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Construction of solutions to a fourth order parabolic obstacle problem via minimizing movements

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1 Introduction

The paper is devoted to a fourth order parabolic obstacle problem. We shall announce a result ([13]) which is a joint work with M. Novaga of Pisa University.

The obstacle problem for elliptic and parabolic PDE's is a topics which attracted a great interest in the past years, and has been widely discussed in the mathematical literature. However, even if many studies are available for second order elliptic and parabolic equations (see for instance [6, 9] and references therein), there are relatively few results for higher order obstacle problems, even in the linear fourth order case. More precisely, the elliptic obstacle problem for the biharmonic operator has been considered in [5, 7, 8, 10, 11, 14]. To the best of our knowledge, there is no analog for the corresponding parabolic obstacle problem. The purpose of this paper is to investigate the regularity of a solution to the obstacle problem for the parabolic biharmonic equation.

In the sequel we let $\Omega \subset \mathbb{R}^N$ be a bounded domain, with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$, and we let $f : \Omega \to \mathbb{R}$ be the obstacle function, satisfying

$$f \in C^2(\bar{\Omega}), \qquad f < 0 \quad \text{on} \quad \partial\Omega. \tag{1.1}$$

We consider an initial datum $u_0 : \Omega \to \mathbb{R}$ such that

$$u_0 \in H_0^2(\Omega), \qquad u_0 \geq f \quad \text{a.e. in} \quad \Omega. \tag{1.2}$$

We recall that $u \in H_0^2(\Omega)$ implies $u = 0$ and $\nabla u \cdot \nu^{\Omega} = 0$ on $\partial\Omega$ (see [7, 8]), that is, u satisfies Dirichlet boundary condition on $\partial\Omega$.

In this paper, we consider the following fourth order parabolic obstacle problem:

$$
\text{(P)} \qquad \begin{cases} \partial_t u(x,t) + \Delta^2 u(x,t) \geq 0 & \text{in} \quad \Omega \times \mathbb{R}_+, \\ \partial_t u(x,t) + \Delta^2 u(x,t) = 0 & \text{in} \quad \{(x,t) \in \Omega \times \mathbb{R}_+ : \ u(x,t) > f(x)\}, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_+, \\ \nabla u(x,t) \cdot \nu^{\Omega}(x) = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}_+, \\ u(x,t) \geq f(x) & \text{in} \quad \Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}
$$

The main result of this paper is the following:

Theorem 1.1. *Let $N \leq 3$, and let f be a function satisfying* (1.1)*. Then, for any initial data u_0 satisfying* (1.2)*, the problem* (P) *has a unique solution*

$$
u \in L^\infty(\mathbb{R}_+; H_0^2(\Omega)) \cap H^1_{loc}(\mathbb{R}_+; L^2(\Omega)), \quad \textit{with} \ \ u_t \in L^2(\mathbb{R}_+ \times \Omega). \tag{1.3}
$$

Furthermore the solution u satisfies the following properties:

(i) *$u \in L^2(0,T; W^{2,\infty}(\Omega))$ for any $T > 0$. In particular, if $N = 1$,*

$$
u \in C^{0,\beta}([0,T]; C^{1,\gamma}(\Omega)) \ \ \textit{with} \ \ 0 < \gamma < \frac{1}{2} \ \ \textit{and} \ \ 0 < \beta < \frac{1-2\gamma}{8}, \tag{1.4}
$$

if $N = 2$, 3,

$$
u \in C^{0,\beta}([0,T]; C^{0,\gamma}(\Omega)) \ \ \textit{with} \ \ 0 < \gamma < \frac{4-N}{2} \ \ \textit{and} \ \ 0 < \beta < \frac{4-N-2\gamma}{8}. \tag{1.5}
$$

(ii) *For a.e. $t \in \mathbb{R}_+$ the quantity*

$$
\mu_t := u_t(\cdot,t) + \Delta^2 u(\cdot,t) \tag{1.6}
$$

defines a Radon measure in Ω, and for any $T > 0$ there exists a constant $C > 0$ such that

$$
\int_0^T \mu_t(\Omega)^2 dt < C. \tag{1.7}
$$

Let us point out that the problem (P) corresponds to the gradient flow of a convex functional defined on the Hilbert space $L^2(\Omega)$, hence we can apply the general theory of maximal monotone operators developed in [4]. Indeed, given f as above, we can define the functional $E_f(u) : L^2(\Omega) \to [0, +\infty]$ as

$$
E_f(u) = \begin{cases} \displaystyle\int_\Omega |\Delta u|^2 & \text{if} \ \ u \in H^2(\Omega) \ \ \text{and} \ \ u \geq f, \\ +\infty & \text{otherwise.} \end{cases}
$$

