap M_2)$. Since $M_1, M_2 \in \mathcal{N}_\lambda \cap N$, the converse holds.

□

4.2 Lemma. Let $\mathcal{N}$ be a matrix that satisfies LD(2) and is complete, then for each cardinal λ with $\omega_2 \leq \lambda \leq \kappa$, there exists a matrix $\mathcal{N}_\lambda$ that satisfies LD(≤ 2) and that if $\lambda < \kappa$, then for all $N \in \mathcal{N}_\lambda$, $\lambda \in N$.

Proof. For the cardinals λ with $\omega_2 < \lambda < \kappa$, we find the subfamily $\mathcal{N}_\lambda$ of $\mathcal{N}$ such that $\mathcal{N}_\lambda$ is a matrix that satisfies LD(≤ 2) and that for all $N \in \mathcal{N}_\lambda$, $\lambda \in N$. If $\lambda = \omega_2$ or $\lambda = \kappa$, let $\mathcal{N}_\lambda = \mathcal{N}$. In either case, $\mathcal{N}_\lambda$ is a matrix that satisfies LD(2). If $\omega_2 \leq \lambda < \kappa$, since ω_2 is definable in H with no parameters, then for all $N \in \mathcal{N}_\lambda$, we have $\lambda \in N$.

□

§5. Many semimorasses by a matrix that satisfies LD(2) and is complete

Koszmider considered a generalization of $(\omega_1, 1)$-morasses ([Ko]), that is (ω_1, λ)-semimorasses. We show a matrix that satisfies LD(2) and is complete entails that for all λ with $\omega_2 \leq \lambda \leq \kappa$, there exists an (ω_1, λ)-semimorass. We follow steps taken by [D] that derives neat morasses from non-neat morasses.

Let $\mathcal{N}$ be a matrix that satisfies LD(2) and is complete. Let λ be a cardinal with $\omega_2 \leq \lambda \leq \kappa$. Let $\mathcal{N}_\lambda$ be a matrix that satisfies LD(≤ 2) as in the previous section. If $\lambda < \kappa$, then for all $N \in \mathcal{N}_\lambda$, $N \in \mathcal{N}_\lambda$.

Since $\mathcal{N}_\lambda$ is a matrix, we have a tree associated with $\mathcal{N}_\lambda$, $(i_0, \overline{\xi_0})$, and a copy K of λ as in the previous section. Now we follow [D].

5.1 Definition. For $i \in I = \{M \cap \omega_1 \mid M \in \mathcal{N}_\lambda\}$ with $i_0 \leq i$, we call i is *redundant*, if there exists $(\underline{N}, N)$ such that $\underline{N}, N \in \mathcal{N}_\lambda$, $i_0 \leq \underline{N} \cap \omega_1 < i = N \cap \omega_1$ and $\underline{N} \cap K = N \cap K$.

5.2 Propoition. Let $i \in I$ with $i_0 \leq i$. The following are equivalent.

(1) i is redundant.

(2) For all $N \in \mathcal{N}_\lambda$ with $N \cap \omega_1 = i$, there exists $\underline{N} \in \mathcal{N}_\lambda \cap N$ such that $i_0 \leq \underline{N} \cap \omega_1$ and $\underline{N} \cap K = N \cap K$.

Proof. It suffices to observe that (1) implies (2). Let $N \in \mathcal{N}_\lambda$ with $N \cap \omega_1 = i$. Since i is redundant, there exists $(\underline{M}, M)$ such that $\underline{M}, M \in \mathcal{N}_\lambda$, $\underline{M} \cap \omega_1 < i = M \cap \omega_1$ and $\underline{M} \cap K = M \cap K$. Pick $N' \in \mathcal{N}_\lambda$ such that $\underline{M} \in N'$, and $N' \cap \omega_1 = i$. Since $N' \cap \omega_1 = M \cap \omega_1$, there exists $\phi : (N', \in) \longrightarrow (M, \in)$. Then $\underline{M} \cap K \subset N' \cap M$ and so $\phi(\underline{M}) \cap K = \phi``(\underline{M} \cap K) = \underline{M} \cap K = M \cap K$. Hence we may assume that $\underline{M} \in M$ with $\underline{M} \cap K = M \cap K$. Let $\phi : (M, \in) \longrightarrow (N, \in)$. Then $\phi(\underline{M}) \in \mathcal{N}_\lambda \cap N$ and $\phi(\underline{M}) \cap K = N \cap K$.

□

5.3 Proposition. (1) If $N \in \mathcal{N}_\lambda$, $\underline{N} \in \mathcal{N}_\lambda \cap N$, $i_0 \leq \underline{N} \cap \omega_1$, and $\underline{N} \cap K = N \cap K$, then for all $M \in \mathcal{N}_\lambda \cap N$ with $\underline{N} \cap \omega_1 \leq M \cap \omega_1$, we have $M \cap K = N \cap K$.

(2) $\{i \in I \mid i_0 \leq i \text{ is not redundant}\}$ is a closed and cofinal subset of ω_1.

Proof. (1): First let $M \in \mathcal{N}_\lambda \cap N$ with $M \cap \omega_1 = \underline{N} \cap \omega_1$. Let $\phi : (M, \in) \longrightarrow (\underline{N}, \in)$. Then $M \cap K \subseteq N \cap K = \underline{N} \cap M$ and so $\underline{N} \cap K = \phi``(M \cap K) = M \cap K$ and so $M \cap K = N \cap K$. Next let $M \in \mathcal{N}_\lambda \cap N$ with $\underline{N} \cap \omega_1 < M \cap \omega_1$. Pick $N' \in \mathcal{N}_\lambda \cap N$ such that $\underline{N} \in N'$ and $N' \cap \omega_1 = M \cap \omega_1$. Then $N' \cap K = N \cap K$. Hence we have seen that $M \cap K = N \cap K$.

(2): (closed) Let $i < \omega_1$ be a limit ordinal such that there are cofinally many j's below i that are not redundant. Since I is closed, we have $i \in I$. If i were redundant, then there exists $(\underline{N}, N)$ such that $\underline{N} \in \mathcal{N}_\lambda \cap N$ such that $N \cap \omega_1 = i$ and $\underline{N} \cap K = N \cap K$. Since $\underline{N} \cap \omega_1 < N \cap \omega_1 = i$, there exists j such that $\underline{N} \cap \omega_1 < j < i$ and j is not redundant. Pick $M \in \mathcal{N}_\lambda \cap N$ such that $\underline{N} \in M$ and $M \cap \omega_1 = j$. Then $M \cap K = \underline{N} \cap K$. Since j is not redundant. This is a contradiction.

(cofinal) Suppose that $\{i \in I \mid i_0 \leq i \text{ is not redundant}\}$ is countable. Since i_0 is not redundant, we have the greatest element i in $\{i \in I \mid i_0 \leq i \text{ is not redundant}\}$. Then for any $N \in \mathcal{N}_\lambda$ with $i < N \cap \omega_1$, there must exist $M \in \mathcal{N}_\lambda \cap N$ such that $M \cap \omega_1 = i$ and $M \cap K = N \cap K$. Hence $\{N' \cap K \mid N' \in \mathcal{N}_\lambda \cap N, i \leq N' \cap \omega_1\}$ is of size one. Since $\mathcal{N}_\lambda$ is $\in$-directed, we conclude $\{N' \cap K \mid N' \in \mathcal{N}_\lambda, i \leq N' \cap \omega_1\}$ is of size one. Thus $K = N \cap K$ for (any) $N \in \mathcal{N}_\lambda$ with $N \cap \omega_1 = i$. Since K is of size greater than or equal to ω_2, this is a contradiction.

□

We introduce semimorasses from [Ko]. We understand that $X_1 \neq X_2$ in the item (5), (b).

5.4 Definition. ([Ko]) Let $\mathcal{F} \subset [\lambda]^\omega$. We call $\mathcal{F}$ is an (ω_1, λ)-*semimorass*, if

(1) $(\mathcal{F}, \subset)$ is well-founded.

(2) For all $X \in \mathcal{F}, \mathcal{F}|X = \{Y \in \mathcal{F} \mid Y \subset X\}$ is of size countable.

(3) For all $X, Y \in \mathcal{F}$, if $\mathrm{rank}(X) = \mathrm{rank}(Y)$, then $\mathrm{o.t.}(X) = \mathrm{o.t.}(Y)$ and $\mathcal{F}|Y = \{f_{XY}``Z \mid Z \in \mathcal{F}|X\}$, where $f_{XY} : X \longrightarrow Y$ is the order isomorphism.

(4) For all $X, Y \in \mathcal{F}$, there exists $Z \in \mathcal{F}$ such that $X, Y \subseteq Z$.

(5) For all $X \in \mathcal{F}$, either (a) || (b).

(a) $\mathcal{F}|X$ is $\subseteq$-directed.

(b) There exist $X_1, X_2 \in \mathcal{F}|X$ such that

- $\mathrm{rank}(X_1) = \mathrm{rank}(X_2)$.
- $X = X_1 \cup X_2$.
- $f_{X_1X_2} : X_1 \longrightarrow X_2$ is the order isomorphism that is the identity on $X_1 \cap X_2$.
- $\mathcal{F}|X = \{X_1, X_2\} \cup \mathcal{F}|X_1 \cup \mathcal{F}|X_2$.

(6) $\bigcup \mathcal{F} = \lambda$.

5.5 Theorem. Let $\mathcal{N}$ be a matrix that satisfy LD(2) and be complete. Then for all λ with $\omega_2 \leq \lambda \leq \kappa$, there exists an (ω_1, λ)-semimorass.

Proof. We use K as a copy of λ. We follow [D] in the rest of the proof. Let $\mathcal{F} = \{N \cap K \mid N \in \mathcal{N}_\lambda, i_0 \leq N \cap \omega_1, N \cap \omega_1 \text{ is not redundant }\}$. Then this $\mathcal{F}$ works. To check 6 items, we prepare

Claim 1. For $X \in \mathcal{F}$, say, $X = N \cap K$,

$$\mathcal{F}|X = \{M \cap K \mid M \in \mathcal{N}_\lambda \cap N, i_0 \leq M \cap \omega_1 \text{ is not redundant}\}.$$

Proof. Let $Y \in \mathcal{F}|X$. Then there exists $\underline{N} \in \mathcal{N}_\lambda$ such that $Y = \underline{N} \cap K$, $i_0 \leq \underline{N} \cap \omega_1$ is not redundant. Since $\underline{N} \cap K = Y \subset X = N \cap K$, we know that $\underline{N} \cap \omega_1 < N \cap \omega_1$. Pick $M' \in \mathcal{N}_\lambda$ such that $\underline{N} \in M'$

and $M' \cap \omega_1 = N \cap \omega_1$. Let $\phi : (M', \in) \longrightarrow (N, \in)$ be the isomorphism that is the identity on $M' \cap N$. Since $\underline{N} \cap K = Y \subset M' \cap X \subset M' \cap N$, $\phi``(\underline{N} \cap K) = \underline{N} \cap K$. Let $M = \phi(\underline{N})$. Then $M \in \mathcal{N}_\lambda \cap N$ and $M \cap \omega_1 = \underline{N} \cap \omega_1$ is not redundant. $M \cap K = \phi``(\underline{N} \cap K) = \underline{N} \cap K = Y$.

□

Claim 2. Let $N, N' \in \mathcal{N}_\lambda$, $i_0 \leq N \cap \omega_1 = N' \cap \omega_1$ be not redundant. Let $\phi : (N, \in) \longrightarrow (N', \in)$ be the isomorphism that is the identity on $N \cap N'$. Then

$$\{\phi``Y \mid Y \in \mathcal{F}|(N \cap K)\} = \mathcal{F}|(N' \cap K).$$

Proof. Since $\mathcal{F}|(N \cap K) = \{M \cap K \mid M \in \mathcal{N}_\lambda \cap N,\ i_0 \leq M \cap \omega_1 \text{ is not redundant}\}$, we have $\{\phi``Y \mid Y \in \mathcal{F}|(N \cap K)\} = \{\phi``(M \cap K) \mid M \in \mathcal{N}_\lambda \cap N,\ i_0 \leq M \cap \omega_1 \text{ is not redundant}\} = \{\phi(M) \cap K \mid M \in \mathcal{N}_\lambda \cap N,\ i_0 \leq M \cap \omega_1 \text{ is not redundant}\} = \{M' \cap K \mid M' \in \mathcal{N}_\lambda \cap N',\ i_0 \leq M' \cap \omega_1 \text{ is not redundant}\} = \mathcal{F}|(N' \cap K)$.

□

Now we begin to check the items in the definition of semimorasses.

(1): Let $\langle N_n \cap K \mid n < \omega\rangle$ be $\subset$ (proper inclusion-ship)-descending elements of $\mathcal{F}$. But $N_{n+1} \cap K \subset N_n \cap K$ entails $N_{n+1} \cap \omega_1 < N_n \cap \omega_1$. This is impossible. Hence $(\mathcal{F}, \subset)$ is well-founded.

(2): Let $X \in \mathcal{F}$, say, $X = N \cap K$. We know that $\mathcal{F}|(N \cap K) = \{M \cap K \mid M \in \mathcal{N}_\lambda \cap N,\ i_0 \leq M \cap \omega_1 \text{ is not redundant}\}$. Hence $\mathcal{F}|X$ is of size countable.

(3): Let $X = N \cap K, Y = M \cap K \in \mathcal{F}$ with $\text{rank}(X) = \text{rank}(Y)$. Then $N \cap \omega_1 = M \cap \omega_1$ holds. Some details follow. If $N \cap \omega_1 < M \cap \omega_1$, then there exists $M' \in \mathcal{N}_\lambda$ such that $N \in M'$ and $M' \cap \omega_1 = M \cap \omega_1$. Let $\phi : (M', \in) \longrightarrow (M, \in)$ be the isomorphism that is the identity on $M' \cap M$. Then $(\mathcal{F}|(M' \cap K), \subset)$ and $(\mathcal{F}|(M \cap K), \subset)$ are isomorphic via $Y \mapsto \phi``Y$. Since $N \in \mathcal{N}_\lambda \cap M'$ and $i_0 \leq N \cap \omega_1$ is not redundant, we have $N \cap K \in \mathcal{F}|(M' \cap K)$. Then $\text{rank}(N \cap K) < \text{rank}(M' \cap K) = \text{rank}(M \cap K)$. This is a contradiction. Similarly, if $M \cap \omega_1 < N \cap \omega_1$, then we would have $\text{rank}(M \cap K) < \text{rank}(N \cap K)$. Hence we must have $N \cap \omega_1 = M \cap \omega_1$.

Since $N \cap \omega_1 = M \cap \omega_1$, we have an isomorphism $\phi : (N, \in) \longrightarrow (M, \in)$ that is the identity on $N \cap M$. Since $\phi``(N \cap K) = M \cap K$, we have $\text{o.t.}(X) = \text{o.t.}(N \cap K) = \text{o.t.}(M \cap K) = \text{o.t.}(Y)$. The order isomorphism $f_{XY}$ from $X$ onto $Y$ is the restriction $\phi \lceil X$. Hence, we know that $\mathcal{F}|Y = \{\phi``Z \mid Z \in \mathcal{F}|X\} = \{f_{XY}``Z \mid Z \in \mathcal{F}|X\}$.

(4): Let $X = N \cap K, Y = M \cap K \in \mathcal{F}$. Since $\mathcal{N}_\lambda$ is $\in$-directed and $\{i \in I \mid i \text{ is not redundant }\}$ is cofinal in ω_1, there exists $M' \in \mathcal{N}_\lambda$ such that $N, M \in M'$ and $M' \cap \omega_1$ is not redundant. Let $Z = M' \cap K$. Then $X, Y \subset Z$ and $Z \in \mathcal{F}$.

(5): Let $X = N \cap K \in \mathcal{F}$. Let $J = \{j \in (N \cap \omega_1) \cap I \mid i_0 \leq j \text{ is not redundant}\}$. We have several cases.

Case 1. $J = \emptyset$: Then $\mathcal{F}|X = \emptyset$ and is vacuously $\subseteq$-directed.

Case 2. J is cofinal below $N \cap \omega_1$: We show $\mathcal{F}|X$ is $\subseteq$-directed. Since $N \cap \omega_1 \in I^*$, we must have $N = \bigcup(\mathcal{N}_\lambda \cap N)$. Let $N' \cap K, N'' \cap K \in \mathcal{F}|X$. Then we may assume that $N', N'' \in N$. Hence there exist $N''' \in \mathcal{N}_\lambda \cap N$ with $N', N'' \in N'''$. Since j that are not redundant are cofinal below $N \cap \omega_1$, we may assume $N''' \cap \omega_1$ is not redundant. Hence $N' \cap K, N'' \cap K \subset N''' \cap K \in \mathcal{F}|X$.

Case 3. J is non-empty and bounded below $N \cap \omega_1$: Let j_1 be the max of J. We have several subcases.

Subcase 1. $N = \bigcup(\mathcal{N}_\lambda \cap N)$: Let us pick any $M \in \mathcal{N}_\lambda \cap N$ with $M \cap \omega_1 = j_1$. Then we know that $\mathcal{F}|X = \{M \cap K\} \cup \mathcal{F}|(M \cap K)$. Hence $\mathcal{F}|X$ is trivially $\subseteq$-directed with the greatest member $M \cap K$.

Subcase 2. $\mathcal{N} \cap N = \{N_1\} \cup (\mathcal{N} \cap N_1)$. Since we have $N \cap K = N_1 \cap K$, $N \cap \omega_1$ is redundant. But $N \cap \omega_1$ is not redundant. Hence, this case does not occur.

Subcase 3. There exist N_1 and N_2 such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, and $\mathcal{N}_\lambda \cap N = \{N_1, N_2\} \cup (\mathcal{N}_\lambda \cap N_1) \cup (\mathcal{N}_\lambda \cap N_2)$: We have several subsubcases.

Subsubcase 1. $N_1 \cap K = N_2 \cap K$: Since $N \cap K = (N_1 \cap K) \cup (N_2 \cap K)$, we have $N \cap K = N_1 \cap K = N_2 \cap K$. Hence $N \cap \omega_1$ is redundant. This case does not occur.

subsubcase 2. $N_1 \cap \omega_1 = N_2 \cap \omega_1 < j_1$: Since there exists no elements of $I \cap (N_1 \cap \omega_1, N \cap \omega_1)$, this case does not occur.

Subsubcase 3. $N_1 \cap K \neq N_2 \cap K$ and $j_1 < N_1 \cap \omega_1 = N_2 \cap \omega_1$: Fix (any) $M_1 \in \mathcal{N}_\lambda \cap N_1$ such that $M_1 \cap \omega_1 = j_1$ and fix (any) $M_2 \in \mathcal{N}_\lambda \cap N_2$ such that $M_2 \cap \omega_1 = j_1$. Since $M_1 \cap \omega_1 = M_2 \cap \omega_1$, there exists an isomorphism $\phi : (M_1, \in) \longrightarrow (M_2, \in)$ that is the identity on $M_1 \cap M_2$. Now we know that $N_1 \cap K = M_1 \cap K$, $\mathcal{F}|(N_1 \cap K) = \mathcal{F}|(M_1 \cap K)$, $N_2 \cap K = M_2 \cap K$, and $\mathcal{F}|(N_2 \cap K) = \mathcal{F}|(M_2 \cap K)$. Let $X_1 = M_1 \cap K$ and $X_2 = M_2 \cap K$. Then $X_1, X_2 \in \mathcal{F}$, $X_1 \neq X_2$, $\text{rank}(X_1) = \text{rank}(X_2)$, $X = X_1 \cup X_2$, $\phi\lceil X_1$ is the isomorphism from X_1 onto X_2, $\phi\lceil X_1$ is the identity on $X_1 \cap X_2$, and $\mathcal{F}|X = \{X_1, X_2\} \cup (\mathcal{F}|X_1) \cup (\mathcal{F}|X_2)$.

Subsubcase 4. $N_1 \cap K \neq N_2 \cap K$ and $j_1 = N_1 \cap \omega_1 = N_2 \cap \omega_1$: Let $X_1 = N_1 \cap K$ and $X_2 = N_2 \cap K$. Then these X_1 and X_2 work.

(6): Since $K = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} = \{(\overline{\xi_0})^N \mid N \in \mathcal{N}_\lambda, N \cap \omega_1 = i_0\} = \bigcup\{N \cap K \mid N \in \mathcal{N}_\lambda, i_0 \leq N \cap \omega_1\}$ and $\{j \in I \mid i_0 \leq j, j \text{ is not redundant}\}$ is a cofinal in ω_1, we have $K = \bigcup \mathcal{F}$.

□

§6. A simplified $(\omega_1, 1)$-morass by a matrix that satisfies LD(2) + Δ

We consider an extra requirement on matrices to get simplifies morasses of [D]. For the sake of convenience, we recall the definition of a simplified $(\omega_1, 1)$-morass, which is not necessarily neat, as a collection of countable subsets of ω_2.

6.1 Definition. ([D]) Let $\mathcal{F} \subset [\omega_2]^{\leq\omega}$. We call $\mathcal{F}$ is a *simplified* $(\omega_1, 1)$*-morass*, if

(1) $(\mathcal{F}, \subset)$ is well-founded.

(2) For all $X \in \mathcal{F}, \mathcal{F}|X = \{Y \in \mathcal{F} \mid Y \subset X\}$ is of size countable.

(3) For all $X, Y \in \mathcal{F}$, if $\text{rank}(X) = \text{rank}(Y)$, then $\text{o.t.}(X) = \text{o.t.}(Y)$ and $\mathcal{F}|Y = \{f_{XY}``Z \mid Z \in \mathcal{F}|X\}$, where $f_{XY} : X \longrightarrow Y$ is the order isomorphism.