Notice that $E_f(u)$ is convex and lower semicontinuous on $L^2(\Omega)$, and (P) corresponds to the gradient flow

$$u_t + \partial E_f(u) \ni 0, \qquad u(0) = u_0, \tag{1.8}$$

where ∂E_f denotes the subdifferential of E_f in $L^2(\Omega)$. In particular, given an initial datum $u_0 \in H_0^2(\Omega)$ with $u_0 \geq f$, by the results in [4] it follows that the evolution problem (1.8) has a unique solution u satisfying

$$u \in L^\infty(\mathbb{R}_+; H_0^2(\Omega)) \cap H^1_{loc}(\mathbb{R}_+; L^2(\Omega)) \quad \textit{with } u_t \in L^2(\mathbb{R}_+ \times \Omega). \tag{1.9}$$

The purpose of this paper is to give an extra regularity of solution to (P). To this aim, we characterize the solution u by means of an implicit variational scheme, corresponding to the minimizing movements introduced by De Giorgi (see e.g. [2]). This approach will allow us to extend some of the arguments in [7], concerning the regularity of the elliptic obstacle problem for the biharmonic operator.

We point out that the method does not rely on the linear structure of the problem and can be applied to more general fourth order parabolic equations. Indeed our motivation for this work rise from an analysis of a motion of planar closed curves which is governed by the straightening flow with obstacle. Curve straightening flow is a L^2 gradient flow for the total squared curvature

$$\mathcal{E}(\gamma) := \frac{1}{2}\int_\gamma \kappa^2\, ds,$$

where γ is a closed planar curve, κ and s denote respectively the curvature and the arc length parameter of γ. Although the flow is a fourth order quasilinear parabolic equation, we expect that the method of this paper will be available for the geometric obstacle problem.

The paper is organized as follows: in Section 2 we introduce the implicit scheme corresponding to problem (P), by means of an appropriate variational problem; in Section 3 we study the regularity of solutions to the variational problem; in Section 4 we pass to the limit in the approximating scheme and prove Theorem 1.1.

2 Preliminary

The equation in (P) is the L^2 gradient flow for the functional

$$E(u) = \frac{1}{2}\int_\Omega |\Delta u(x)|^2\, dx.$$

Let $T > 0$, $n \in \mathbb{N}$, and set

$$\tau_n = \frac{T}{n}.$$

Let us set $u_{0,n} = u_0$. For $i = 1, \cdots, n$, we define inductively $u_{i,n}$ as a solution of the minimum problem

$$(M_{i,n}) \qquad \min\{G_{i,n}(u) : \ u \in K\},$$

where

$$G_{i,n}(u) := E(u) + P_{i,n}(u) \tag{2.1}$$

with

$$P_{i,n}(u) := \frac{1}{2\tau_n}\int_\Omega (u - u_{i-1,n})^2\, dx, \tag{2.2}$$

and K is a convex set given by

$$K := \{u \in H_0^2(\Omega) : \ u(x) \geq f(x) \text{ a.e. in } \Omega\}.$$

In the following, we let

$$V_{i,n}(x) := \frac{u_{i,n}(x) - u_{i-1,n}(x)}{\tau_n}. \tag{2.3}$$

We give a definition of a piecewise linear interpolations of $\{u_{i,n}\}$:

Definition 2.1. (Piecewise linear interpolation) *Let f be a function satisfying* (1.1). *Let $u_0 \in H_0^2(\Omega)$ with $u_0(x) \geq f(x)$ a.e. in Ω. Define $u_n : \Omega \times [0,T] \to \mathbb{R}$ as*

$$u_n(x,t) := u_{i-1,n}(x) + (t - (i-1)\tau_n)V_{i,n}(x) \tag{2.4}$$

if $(x,t) \in \Omega \times [(i-1)\tau_n, i\tau_n]$ for $i = 1, \cdots, n$.

By a technical reason, additionally we need a piecewise constant interpolations of $\{u_{i,n}\}$ and $\{V_{i,n}\}$.