(4) For all $X, Y \in \mathcal{F}$, there exists $Z \in \mathcal{F}$ such that $X, Y \subseteq Z$.

(5) For all $X \in \mathcal{F}$, either (a) || (b).

(a) $\mathcal{F}|X$ is $\subseteq$-directed.

(b) There exist $X_1, X_2 \in \mathcal{F}|X$ such that

- $\text{rank}(X_1) = \text{rank}(X_2)$.
- $X_1 \cap X_2$ is a proper initial segment of both X_1 and X_2.
- $X_1 \subseteq \min(X_2 \setminus (X_1 \cap X_2))$.
- For all $W \in \mathcal{F}|X$, either $W \subseteq X_1$ || $W \subseteq X_2$.

(6) $\bigcup \mathcal{F} = \omega_2$.

Since morasses require Δ-systems, we need to strengthen our matrices.

6.2 Theorem. Let $\mathcal{N}$ be a matrix that satisfy LD(2) + Δ. Then there exists a simplified $(\omega_1, 1)$-morass.

Proof. The proof is identical to the one for semimorasses. We consider a copy K of ω_2. Hence,

(1) $i_0 \in I = \{M \cap \omega_1 \mid M \in \mathcal{N}\}$, (if $\omega_2 < \kappa$, then $\overline{\xi_0} < (\omega_2)^{\overline{N_{i_0}}}$), (if $\omega_2 = \kappa$, then $\overline{\xi_0} < \text{o}(\overline{N_{i_0}})$), and

$$K = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \in [\omega_2]^{\omega_2}.$$

(2) If $N \in \mathcal{N}$ with $N \cap \omega_1 = i_0$, then $N \cap K = \{\overline{\xi_0}^N\}$, where $\overline{\xi_0}^N = (c_N)^{-1}(\overline{\xi_0})$. (one-point)

(3) For all $N \in \mathcal{N}$ with $i_0 < i = N \cap \omega_1$,

$$N \cap K = \{(c_N)^{-1} \circ f\,(\overline{\xi_0}) \mid f \in F_{i_0 i}\} = \{(\overline{\xi_0})^{N_0} \mid N_0 \in \mathcal{N} \cap N, N_0 \cap \omega_1 = i_0\}$$

$$= \bigcup\{\underline{N} \cap K \mid \underline{N} \in \mathcal{N} \cap N, i_0 \leq \underline{N} \cap \omega_1\}.$$

(4) For all $N, N' \in \mathcal{N}$, if $i_0 \leq N \cap \omega_1 = N' \cap \omega_1$, then $(N, \in, K \cap N)$ and $(N', \in, K \cap N')$ are isomorphic.

(5) For all $N, N' \in \mathcal{N}$ with $i_0 \leq N \cap \omega_1, N' \cap \omega_1$, if $N \cap K \subset N' \cap K$ (proper inclusion-ship), then $N \cap \omega_1 < N' \cap \omega_1$.

6.3 Definition. For $i \in I = \{N \cap \omega_1 \mid N \in \mathcal{N}\}$ with $i_0 \leq i$, we call i is *redundant*, if there exists $(\underline{N}, N)$ such that $\underline{N}, N \in \mathcal{N}$, $i_0 \leq \underline{N} \cap \omega_1 < i = N \cap \omega_1$ and $\underline{N} \cap K = N \cap K$.

We observed

6.4 Proposition. Let $i \in I$ with $i_0 \leq i$. The following are equivalent.

(1) i is redundant.

(2) For all $N \in \mathcal{N}$ with $N \cap \omega_1 = i$, there exists $\underline{N} \in \mathcal{N} \cap N$ such that $i_0 \leq \underline{N} \cap \omega_1$ and $\underline{N} \cap K = N \cap K$.

□

6.5 Proposition. (1) If $N \in \mathcal{N}$, $\underline{N} \in \mathcal{N} \cap N$, $i_0 \leq \underline{N} \cap \omega_1$, and $\underline{N} \cap K = N \cap K$, then for all $M \in \mathcal{N} \cap N$ with $\underline{N} \cap \omega_1 \leq M \cap \omega_1$, we have $M \cap K = N \cap K$.

(2) $\{i \in I \mid i_0 \leq i$ is not redundant$\}$ is a closed and cofinal subset of ω_1.

□

Let $\mathcal{F} = \{N \cap K \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1$ is not redundant$\}$. We observe that this $\mathcal{F}$ is a simplified $(\omega_1, 1)$-morass on K. We prepared

Claim 1. For $X \in \mathcal{F}$, say, $X = N \cap K$,

$$\mathcal{F}|X = \{M \cap K \mid M \in \mathcal{N} \cap N, i_0 \leq M \cap \omega_1 \text{ is not redundant}\}.$$

□

Claim 2. Let $N, N' \in \mathcal{N}$, $i_0 \leq N \cap \omega_1 = N' \cap \omega_1$ be not redundant. Let $\phi : (N, \in) \longrightarrow (N', \in)$ be the isomorphism that is the identity on $N \cap N'$. Then

$$\{\phi``Y \mid Y \in \mathcal{F}|(N \cap K)\} = \mathcal{F}|(N' \cap K).$$

□

Now we begin to repeat checking 6 items. In item (5), it gets a little new.

(1): $\mathcal{F} \subset [K]^{\leq\omega}$ is well-founded with respect to the proper inclusion-ship $\subset$. To see this, let $\langle N_n \cap K \mid n < \omega\rangle$ be a $\subset$-descending sequence. But $N_{n+1} \cap K \subset N_n \cap K$ entails $N_{n+1} \cap \omega_1 < N_n \cap \omega_1$. This is a contradiction.

(2): For $X \in \mathcal{F}$, say, $X = N \cap K$, we have $\mathcal{F}|X = \{M \cap K \mid M \in \mathcal{F} \cap N, i_0 \leq M \cap \omega_1$ is not redundant$\}$. Hence $\mathcal{F}|X$ is of size countable.

(3): Let $X = N \cap K, Y = M \cap K \in \mathcal{F}$ with $\text{rank}(X) = \text{rank}(Y)$. Then $N \cap \omega_1 = M \cap \omega_1$ holds. Some details follow. If $N \cap \omega_1 < M \cap \omega_1$, then there exists $M' \in \mathcal{N}$ such that $N \in M'$ and $M' \cap \omega_1 = M \cap \omega_1$. Let $\phi : (M', \in) \longrightarrow (M, \in)$ be the isomorphism that is the identity on $M' \cap M$. Then $(\mathcal{F}|(M' \cap K), \subset)$ and $(\mathcal{F}|(M \cap K), \subset)$ are isomorphic via $Y \mapsto \phi``Y$. Since $N \in \mathcal{N} \cap M'$ and $i_0 \leq N \cap \omega_1$ is not redundant, we have $N \cap K \in \mathcal{F}|(M' \cap K)$. Then $\text{rank}(N \cap K) < \text{rank}(M' \cap K) = \text{rank}(M \cap K)$. This is a contradiction. Similarly, if $M \cap \omega_1 < N \cap \omega_1$, then we would have $\text{rank}(M \cap K) < \text{rank}(N \cap K)$. Hence we must have $N \cap \omega_1 = M \cap \omega_1$.

Since $N \cap \omega_1 = M \cap \omega_1$, we have an isomorphism $\phi : (N, \in) \longrightarrow (M, \in)$ that is the identity on $N \cap M$. Since $\phi``(N \cap K) = M \cap K$, we have $\text{o.t.}(X) = \text{o.t.}(N \cap K) = \text{o.t.}(M \cap K) = \text{o.t.}(Y)$. The order isomorphism $f_{XY}$ from $X$ onto $Y$ is the restriction $\phi\lceil X$. Hence, we know that $\mathcal{F}|Y = \{\phi``Z \mid Z \in \mathcal{F}|X\} = \{f_{XY}``Z \mid Z \in \mathcal{F}|X\}$.

(4): Let $X = N \cap K, Y = M \cap K \in \mathcal{F}$. Since $\mathcal{N}$ is $\in$-directed and $\{i \in I \mid i_0 \leq i, i \text{ is not redundant}\}$ is cofinal in ω_1, there exists $M' \in \mathcal{N}$ such that $N, M \in M'$ and $M' \cap \omega_1$ is not redundant. Let $Z = M' \cap K$. Then $X, Y \subset Z$ and $Z \in \mathcal{F}$.

(5): Let $X = N \cap K \in \mathcal{F}$. Let $J = \{j \in (N \cap \omega_1) \cap I \mid i_0 \leq j \text{ is not redundant}\}$. We have several cases.

Case 1. $J = \emptyset$: Then $\mathcal{F}|X = \emptyset$ and is vacuously $\subseteq$-directed.

Case 2. J is cofinal below $N \cap \omega_1$: We show $\mathcal{F}|X$ is $\subseteq$-directed. Since $N \cap \omega_1 \in I^*$, we must have $N = \bigcup(\mathcal{N} \cap N)$. Let $N' \cap K, N'' \cap K \in \mathcal{F}|X$. Then we may assume that $N', N'' \in N$. Hence there exist $N''' \in \mathcal{N} \cap N$ with $N', N'' \in N'''$. Since j that are not redundant are cofinal below $N \cap \omega_1$, we may assume $N''' \cap \omega_1$ is not redundant. Hence $N' \cap K, N'' \cap K \subset N''' \cap K \in \mathcal{F}|X$.

Case 3. J is non-empty and bounded below $N \cap \omega_1$: Let j_1 be the max of J. We have several subcases.

Subcase 1. $N = \bigcup(\mathcal{N} \cap N)$: Let us pick any $M \in \mathcal{N} \cap N$ with $M \cap \omega_1 = j_1$. Then we know that $\mathcal{F}|X = \{M \cap K\} \cup \mathcal{F}|(M \cap K)$. Hence $\mathcal{F}|X$ is trivially $\subseteq$-directed with the greatest member $M \cap K$.

Subcase 2. There exist N_1 and N_2 such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$, and $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system: We have several subsubcases.

Subsubcase 1. $N_1 \cap K = N_2 \cap K$: Since $N \cap K = (N_1 \cap K) \cup (N_2 \cap K)$, we have $N \cap K = N_1 \cap K = N_2 \cap K$. Hence $N \cap \omega_1$ is redundant. This case does not occur.

subsubcase 2. $N_1 \cap \omega_1 = N_2 \cap \omega_1 < j_1$: Since there exists no elements of $I \cap (N_1 \cap \omega_1, N \cap \omega_1)$, this case does not occur.

Subsubcase 3. $N_1 \cap K \neq N_2 \cap K$ and $j_1 < N_1 \cap \omega_1 = N_2 \cap \omega_1$: Fix (any) $M_1 \in \mathcal{N} \cap N_1$ such that $M_1 \cap \omega_1 = j_1$ and fix (any) $M_2 \in \mathcal{N} \cap N_2$ such that $M_2 \cap \omega_1 = j_1$. Since $M_1 \cap \omega_1 = M_2 \cap \omega_1$, there exists an isomorphism $\phi : (M_1, \in) \longrightarrow (M_2, \in)$ that is the identity on $M_1 \cap M_2$. Now we know that $N_1 \cap K = M_1 \cap K$, $N_2 \cap K = M_2 \cap K$, and so $M_1 \cap K \neq M_2 \cap K$. Since $M_1 \cap K = N_1 \cap K = (N_1 \cap \omega_2) \cap K$ and $M_2 \cap K = N_2 \cap K = (N_2 \cap \omega_2) \cap K$, $\{M_1 \cap K, M_2 \cap K\}$ forms a Δ-system with the non-empty tails. Let $X_1 = M_1 \cap K$ and $X_2 = M_2 \cap K$. Then $X_1, X_2 \in \mathcal{F}$, $\text{rank}(X_1) = \text{rank}(X_2)$, $X = X_1 \cup X_2$, $\phi\lceil X_1$ is the isomorphism from X_1 onto X_2, $\phi\lceil X_1$ is the identity on $X_1 \cap X_2$, $\{X_1, X_2\}$ forms a Δ-system, and $\mathcal{F}|X = \{X_1, X_2\} \cup (\mathcal{F}|X_1) \cup (\mathcal{F}|X_2)$.

Subsubcase 4. $N_1 \cap K \neq N_2 \cap K$ and $j_1 = N_1 \cap \omega_1 = N_2 \cap \omega_1$: Let $X_1 = N_1 \cap K$ and $X_2 = N_2 \cap K$. Then these X_1 and X_2 work.

(6): Since $K = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} = \{(\overline{\xi_0})^N \mid N \in \mathcal{N}, N \cap \omega_1 = i_0\} = \bigcup\{N \cap K \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1\}$ and $\{j \in I \mid i_0 \leq j, j \text{ is not redundant}\}$ is a cofinal in ω_1, we have $K = \bigcup \mathcal{F}$.

□

§7. Forcing a matrix $\dot{\mathcal{N}}$

We force a matrix $\dot{\mathcal{N}}$ that satisfies LD(2) + Δ and is Δ-complete.

7.1 Premise. In the ground model V, let κ be a regular cardinal with $\kappa \geq \omega_2$. We assume the continuum hypothesis (CH) in V.

Our p.o. set P is σ-Baire and has (CH) the ω_2-c.c. For $p \in P$, p is of size countable and consists of countable subsets N of H_κ such that $(N, \in)$ are elementary substructures of the structure $(H_\kappa, \in)$. We require that each $(N, \in)$ has an isomorphic copy $(N', \in)$ such that $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system and the (necessarily unique) isomorphism is the identity on the intersection $N \cap N'$.

7.2 Proposition. (CH) Let $\mathcal{M} = \{N \in [H_\kappa]^\omega \mid$ there exists $N' \neq N$ such that $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system, $(N, \in)$ and $(N', \in)$ are isomorphic countable elementary substructures of $(H_\kappa, \in)$ and the isomorphism ϕ is the identity on $N \cap N'\}$. Then $\mathcal{M}$ is stationary in $[H_\kappa]^\omega$.

Proof. Let $F : [H_\kappa]^{<\omega} \longrightarrow H_\kappa$. For each $i < \omega_2$, fix a countable elementary substructure $(N_i, \in)$ of $(H_\kappa, \in)$ such that $i \in N_i$ is closed under F. By CH, we may assume that $\langle N_i \cap \omega_2 \mid i < \omega_2\rangle$ forms a

Δ-system. We may also assume that for any $i \neq j$, $(N_i, \in, i)$ and $(N_j, \in, j)$ are pair-wise isomorphic and that the isomorphism is the identity on $N_i \cap N_j$. Pick any $i < j$ and set $N = N_i$ and $N' = N_j$.

□

As in [B-S] and [Ko], we require that the conditions are the possible initial segments of $\dot{\mathcal{N}}$ in the following sense.

7.3 Definition. Let $p \in P$, if

(1) p is a countable subset of $\mathcal{M}$ such that

- There exists the top element $N^p \in p$ such that $p = \{N^p\} \cup (p \cap N^p)$.
- For all $N \in p \cap N^p$, there exist $N' \in p \cap N^p$ such that $N' \cap \omega_1 = N \cap \omega_1$ and $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system.

(2) For all $N, N' \in p$, if $N \cap \omega_1 = N' \cap \omega_1$, then two structures $(N, \in, p \cap N)$ and $(N', \in, p \cap N')$ are isomorphic and the isomorphism ϕ is the identity on the intersection $N \cap N'$.

(3) For all $\underline{N}, N' \in p$, if $\underline{N} \cap \omega_1 < N' \cap \omega_1$, then there exists $N \in p$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$.

(4) For all $N \in p$, (exclusively) either (0) || (limit) || (suc) holds, where

(0) $p \cap N = \emptyset$.

(limit) $N = \bigcup(p \cap N)$.

(suc) There exist $N_1 \neq N_2$ such that $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $p \cap N = \{N_1, N_2\} \cup (p \cap N_1) \cup (p \cap N_2)$, and $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system.

For $p, q \in P$, we set $q \leq p$, if $N^p \in q$ and $q \cap N^p = p \cap N^p$.

7.4 Proposition. Let $p \in P$ and $N, N' \in p$.

(1) If $N' \cap \omega_1 < N \cap \omega_1$, then there exists $\underline{N} \in p$ such that $\underline{N} \in N$ and $\underline{N} \cap \omega_1 = N' \cap \omega_1$.

(2) If $N \in N'$ and there exists $N'' \in p$ with $N \cap \omega_1 < N'' \cap \omega_1 < N' \cap \omega_1$, then there exists $\underline{N} \in p$ such that $N \in \underline{N} \in N'$ and $\underline{N} \cap \omega_1 = N'' \cap \omega_1$.

Proof. (1): Take $M \in p$ such that $N' \in M$ and $M \cap \omega_1 = N \cap \omega_1$. Let $\phi : (M, \in) \longrightarrow (N, \in)$ be the isomorphism and set $\underline{N} = \phi(N')$. Since $N' \in M$ and N' is countable, we have $N' \subset M$. Then $N' \cap \omega_1 \subset M \cap N$ and so $N' \cap \omega_1 = \phi``(N' \cap \omega_1) = \phi(N' \cap \omega_1) = \phi(N') \cap \omega_1 = \underline{N} \cap \omega_1$. Hence $\underline{N} \cap \omega_1 = N' \cap \omega_1$ and $\underline{N} = \phi(N') \in N$.

(2): Take $M \in p$ such that $N \in M$ and $M \cap \omega_1 = N'' \cap \omega_1$. Then take $M' \in p$ such that $M \in M'$ and $M' \cap \omega_1 = N' \cap \omega_1$. Let $\phi : (M', \in) \longrightarrow (N', \in)$ be the isomorphism. Let $\underline{N} = \phi(M)$. Then $\underline{N} \in p \cap N'$. Notice that $N \in N' \cap M'$ holds and so $\phi(N) = N$. Now it is routine to show this $\underline{N}$ works.

□

It is straightforward to observe the following.

7.5 Proposition. Let $p, q \in P$.

(1) $\{N \cap \omega_1 \mid N \in p\}$ is a countable closed subset of ω_1 with the max $N^p \cap \omega_1$.

(2) If $q \leq p$, then $\{N \cap \omega_1 \mid N \in p\}$ is an initial segment of $\{N \cap \omega_1 \mid N \in q\}$.

Proof. (1): Let $i < \omega_1$ be a limit ordinal such that $\{N \cap \omega_1 \mid N \in p\} \cap i$ is cofinal below i. Since $i \leq N^P \cap \omega_1$, we can fix $N \in p$ such that $N \cap \omega_1$ is the least with $i \leq N \cap \omega_1$. Then we may show that $i = N \cap \omega_1$ as follows. On the contrary, suppose $i < N \cap \omega_1$. Then check in three cases (0) || (limit) || (suc) with respect to this N. In either case, we have a contradiction. Hence $i = N \cap \omega_1$.

(2): We get no new ordinals $N \cap \omega_1$ with $N \in q$ strictly below $N^p \cap \omega_1$.

□

We record a typical construction of conditions in P.

7.6 Lemma. Let $p \in P$. Let $M = N^p$ and let M' be such that $M' \neq M$, $\{M \cap \omega_2, M' \cap \omega_2\}$ forms a Δ-system, two elementary substructures $(M, \in)$ and $(M', \in)$ of $(H_\kappa, \in)$ are isomorphic and the isomorphism $\phi : (M, \in) \longrightarrow (M', \in)$ is the identity on $M' \cap M$. Extend $\phi(M) = M'$ and $\phi(p) = \phi``(p \cap N^p) \cup \{M'\}$.

(1) $\phi(p) \in P$ and $p \cap M' = \phi(p) \cap M$.

(2) If $M'' \in \mathcal{M}$ such that $p, \phi(p) \in M''$, then $p \cup \phi(p) \cup \{M''\} \in P$.

Proof. First note that $M' \in \mathcal{M}$ by assumption. Since $M \notin M$ and $p = \{M\} \cup (p \cap M) \notin M \cup \{M\}$, $\phi(M)$ and $\phi(p)$ are well-defined. We can check that $\phi(p) = \phi``p$ and that for all $N \in M \cup \{M\}$, $\phi(p) \cap \phi(N) = \phi``(p \cap N)$ as follows.

$$\phi(p) = \phi``(p \cap N^p) \cup \{M'\} = \phi``(p \cap M) \cup \{\phi(M)\} = \phi``(\{M\} \cup (p \cap M)) = \phi``p.$$

For $N = M$,

$$\phi(p) \cap \phi(N) = \phi(p) \cap M' = (\phi``(p \cap N^p) \cup \{M'\}) \cap M' = \phi``(p \cap N^p) = \phi``(p \cap N).$$

For $N \in M = N^p$,

$$\phi(p) \cap \phi(N) = (\phi``(p \cap N^p) \cup \{M'\}) \cap \phi(N) = \phi``(p \cap N^p) \cap \phi(N) = \phi``(p \cap N).$$

(1): We check 4 items to conclude $\phi(p) \in P$.