Definition 2.2. (Piecewise constant interpolation) *Define $\tilde{u}_n : \Omega \times [0,T] \to \mathbb{R}$ as*

$$\tilde{u}_n(x,t) := u_{i,n}(x), \tag{2.5}$$

$$V_n(x,t) := V_{i,n}(x), \tag{2.6}$$

if $(x,t) \in \Omega \times [(i-1)\tau_n, i\tau_n)$ for $i = 1, \cdots, n$.

3 Existence and regularity of minimizers of $(M_{i,n})$

We first mention a well-known compactness result in $H_0^2(\Omega)$ ([1]).

Proposition 3.1. *The following embedding is compact*:

$$
H_0^2(\Omega) \hookrightarrow \begin{cases} C^{1,\gamma}(\overline{\Omega}) & \textit{for} \quad 0<\gamma<\dfrac{1}{2} & \textit{if} \quad N=1, \\ C^{0,\gamma}(\Omega) & \textit{for} \quad 0<\gamma<2-\dfrac{N}{2} & \textit{if} \quad N=2,3, \\ L^q(\Omega) & \textit{for} \quad 1 \le \forall q < +\infty & \textit{if} \quad N=4, \\ L^q(\Omega) & \textit{for} \quad 1 \le \forall q < \dfrac{2N}{N-4} & \textit{if} \quad N \ge 5. \end{cases} \tag{3.1}
$$

We start with the existence of minimizers of $(M_{i,n})$.

Theorem 3.1. (Existence of minimizers) *Let f be a function satisfying* (1.1). *Let $u_0 \in H_0^2(\Omega)$ with $u_0(x) \ge f(x)$ a.e. in Ω. Then the problem $(M_{i,n})$ possesses a unique solution $u_{i,n} \in H_0^2(\Omega)$ with $u_{i,n}(x) \ge f(x)$ a.e. in Ω for each $i = 1, \cdots, n$.*

Proof. Fix $n \in \mathbb{N}$, $T > 0$, and $i = 1, \cdots, n$. From (2.1)-(2.2) and the minimality of a solution u to $(M_{i,n})$, we obtain that

$$E(u) \le G_{i,n}(u) \le G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}),$$

and then

$$0 \le \inf_{H_0^2(\Omega)} G_{i,n}(u) \le G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}) \le \cdots \le E(u_0).$$

Thus we can take a minimizing sequence $\{u_j\} \subset H_0^2(\Omega)$ for $(M_{i,n})$ such that $u_j(x) \ge f(x)$ a.e. in Ω for each $j \in \mathbb{N}$ and $\sup_j G_{i,n}(u_j) < \infty$.

Observing that the norm $\|\Delta u\|_{L^2(\Omega)}$ is equivalent to $\|u\|_{H_0^2(\Omega)}$ (e.g., see [12]), it follows from

$$\|\Delta u_j\|_{L^2(\Omega)} = \sqrt{2E(u_j)} \le \sqrt{2E(u_0)} = \|\Delta u_0\|_{L^2(\Omega)}$$

that $\{u_j\}$ is uniformly bounded in $H_0^2(\Omega)$. Thus there exists $u \in H_0^2(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{in} \quad H_0^2(\Omega), \tag{3.2}$$

in particular,

$$\Delta u_j \rightharpoonup \Delta u \quad \text{in} \quad L^2(\Omega), \tag{3.3}$$

up to a subsequence. Thenks to Proposition 3.1, we obtain that

$$
u_j \to u \quad \text{in} \quad \begin{cases} C^{1,\gamma}(\bar{\Omega}) & \text{for} \quad 0<\gamma<\dfrac{1}{2} & \text{if} \quad N=1, \\ C^{0,\gamma}(\Omega) & \text{for} \quad 0<\gamma<2-\dfrac{N}{2} & \text{if} \quad N=2,3, \\ L^q(\Omega) & \text{for} \quad 1 \le \forall q < +\infty & \text{if} \quad N=4, \\ L^q(\Omega) & \text{for} \quad 1 \le \forall q < \dfrac{2N}{N-4} & \text{if} \quad N \ge 5. \end{cases}
$$

In particular, for the case of $N \geq 4$,

$$u_j \to u \quad \text{a.e. in} \quad \Omega \quad \text{up to a subsequence.} \tag{3.4}$$