- $\phi(p) = \phi``p$ the set of point-wise images is countable. For any $N \in p$, $(\phi(N), \in)$ is a countable elementary substructure of $(H_\kappa, \in)$. The restriction $\phi\lceil N : (N, \in) \longrightarrow (\phi(N), \in)$ is an isomorphism that is the identity on $N \cap \phi(N)$. By assumption, we have $\phi(N^p) = M' \in \mathcal{M}$. For $\phi(N) \in \phi(p) \cap \phi(N^p)$, we have $N' \in p \cap N^p$ such that $N' \cap \omega_1 = N \cap \omega_1$ and $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system. Let $\sigma : (N, \in) \longrightarrow (N', \in)$ be the unique isomorphism that is the identity on $N \cap N'$. Notice that $\sigma \in M$. Then $\phi(N') \neq \phi(N)$, $\{\phi(N) \cap \omega_2, \phi(N') \cap \omega_2\}$ forms a Δ-system, and $\phi(\sigma) : (\phi(N), \in) \longrightarrow (\phi(N'), \in)$ is an isomorphism that is the identity on $\phi(N) \cap \phi(N')$. Hence $\phi(N) \in \mathcal{M}$. And so $\phi(p) = \{\phi(N^p)\} \cup (\phi(p) \cap \phi(N^p)) \subset \mathcal{M}$. For all $\phi(N) \in \phi(p) \cap \phi(N^p)$, we have seen that there exist $\phi(N') \in \phi(p) \cap \phi(N^p)$ such that $\phi(N) \cap \omega_1 = \phi(N') \cap \omega_1$ and $\{\phi(N) \cap \omega_2, \phi(N') \cap \omega_2\}$ forms a Δ-system.
- For $\phi(N), \phi(N') \in \phi(p)$ with $\phi(N) \cap \omega_1 = \phi(N') \cap \omega_1$, we have seen that $\phi(\sigma) : (\phi(N), \in) \longrightarrow (\phi(N'), \in)$ is an isomorphism that is the identity on $\phi(N) \cap \phi(N')$. Since $\phi(\sigma)``(\phi(p) \cap \phi(N)) = \phi``(\sigma``(p \cap N)) = \phi``(p \cap N') = \phi(p) \cap \phi(N')$, we conclude $\phi(\sigma) : (\phi(N), \in, \phi(p) \cap \phi(N)) \longrightarrow (\phi(N'), \in, \phi(p) \cap \phi(N'))$ is an isomorphism that is the identity on $\phi(N) \cap \phi(N')$.
- Let $\phi(\underline{N}) \cap \omega_1 < \phi(N') \cap \omega_1$. Then $\underline{N} \cap \omega_1 < N' \cap \omega_1$. Hence there exists $N \in p$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$. Hence $\phi(N) \in \phi(p)$, $\phi(\underline{N}) \in \phi(N)$, and $\phi(N) \cap \omega_1 = \phi(N') \cap \omega_1$.
- Let $\phi(N) \in \phi(p)$. If $N \cap p = \emptyset$, then $\phi(N) \cap \phi(p) = \emptyset$. If $N = \bigcup(p \cap N)$, then $\phi(N) = \bigcup(\phi(p) \cap \phi(N))$. If there exist $N_1 \neq N_2$ such that $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system, and $p \cap N = \{N_1, N_2\} \cup (p \cap N_1) \cup (p \cap N_2)$, then $\phi(N_1) \neq \phi(N_2)$, $\phi(N_1) \cap \omega_1 = \phi(N_2) \cap \omega_1$, $\{\phi(N_1) \cap \omega_2, \phi(N_2) \cap \omega_2\}$ forms a Δ-system, and $\phi(p) \cap \phi(N) = \{\phi(N_1), \phi(N_2)\} \cup (\phi(p) \cap \phi(N_1)) \cup (\phi(p) \cap \phi(N_2))$. Next, since $p = \{M\} \cap (p \cap M)$, $\phi(p) = \{M'\} \cup (\phi(p) \cap M')$, and $M \cap \omega_1 = M' \cap \omega_1$, we have $p \cap M' = p \cap M \cap M' = (\phi``p) \cap M \cap M' = \phi(p) \cap M \cap M' = \phi(p) \cap M$.

(2): We check 4 items to conclude $q = p \cup \phi(p) \cup \{M''\} \in P$.

- Since $p \subset \mathcal{M}$, $\phi(p) \subset \mathcal{M}$, and $M'' \in \mathcal{M}$, we have $q \subset \mathcal{M}$. Since $q \cap M'' = p \cup \phi(p)$, we have $q = \{M''\} \cup (q \cap M'')$. For all $N \in q \cap M'' = p \cup \phi(p)$, we know that there exists $N' \in q \cap M''$ such that $N' \cap \omega_1 = N \cap \omega_1$ and $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system.
- Since $q \cap M = (p \cup \phi(p)) \cap M = (p \cap M) \cup (\phi(p) \cap M) = (p \cap M) \cup (p \cap M' \cap M) = p \cap M$ and $q \cap M' = (p \cup \phi(p)) \cap M' = (p \cap M') \cup (\phi(p) \cap M') = (\phi(p) \cap M \cap M') \cup (\phi(p) \cap M') = \phi(p) \cap M' = \phi``(p \cap M)$, we have
$$\phi : (M, \in, q \cap M) \longrightarrow (M', \in, q \cap M')$$

is an isomorphism that is the identity on $M \cap M'$. For $N \in p \cap M$, we have $q \cap N = (q \cap M) \cap N = (p \cap M) \cap N = p \cap N$ and $q \cap \phi(N) = (q \cap M') \cap \phi(N) = (\phi(p) \cap M') \cap \phi(N) = \phi(p) \cap \phi(N)$. Hence for all $N, N' \in p \cap M$ with $N \cap \omega_1 = N' \cap \omega_1$, the maps

$$\sigma : (N, \in, q \cap N) \longrightarrow (N', \in, q \cap N'),$$

$$\phi(\sigma) : (\phi(N), \in, q \cap \phi(N)) \longrightarrow (\phi(N'), \in, q \cap \phi(N')),$$

$$\phi\lceil N : (N, \in, q \cap N) \longrightarrow (\phi(N), \in, q \cap \phi(N)),$$

$$\phi(\sigma) \circ \phi\lceil N : (N, \in, q \cap N) \longrightarrow (\phi(N'), \in, q \cap \phi(N'))$$

are all isomorphisms that are the identities on the intersections.

- $M'' \cap \omega_1$ is the max in $\{N \cap \omega_1 \mid N \in q\}$ and $p \cup \phi(p) \subset M''$.
- For any $N \in p$, we have seen that $q \cap N = p \cap N$ and $q \cap \phi(N) = \phi(p) \cap \phi(N)$. Hence if $p \cap N = \emptyset$, then $q \cap N = q \cap \phi(N) = \emptyset$. If $N = \bigcup(p \cap N)$, then $N = \bigcup(q \cap N)$ and $\phi(N) = \bigcup(\phi(p) \cap \phi(N)) = \bigcup(q \cap \phi(N))$. If $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system, and $p \cap N = \{N_1, N_2\} \cup (p \cap N_1) \cup (p \cap N_2)$, then $q \cap N = p \cap N = \{N_1, N_2\} \cup (q \cap N_1) \cup (q \cap N_2)$ and $q \cap \phi(N) = \phi(p) \cap \phi(N) = \{\phi(N_1), \phi(N_2)\} \cup (\phi(p) \cap \phi(N_1)) \cup (\phi(p) \cap \phi(N_2)) = \{\phi(N_1), \phi(N_2)\} \cup (q \cap \phi(N_1)) \cup (q \cap \phi(N_2))$. Since $N^p \cap \omega_1 = \phi(N^p) \cap \omega_1$, $\{N^p \cap \omega_2, \phi(N^p) \cap \omega_2\} = \{M \cap \omega_2, M' \cap \omega_2\}$ forms a Δ-system, and $q \cap M'' = (p \cap M'') \cup (\phi(p) \cap M'') = p \cup \phi(p) = \{N^p\} \cup (p \cap N^p) \cup \{\phi(N^p)\} \cup (\phi(p) \cap \phi(N^p)) = \{N^P, \phi(N^p)\} \cup (q \cap N^p) \cup (q \cap \phi(N^p))$. Thus M'' satisfies (suc).

□

7.7 Lemma. For any $p \in P$ and $e \in H_\kappa$, there exists $q \leq p$ such that $e \in N^q$.

Proof. Since $N^p \in \mathcal{M}$, there exists M' such that $M' \cap \omega_1 = N^p \cap \omega_1$, $\{N^p \cap \omega_2, M' \cap \omega_2\}$ forms a Δ-system, $(M', \in)$ and $(N^p, \in)$ are two isomorphic countable elementary substructures of $(H_\kappa, \in)$ and the isomorphism $\phi : (N^p, \in) \longrightarrow (M', \in)$ is the identity on $N^p \cap M'$. Let $M'' \in \mathcal{M}$ with $e, p, \phi``(p \cap N^p) \cup \{M'\} \in M''$. Then we have seen that $q = p \cup (\phi``(p \cap N^p) \cup \{M'\}) \cup \{M''\} \in P$. This q works.

□

7.8 Lemma. P is σ-Baire.

Proof. Let $p \in P$. Let $p, P, H_\kappa \in M \prec H_\theta$, where θ be a sufficiently large regular cardinal and $|M| = \omega$. We may assume that $M \cap H_\kappa \in \mathcal{M}$. Let $\langle p_n \mid n < \omega \rangle$ be a (P, M)-generic sequence with $p_0 = p$. Hence $H_\kappa \cap M = \bigcup\{N^{p_n} \mid n < \omega\}$ and so $N^{p_n} \cap \omega_1$'s are cofinal below $M \cap \omega_1$. Let $q = \{M \cap H_\kappa\} \cup \bigcup\{p_n \mid n < \omega\}$. Then this $q \in P$ and so P is σ-Baire. Some details follow.

- For all $n < \omega$, $p_n \subset \mathcal{M}$ and $M \cap H_\kappa \in \mathcal{M}$. Hence $q \subset \mathcal{M}$. $M \cap H_\kappa$ is the top element of q, as $q \cap (M \cap H_\kappa) = \bigcup\{p_n \mid n < \omega\}$ and so $q = \{M \cap H_\kappa\} \cup (q \cap (M \cap H_\kappa))$. For all elements $N \in \bigcup\{p_n \mid n < \omega\}$, say, $N \in p_n \cap N^{p_n}$, there exists $N' \in p_n \cap N^{p_n}$ such that $N' \cap \omega_1 = N \cap \omega_1$ and $\{N \cap \omega_2, N' \cap \omega_2\}$ forms a Δ-system.
- Let $N, N' \in q \cap (M \cap H_\kappa)$, say, $N, N' \in p_n$. If $N \cap \omega_1 = N' \cap \omega_1$, then $(N, \in, p_n \cap N)$ and $(N', \in, p_n \cap N')$ are isomorphic and that the isomorphism $(N, \in, p_n \cap N) \longrightarrow (N', \in, p_n \cap N')$ is the identity on $N \cap N'$. Since $q \cap N = p_n \cap N$ and $q \cap N' = p_n \cap N'$, the map ϕ is an isomophism from $(N, \in, q \cap N)$ onto $(N', \in, q \cap N')$.
- Let $\underline{N}, N' \in q$. If $\underline{N} \cap \omega_1 < N' \cap \omega_1 < M \cap \omega_1$, then $\underline{N}, N \in p_n$ for some $n < \omega$. Hence there exists $N \in p_n$ such that $\underline{N} \in N$ and $N \cap \omega_1 = N' \cap \omega_1$. If $\underline{N} \cap \omega_1 < M \cap \omega_1$, then $\underline{N} \in M \cap H_\kappa$.
- Let $N \in q$. Suppose first $N \in p_n$ for some $n < \omega$. If $p_n \cap N = \emptyset$, then $q \cap N = p_n \cap N = \emptyset$. If $N = \bigcup(p_n \cap N)$, then $N = \bigcup(p_n \cap N) = \bigcup(q \cap N)$. If there are $N_1 \neq N_2$ such that $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system, and $p_n \cap N = \{N_1, N_2\} \cup (p_n \cap N_1) \cup (p_n \cap N_2)$, then $q \cap N = p_n \cap N = \{N_1, N_2\} \cup (p_n \cap N_1) \cup (p_n \cap N_2) = \{N_1, N_2\} \cup (q \cap N_1) \cup (q \cap N_2)$. Suppose next $N = M \cap H_\kappa$. Then $N = \bigcup\bigcup\{p_n \mid n < \omega\} = \bigcup(q \cap N)$. Namely $M \cap H_\kappa$ satisfies (limit).

□

7.9 Lemma. (CH) P has the ω_2-c.c.

Proof. Let $\langle p_i \mid i < \omega_2\rangle$ be a list of conditions. We may assume that $\langle N^{p_i} \cap \omega_2 \mid i < \omega_2\rangle$ forms a Δ-system. Hence for all $i \neq j$, $N^{p_i} \neq N^{p_j}$. We may also assume that for all $i \neq j$, $(N^{p_i}, \in, p_i \cap N^{p_i})$ and $(N^{p_j}, \in, p_j \cap N^{p_j})$ are isomorphic and the isomorphisms are the identities on the intersections $N^{p_i} \cap N^{p_j}$. Let $M \in \mathcal{M}$ such that $p_i, p_j \in M$. Let $q = p_i \cup p_j \cup \{M\}$. Then $q \in P$ and $q \leq p_i, p_j$.

□

7.10 Lemma. Let G be P-generic over the ground model V. In the generic extension $V[G]$, let $\dot{\mathcal{N}} = \bigcup G$. Then this $\dot{\mathcal{N}}$ is a matrix that satisfies LD(2) + Δ with $H = H^V_\kappa$.

Proof. By genericity. We mention that for any $N \in \dot{\mathcal{N}}$, say, $N \in p \in G$, we have $\dot{\mathcal{N}} \cap N = p \cap N$.

□

The matrix $\dot{\mathcal{N}}$ is Δ-complete.

7.11 Lemma. In $V[G]$, let $\langle e_i \mid i < \omega_2\rangle$ be a sequence of elements of $H = H^V_\kappa$. Then there exist $N, N_1, N_2 \in \dot{\mathcal{N}}$ and $i < j < \omega_2$ such that

(1) $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system, and $\dot{\mathcal{N}} \cap N = \{N_1, N_2\} \cup (\dot{\mathcal{N}} \cap N_1) \cup (\dot{\mathcal{N}} \cap N_2)$.

(2) $e_i \in N_1$, $e_j \in N_2$, and $(N_1, \in, e_i)$ and $(N_2, \in, e_j)$ are isomorphic.

Proof. Repeat the proof of the ω_2-c.c.

□

§8. A construction along a matrix

We present a construction along a matrix $\mathcal{N}$ that is a complex next to the ordinals. While our construction is a direct one, but it is weaker than [D] and [I], since we assume that the relevant matrix is Δ-complete. We make use of this Δ-completeness to force a sort of diamond by the Cohen forcing $\mathrm{Fn}(\omega_1, \omega)$ that adds a new subset of ω_1 by the finite conditions.

8.1 Lemma. Let $\mathcal{N}$ be a matrix that satisfy LD(2) + Δ and be Δ-complete. Let $K_2 = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \subseteq \omega_2$ be a copy of ω_2 along $\mathcal{N}$. Then in the generic extension by the Cohen forcing $\mathrm{Fn}(\omega_1, \omega)$, we have a choice function $F : \mathcal{N} \longrightarrow K_2$ and a flag $E : \mathcal{N} \longrightarrow 2$ such that

(1) For any $N, M \in \mathcal{N}$, if $i_0 \leq N \cap \omega_1 = M \cap \omega_1$, then $(N, \in, F(N), E(N))$ and $(M, \in, F(M), E(M))$ are isomorphic.

(2) For any one-to-one list $\langle \xi_i \mid i < \omega_2\rangle$ of elements of K_2 and $e \in 2$, there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that

- $N_1 \neq N_2$, $i_0 \leq N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
- $F(N_1) = \xi_i$, $F(N_2) = \xi_j$, and $E(N) = e$.

Proof. Let $p \in P$, if $p = (F^p, E^p)$ such that

(1) F^p and E^p are finite functions of the same domain that is included in $\{N \cap \omega_1 \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1\}$.

(2) For all $i \in \mathrm{dom}(F^p) = \mathrm{dom}(E^p)$, $F^p(i) \in \overline{N_i \cap K_2} = \{f(\overline{\xi_0}) \mid f \in F_{i_0 i}\}$ and $E^p(i) \in 2$.

For $p, q \in P$, we set $q \leq p$, if $F^q \supseteq F^p$ and $E^q \supseteq E^p$.

Let G be P-generic over V. In $V[G]$, let $F^G = \bigcup\{F^p \mid p \in G\}$ and $E^G = \bigcup\{E^p \mid p \in G\}$. Then F^G and E^G are total functions with the domain $\{N \cap \omega_1 \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1\}$. For $N \in \mathcal{N}$ with $i_0 \leq N \cap \omega_1$, let $F(N) \in N \cap K_2$ and $E(N) \in 2$ such that $(N, \in, F(N), E(N))$ is isomorphic with $(\overline{N}, \in$

$, F^G(N \cap \omega_1), E^G(N \cap \omega_1))$, where $\overline{N}$ denotes the transitive collapse of N. Then for all N and M such that $N, M \in \mathcal{N}$ and $i_0 \leq N \cap \omega_1 = M \cap \omega_1$, by definition, the two structures $(N, \in, F(N), E(N))$ and $(M, \in, F(M), E(M))$ are isomorphic.

Claim. For any one-to-one list $\langle \xi_i \mid i < \omega_2 \rangle$ of elements of K_2 and $e \in 2$, there exists $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that

- $N_1 \neq N_2$, $i_0 \leq N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
- $F(N_1) = \xi_i$, $F(N_2) = \xi_j$, and $E(N) = e$.

Proof. Let $p \Vdash_P$ "$\langle \dot{\xi}_i \mid i < \omega_2 \rangle$ be a one-to-one list in K_2 and $\dot{e} \in 2$". For each $i < \omega_2$, let $p_i \leq p$, $\xi_i \in K_2$ and $e_i \in 2$ such that $p_i \Vdash_P$ "$\dot{\xi}_i = \xi_i$ and $\dot{e} = e_i$". Consider a sequence $\langle (p_i, \xi_i, e_i) \mid i < \omega_2 \rangle$ of elements of H. Since $\mathcal{N}$ is Δ-complete, there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that

(1) $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_2$, $\{N_1 \cap \omega_2, N_2 \cap \omega_2\}$ forms a Δ-system, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.

(2) $p_i = p_j = r$, $e_i = e_j = e$, $p_i, \xi_i \in N_1$, $p_j, \xi_j \in N_2$, and $(N_1, \in, \xi_i)$ and $(N_2, \in, \xi_j)$ are isomorphic.

Since $r \Vdash_P$ "$\xi_i = \dot{\xi}_i \neq \dot{\xi}_j = \xi_j$", we must have $\xi_i \neq \xi_j$. Since $(N_1, \in, N_1 \cap K_2)$ and $(N_2, \in, N_2 \cap K_2)$ are isomorphic and the isomorphism is the identity on $N_1 \cap N_2$, we must have $\xi_i \in (N_1 \cap K_2) \setminus N_2$ and $\xi_j \in (N_2 \cap K_2) \setminus N_1$. Hence, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system. Let $q = (F^q, E^q)$, where $F^q = F^r \cup \{(N_1 \cap \omega_1, \text{the } N_1\text{-collapse of } \xi_i), (N \cap \omega_1, \text{Free})\}$ and $E^q = E^r \cup \{(N_1 \cap \omega_1, \text{Free}), (N \cap \omega_1, e)\}$. Then $q \Vdash_P$ "$\dot{F}(N_1) = \xi_i, \dot{F}(N_2) = \xi_j$, and $\dot{E}(N) = e$".

□

8.2 Theorem. Let $\mathcal{N}$ be a matrix that satisfy LD(2) + Δ. Let $K_2 = \{f(\overline{\xi_0}) \mid f \in F_{i_0\omega_1}\} \subseteq \omega_2$ be a copy of ω_2 along $\mathcal{N}$. Let there exist a choice function $F : \mathcal{N} \longrightarrow K_2$ and a flag $E : \mathcal{N} \longrightarrow 2$ such that

(1) For any $N, M \in \mathcal{N}$, if $i_0 \leq N \cap \omega_1 = M \cap \omega_1$, then $(N, \in, F(N), E(N))$ and $(M, \in, F(M), E(M))$ are isomorphic.

(2) For any one-to-one list $\langle \xi_i \mid i < \omega_2 \rangle$ of elements of K_2 and $e \in 2$, there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that
   - $N_1 \neq N_2$, $i_0 \leq N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
   - $F(N_1) = \xi_i$, $F(N_2) = \xi_j$, and $E(N) = e$.

Then there exists an ω_2-Souslin tree.

Proof. We first provide an out-line.

Step 1. Construct $\langle N \mapsto T^N = (N \cap K_2, <^N) \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1 \rangle$ such that

(1) T^N is a countable tree such that if $\alpha <^N \beta$, then $\alpha < \beta$.

(2) If $\phi : (N, \in) \longrightarrow (N', \in)$ is the isomorphism, then $N' \cap K_2 = \phi``(N \cap K_2)$ and for all $\alpha, \beta \in N \cap K_2$, $\phi(\alpha) <^{N'} \phi(\beta)$ iff $\alpha <^N \beta$.

(3) If $N' \subseteq N$, then $T^{N'}$ is a subtree of T^N (for all $\alpha, \beta \in N' \cap K_2$, $\alpha <^N \beta$ iff $\alpha <^{N'} \beta$).

(4) Exclusively either (0) || (limit) || (suc) holds.

   (0) If $N_0 \in \mathcal{N}$ with $N_0 \cap \omega_1 = i_0$, then $T^{N_0} = (\{(\overline{\xi_0})^{N_0}\}, \emptyset)$.

   (limit) If $N = \bigcup(\mathcal{N} \cap N)$, then $T^N = (N \cap K_2, \bigcup\{<^{\underline{N}} \mid \underline{N} \in \mathcal{N} \cap N, i_0 \leq \underline{N} \cap \omega_1\})$.

   (suc) Let there exist $N_1, N_2 \in \mathcal{N} \cap N$ such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system with possibly empty tails, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. We are interested in the case $F(N_1) \neq F(N_2)$, where we may assume $F(N_1) < F(N_2)$ as two ordinals. Now we have two cases. If $E(N) = 1$, then $F(N_1) <^N F(N_2)$. If $E(N) = 0$, then $F(N_1) \not<^N F(N_2)$ and $F(N_2) \not<^N F(N_1)$.

Step 2. Let $T = (K_2, <^T) = (K_2, \bigcup\{<^N \mid N \in \mathcal{N}, i_0 \leq N \cap \omega_1\})$. Then this T is an ω_2-Souslin tree.

Assuming Step 1, we show Step 2.

Claim. $T = (K_2, <^T)$ is a tree that is embeddable into the ordinals.

Proof. (irreflexive) If $\alpha <^T \alpha$, then $\alpha <^N \alpha$ for some $N \in \mathcal{N}$ with $i_0 \leq N \cap \omega_1$. But then $\alpha < \alpha$ as an ordinal. This is a contradiction.

(transitive) Let $\alpha <^T \beta <^T \gamma$. Then $\alpha <^N \beta <^{N'} \gamma$. Let $N, N' \in N'' \in \mathcal{N}$. We have $\alpha <^{N''} \beta <^{N''} \gamma$ and so $\alpha <^{N''} \gamma$. Hence $\alpha <^T \gamma$.

(comparison below a node) Let $\alpha <^T \gamma$ and $\beta <^T \gamma$. Then $\alpha <^N \gamma$ and $\beta <^{N'} \gamma$. Let $N, N' \in N'' \in \mathcal{N}$. Then $\alpha <^{N''} \gamma$ and $\beta <^{N''} \gamma$ and so α, β are comparable in $T^{N''}$ and so are in T.

(embeddable into the ordinals) Let $\alpha <^T \beta$. Then $\alpha <^N \beta$ and so $\alpha < \beta$.

Hence $T = (K_2, <^T)$ is a tree.

□

Claim. If $A \subset T$ is an antichain, then $|A| < \omega_2$.

Proof. To the contrary, assume that A is of size ω_2. Let $\langle \xi_i \mid i < \omega_2 \rangle$ be a one-to-one enumeration of A. Let $e = 1$. Then there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that

- $N_1, N_2 \in \mathcal{N} \cap N$, $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
- $F(N_1) = \xi_i$, $F(N_2) = \xi_j$, and $E(N) = e = 1$.

By Step 1, since $F(N_1) = \xi_i \neq \xi_j = F(N_2)$ and $E(N) = 1$, we have $F(N_1) <^N F(N_2)$ and so $\xi_i <^T \xi_j$. Since A is an antichain, this is a contradiction.

□

Claim. If $B \subset T$ is a chain, then $|B| < \omega_2$.

Proof. To the contrary, assume that B is of size ω_2. Let $\langle \xi_i \mid i < \omega_2 \rangle$ be a one-to-one enumeration of B. Let $e = 0$. Then there exist $N, N_1, N_2 \in \mathcal{N}$ and $i < j < \omega_2$ such that

- $N_1, N_2 \in \mathcal{N} \cap N$, $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$.
- $F(N_1) = \xi_i$, $F(N_2) = \xi_j$, and $E(N) = e = 0$.

By Step 1, since $F(N_1) = \xi_i \neq \xi_j = F(N_2)$ and $E(N) = 0$, two different nodes ξ_i and ξ_j are incomparable in the tree T^N. Then we conclude ξ_i and ξ_j are incomparable in the tree T. This is because, say, if $\xi_i <^T \xi_j$, then $\xi_i <^{N'} \xi_j$ for some $N' \in \mathcal{N}$. Let $N, N' \in N'' \in \mathcal{N}$. Then $\xi_i <^{N''} \xi_j$. Since T^N is a subtree of $T^{N''}$, this would be a contradiction. Since B is a chain, this is a contradiction.

□

Claim. T is an ω_2-Souslin tree.

Proof. Let T_α denote the α-th level of T. If $T_{\omega_2} \neq \emptyset$, then any element of T_{ω_2} gives rise to a chain B of size ω_2. Hence $T_{\omega_2} = \emptyset$. Hence T is of height $\leq \omega_2$. For each $T_\alpha \neq \emptyset$, since T_α is an antichain, T_α is of size at most ω_1. Since T is of size ω_2, we conclude that T is an ω_2-tree. Namely, T is a tree of height ω_2 with each level of size at most ω_1. Since T has no antichain of size ω_2 and no chain of size ω_2, T is an ω_2-Souslin tree.

□

Proof of Step 1. Construct $T^N = (N \cap K_2, <^N)$ by recursion on $i_0 \leq N \cap \omega_1$ such that

(1) T^N is a countable tree that is embeddable into the ordinals.

(2) If $N \cap \omega_1 = N' \cap \omega_1$, then $(N, \in, T \cap K_2, <^N)$ and $(N', \in, N' \cap K_2, <^{N'})$ are isomorphic.

(3) If $N' \subseteq N$, then $T^{N'}$ is a subtree of T^N.

(4) Exclusively either (0) || (limit) || (suc) holds.

(0) If $N_0 \in \mathcal{N}$ with $N_0 \cap \omega_1 = i_0$, then $T^{N_0} = (\{(\overline{\xi_0})^{N_0}\}, \emptyset)$.

(limit) If $N = \bigcup(\mathcal{N} \cap N)$, then $T^N = (N \cap K_2, \bigcup\{<^{\underline{N}} \mid \underline{N} \in \mathcal{N} \cap N, i_0 \leq \underline{N} \cap \omega_1\})$.

(suc) Let there exist $N_1, N_2 \in \mathcal{N} \cap N$ such that $N_1 \neq N_2$, $N_1 \cap \omega_1 = N_2 \cap \omega_1$, $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system with possibly empty tails, and $\mathcal{N} \cap N = \{N_1, N_2\} \cup (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2)$. If $F(N_1) \neq F(N_2)$, where we may assume $F(N_1) < F(N_2)$, and $E(N) = 1$, then $T^N = (N \cap K_2, <^{N_1} \cup <^{N_2} \cup <_3)$. Otherwise, $T^N = (N \cap K_2, <^{N_1} \cup <^{N_2})$. Here

- $<_3 = O(F(N_1)) \times ([F(N_2)']_{<^{N_2}} \setminus N_1)$.
- $O(F(N_1)) = \{\xi \in N_1 \cap K_2 \mid \xi \leq^{N_1} F(N_1)\}$.
- $F(N_2)'$ is the $\leq^{N_2}$-least element ξ such that $\xi \in (N_2 \cap K_2) \setminus N_1$ and $\xi \leq^{N_2} F(N_2)$.
- $[F(N_2)']_{<^{N_2}} = \{\xi \in N_2 \cap K_2 \mid \xi \leq^{N_2} F(N_2)' \mid\mid F(N_2)' \leq^{N_2} \xi\}$.

Case (0): Let $N_0 \in \mathcal{N}$ with $N_0 \cap \omega_1 = i_0$. We set $T^{N_0} = (\{(\overline{\xi_0})^{N_0}\}, \emptyset)$, where $N_0 \cap K_2 = \{(\overline{\xi_0})^{N_0}\}$. Hence T^{N_0} consists of the single element. The T^{N_0}'s satisfy the induction hypothesis.

Case (limit): Let $N = \bigcup(\mathcal{N} \cap N)$. Set $T^N = (N \cap K_2, \bigcup\{<^{\underline{N}} \mid \underline{N} \in \mathcal{N} \cap N\ , i_0 \leq \underline{N} \cap \omega_1\})$.

Claim. T^N is a tree that is embeddable into the ordinals.

Proof. (irreflexive) If $\alpha <^N \alpha$, then $\alpha <^{\underline{N}} \alpha$ and so $\alpha < \alpha$ as an ordinal. This is a contradiction.

(transitive) Let $\alpha <^N \beta <^N \gamma$. Then $\alpha <^{\underline{N}} \beta$ and $\beta <^{\underline{\underline{N}}} \gamma$. Let $\underline{N}, \underline{\underline{N}} \in N' \in \mathcal{N} \cap N$. Then $\alpha <^{N'} \beta <^{N'} \gamma$ and so $\alpha <^{N'} \gamma$. Hence $\alpha <^N \gamma$.

(comparison below a node) Let $\alpha <^N \gamma$ and $\beta <^N \gamma$. Then $\alpha <^{\underline{N}} \gamma$ and $\beta <^{\underline{\underline{N}}} \gamma$. Let $\underline{N}, \underline{\underline{N}} \in N' \in \mathcal{N} \cap N$. Then $\alpha <^{N'} \gamma$ and $\beta <^{N'} \gamma$ and so α and β are comparable in $T^{N'}$ and so are in T^N.

(embeddable into the ordinals) Let $\alpha <^N \beta$. Then $\alpha <^{\underline{N}} \beta$ and so $\alpha < \beta$.

□

Claim. If $N \cap \omega_1 = N' \cap \omega_1$, then $(T^N, <^N)$ and $(T^{N'}, <^{N'})$ are isomorphic under the isomorphism $\phi : (N, \in) \longrightarrow (N', \in)$.

Proof. We know that $\phi``(N \cap K_2) = N' \cap K_2$. Let $\alpha, \beta \in N \cap K_2$ and let $\underline{N} \in \mathcal{N} \cap N$. Then $\phi\lceil\underline{N} : (\underline{N}, \in) \longrightarrow (\phi(\underline{N}), \in)$ is the unique isomorphism. Thus by induction, $\alpha <^{\underline{N}} \beta$ iff $\phi(\alpha) <^{\phi(\underline{N})} \phi(\beta)$ and so $\alpha <^N \beta$ iff $\phi(\alpha) <^{N'} \phi(\beta)$.

□

Claim. If $\underline{N} \subset N$ (proper inclusion), then $T^{\underline{N}}$ is a subtree of T^N.

Proof. We must have $\underline{N} \cap \omega_1 < N \cap \omega_1$ and $\underline{N} \in N$. This is because, if $N \cap \omega_1 < \underline{N} \cap \omega_1$, then there exists $N' \in \mathcal{N}$ such that $N \in N'$ and $N' \cap \omega_1 = \underline{N} \cap \omega_1$. Let $\phi : (\underline{N}, \in) \longrightarrow (N', \in)$ be the isomorphism. Since $\underline{N} \subset \underline{N} \cap N \subset \underline{N} \cap N'$, we have $\underline{N} = \phi``\underline{N} = N'$ and so $\underline{N} \subset N \subset N' = \underline{N}$. This is a contradiction. If $\underline{N} \cap \omega_1 = N \cap \omega_1$. Then let $\phi : (\underline{N}, \in) \longrightarrow (N, \in)$ be the isomorphism. Then $\underline{N} \subset \underline{N} \cap N$ and so $\underline{N} = \phi``\underline{N} = N$. This is a contradiction. Hence $\underline{N} \cap \omega_1 < N \cap \omega_1$. Then there exists $N' \in \mathcal{N}$ such that $\underline{N} \in N'$ and $N' \cap \omega_1 = N \cap \omega_1$. Let $\phi : (N', \in) \longrightarrow (N, \in)$ be the isomorphim. Since $\underline{N} \subset N' \cap N$, we have $\underline{N} = \phi``\underline{N} = \phi(\underline{N}) \in N$.

Let $\alpha, \beta \in \underline{N} \cap K_2$. If $\alpha <^{\underline{N}} \beta$, then by definition, we have $\alpha <^N \beta$. Conversely, if $\alpha <^N \beta$, then $\alpha <^{\underline{\underline{N}}} \beta$ for some $\underline{\underline{N}} \in \mathcal{N} \cap N$. Let $\underline{N}, \underline{\underline{N}} \in N' \in \mathcal{N} \cap N$. Then $\alpha <^{N'} \beta$ and it in turn, by induction, entails $\alpha <^{\underline{N}} \beta$.

□

Case (suc). Let $(T^N, <^N) = (N \cap K_2, <^{N_1} \cup <^{N_2} \cup <_3)$, if $E(N) = 1$ and $F(N_1) \neq F(N_2)$, where we may assume $F(N_1) < F(N_2)$. Otherwise, let $(T^N, <^N) = (N \cap K_2, <^{N_1} \cup <^{N_2})$.

Claim. T^N is a tree that is embeddable into the ordinals.

Proof. (irreflexive) If $\alpha <^N \alpha$, then $\alpha <^{N_1} \alpha \mid\mid \alpha <^{N_2} \alpha \mid\mid \alpha <_3 \alpha$. Then $\alpha < \alpha$ as an ordinal. This is a contradiction.

(transitive) Let $\alpha <^N \beta <^N \gamma$. Want to show $\alpha <^N \gamma$.

Case. $<_3$ is not relevant: We provide details of two subcases.

Subcase 1. $\alpha <^{N_1} \beta <^{N_2} \gamma$. Since $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system with possible empty tails, we have $\alpha, \beta \in N_1 \cap N_2$ and so, via the isomorphism, $\alpha <^{N_2} \beta <^{N_2} \gamma$. Hence $\alpha <^{N_2} \gamma$ and so $\alpha <^N \gamma$.

Subcase 2. $\alpha <^{N_2} \beta <^{N_1} \gamma$. We have $\alpha, \beta \in N_1 \cap N_2$ and $\alpha <^{N_1} \beta <^{N_1} \gamma$. Hence $\alpha <^{N_1} \gamma$ and so $\alpha <^N \gamma$.

□

Case. $<_3$ is relevant: We have several subcases. The point is that $\{N_1 \cap K_2, N_2 \cap K_2\}$ forms a Δ-system with possible empty tails.

Subcase 1. $\alpha <_3 \beta <_3 \gamma$. By the definition of $<_3$, we have $\alpha \in N_1$, $\beta \in N_2 \setminus N_1$, $\beta \in N_1$, and $\gamma \in N_2 \setminus N_1$. This case does not occur.

Subcase 2. $\alpha <_3 \beta <^{N_1} \gamma$. This case does not occur.

Subcase 3. $\alpha <_3 \beta <^{N_2} \gamma$. Then $\alpha <_3 \gamma$ holds.

Subcase 4. $\alpha <^{N_1} \beta <_3 \gamma$. Then $\alpha <_3 \gamma$ holds.

Subcase 5. $\alpha <^{N_2} \beta <_3 \gamma$. Then $\alpha <_3 \gamma$ holds.

(comparison below a node) If $\alpha <^\gamma$ and $\beta <^N \gamma$, then α and β are comparable in T^N.

Case. $<_3$ is irrelevant: We provide details when $\alpha <^{N_1} \gamma$ and $\beta <^{N_2} \gamma$. In this case, α, β, and γ are all in $N_1 \cap N_2$. Hence $\beta <^{N_1} \gamma$ and so α and β are comparable in T^{N_1}. Hence so are in T^N.

Case. $<_3$ is relevant: We have several subcases.

Subcase 1. $\alpha <_3 \gamma$ and $\beta <_3 \gamma$. Then α and β are comparable in T^{N_1} and so are in T^N.

Subcase 2. $\alpha <_3 \gamma$ and $\beta <^{N_1} \gamma$. Since $\gamma \in N_2 \setminus N_1$, this case does not occur.

Subcase 3. $\alpha <_3 \gamma$ and $\beta <^{N_2} \gamma$. If $\beta \in N_2 \setminus N_1$, then $\alpha <_3 \beta$. If $\beta \in N_1 \cap N_2$, then $\beta <^{N_1} \alpha \mid\mid \alpha = \beta \mid\mid \alpha <^{N_1} \beta$. In any case, α and β are comparable in T^N.

(embeddable into the ordinals) Let $\alpha <^N \beta$. Then in either case $\alpha <^{N_1} \beta \mid\mid \alpha <^{N_2} \beta \mid\mid \alpha <_3 \beta$, we have $\alpha < \beta$.

□

Claim. $<^N \cap N_1 =<^{N_1}$ and $<^N \cap N_2 =<^{N_2}$.

Proof. We have several cases.

Case 1. $\alpha, \beta \in N_1$ and $\alpha <^{N_2} \beta$. Then, via isomorphism, $\alpha <^{N_1} \beta$.

Case 2. $\alpha, \beta \in N_2$ and $\alpha <^{N_1} \beta$. Then, via isomorphism, $\alpha <^{N_2} \beta$.

Case 3. $\alpha, \beta \in N_1$ and $\alpha <_3 \beta$. This case does not occur.

Case 4. $\alpha, \beta \in N_2$ and $\alpha <_3 \beta$. Then, via isomorphism, $\alpha <^{N_2} \beta$.

□

Claim. Let $\underline{N} \in \mathcal{N} \cap N$. Then $T^{\underline{N}}$ is a subtree of T^N.

Proof. We have several cases.

Case 1. $\underline{N} = N_1$. We have seen $<^N \cap N_1 =<^{N_1}$.
Case 2. $\underline{N} = N_2$. We have seen $<^N \cap N_2 =<^{N_2}$.
Case 3. $\underline{N} \in N_1$. We calculate $<^{\underline{N}}=<^{N_1} \cap \underline{N} = (<^N \cap N_1) \cap \underline{N} =<^N \cap \underline{N}$.

Case 4. $\underline{N} \in N_2$. We calculate $<^{\underline{N}}=<^{N_2} \cap \underline{N} = (<^N \cap N_2) \cap \underline{N} =<^N \cap \underline{N}$.

□

Claim. Let $\phi : (N, \in, N_1, N_2) \longrightarrow (N', \in, N_1', N_2')$ be the isomorphism. We have

$$\phi : (N, \in, T^N, <^N) \longrightarrow (N', \in, T^{N'}, <^{N'}).$$

Proof. Since $\phi\lceil N_1$ and $\phi\lceil N_2$ are the isomorphisms from N_1 (N_2) onto N_1' (N_2'), respectively, we have

$$\phi : (N, \in, F(N_1), F(N_2), E(N)) \longrightarrow (N', \in, F(N_1'), F(N_2'), E(N')).$$

Then $F(N_1) \neq F(N_2)$ and $E(N) = 1$ iff $F(N_1') \neq F(N_2')$ and $E(N') = 1$. By induction,

$$\phi\lceil N_1 : (N_1, T^{N_1}, <^{N_1}) \longrightarrow (N_1', T^{N_1'}, <^{N_1'}),$$

$$\phi\lceil N_2 : (N_2, T^{N_2}, <^{N_2}) \longrightarrow (N_2', T^{N_2'}, <^{N_2'}),$$

and

$$\phi : (N, \in, N \cap K_2, <_3) \longrightarrow (N', \in, N' \cap K_2, <_3')$$

are isomorphisms, where $<_3'$ considered with respect to $F(N_1')$, $F(N_2')$, $T^{N_1'}$, and $T^{N_2'}$. Hence $(N, \in, T^N, <^N)$ and $(N', \in, T^{N'}, <^{N'})$ are isomorphic via $\phi : (N, \in) \longrightarrow (N', \in)$.

□

This completes Step 1.

□

8.3 Question. (1) Develop a theory of finitely many sorted matrices that would serve as alternative structures to some of higher gap morasses.

(2) Do we have any theory of non-homogeneous matrices, where no ω_2-c.c. is expected, that would provide another view to [F], [Kr], [Mit], and [Mo] ?

(3) What kind of directions do [N] and [V-V] suggest, say, with respect to question (1) ?

References

[A-M] D. Aspero, M. Mota, A generalization of Martin's Axiom, preprint, 2012. http://arxiv.org/pdf/1206.6724.pdf

[B-S] J. Baumgartner, S. Shelah, Remarks on superatomic Boolean algebras, *Annals of Pure and Applied Logic*, 33 (1987) pp.109–129.

[F] S. Friedman, Forcing with finite conditions, preprint, October 6, 2005.

[I] B. Irrgang, Morasses and finite support iterations, *Proceedings of the American Mathematical Society*, vol.137 (2009) no.3, pp.1103–1113.

[Ka] A. Kanamori, Morasses in combinatorial set theory, Surveys in Set Theory, *London Mathematical Society Lecture Notes Series*, 87 (1983) pp.167–196.

[Ko] P. Koszmider, Semimorasses and nonreflection at singular cardinals, *Annals of Pure and Applied Logic*, vol.72 (1995) pp.1–23.

[Kr] J. Krueger, Strongly adequate sets and adding a club with finite conditions, preprint.

[Mit] W. Mitchell, Adding closed unbounded subsest of ω_2 with finite forcing, *Notre Dame Journal of Formal Logic*, Vol.46, No.3 (2005) pp.357–371.

[M] T. Miyamoto, Forcing a quagmire via matrices of models, preprint, 2013.

[Mo] C. Morgan, Adding club subsets of ω_2 using conditions with finite working parts, preprint.

[N] I. Neeman, Analogues of properness, and applications, slides. October 2013.

[T1] S. Todorcevic, A note on the proper forcing axiom, *Contemporary Mathematics*, vol.31 (1984) pp.209–218.

[T2] S. Todorcevic, Directed sets and cofinal types, *Transactions of the American Mathematical Society*, vol.290 (1985) no.2, pp.711–723.

[V-V] B. Velickovic, G. Venturi, Proper forcing remastered, preprint

[V] D. Velleman, Simplified morasses, *Journal of Symbolic Logic*, vol. 49 (1984) no.1, pp.257–271.

miyamoto@nanzan-u.ac.jp

Mathematics

Nanzan University

18 Yamazato-cho, Showa-ku, Nagoya

466-8673 Japan

南山大学　数学　宮元忠敏