Recalling $u_j \geq f$ a.e. in Ω for each $j \in \mathbb{N}$, (3.4) yields that $u \geq f$ a.e. in Ω. Making use of Fatou's Lemma, we conclude that

$$P_{i,n}(u) \leq \liminf_{j\to\infty} P_{i,n}(u_j). \tag{3.5}$$

Furthermore (3.3) implies

$$E(u) = \frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)} \leq \frac{1}{2}\liminf_{j\to\infty}\|\Delta u_j\|^2_{L^2(\Omega)} = \liminf_{j\to\infty} E(u_j). \tag{3.6}$$

Combining (3.5) with (3.6), we see that $u \in H^2_0(\Omega)$ is the minimizer of $(M_{i,n})$ with $u \geq f$ a.e. in Ω. The uniqueness follows from the fact that the functional $G_{i,n}(\cdot)$ is strictly convex. □

Regarding the regularity of the minimizer $u_{i,n}$ obtained in Theorem 3.1, we show the following:

Theorem 3.2. *Let $u_{i,n}$ be the solution of $(M_{i,n})$ obtained by Theorem* 3.1. *Then, for any $n \in \mathbb{N}$, it holds that*

$$\int_0^T\int_\Omega |V_n(x,t)|^2\,dxdt \leq 2E(u_0), \tag{3.7}$$

$$\sup_i \|\Delta u_{i,n}\|_{L^2(\Omega)} \leq \sqrt{2E(u_0)}. \tag{3.8}$$

Proof. Fix $T > 0$ and $n \in \mathbb{N}$. For each $i = 1, \cdots, n$, it follows from (2.1)-(2.2) and the minimality of $u_{i,n}$ that

$$G_{i,n}(u_{i,n}) \leq G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}). \tag{3.9}$$

Hence we get

$$P_{i,n}(u_{i,n}) \leq E(u_{i-1,n}) - E(u_{i,n}),$$

i.e.,

$$\frac{1}{2\tau_n}\int_\Omega (u_{i,n} - u_{i-1,n})^2\,dx \leq E(u_{i-1,n}) - E(u_{i,n}). \tag{3.10}$$

Combining (3.10) with definitions (2.3) and (2.6), we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega |V_n(x,t)|^2\,dxdt &= \frac{1}{2}\sum_{i=1}^n\int_{(i-1)\tau_n}^{i\tau_n}\int_\Omega |V_{i,n}(x)|^2\,dxdt \\ &\leq \sum_{i=1}^n (E(u_{i-1,n}) - E(u_{i,n})) = E(u_0) - E(u_{n,n}) \leq E(u_0),\end{aligned}$$

i.e., (3.7).

By (3.9), we obtain that $E(u_{i,n}) \le E(u_{i-1,n})$ for each $i = 1, \cdots, n$, and then

$$\frac{1}{2}\int_\Omega (\Delta u_{i,n})^2\,dx = E(u_{i,n}) \le E(u_0). \tag{3.11}$$

It is clear that (3.11) is equivalent to (3.8). □

By the definition of $u_{i,n}$, we see that

$$\int_\Omega |\Delta(u_{i,n} + \varepsilon\zeta)|^2\,dx + \frac{1}{2\tau_n}\int_\Omega (u_{i,n} - u_{i-1,n} + \varepsilon\zeta)^2\,dx$$
$$\ge \int_\Omega |\Delta u_{i,n}|^2\,dx + \frac{1}{2\tau_n}\int_\Omega (u_{i,n} - u_{i-1,n})^2\,dx$$

for any $\varepsilon > 0$ and $\zeta \in H_0^2(\Omega)$ with $\zeta \ge 0$. This implies

$$\int_\Omega \Delta u_{i,n}\Delta\zeta\,dx + \frac{1}{\tau_n}\int_\Omega (u_{i,n} - u_{i-1,n})\zeta\,dx \ge 0,$$

so that

$$\mu_{i,n} := \Delta^2 u_{i,n} + V_{i,n} \ge 0 \tag{3.12}$$

in the sense of the distribution. Hence $\mu_{i,n}$ is a measure in Ω (e.g., see [15]).

When we restrict dimensions to $N \le 3$, Proposition 3.1 implies that $u_{i,n}$ is continuous. Under such restriction, we define

$$\mathcal{C}_{i,n} := \{x \in \Omega : \ u_{i,n}(x) = f(x)\}, \tag{3.13}$$
$$\mathcal{N}_{i,n} := \{x \in \Omega : \ u_{i,n}(x) > f(x)\}. \tag{3.14}$$

It is clear that $\mathcal{C}_{i,n} \cup \mathcal{N}_{i,n} = \Omega$. We can show a relation between the support of $\mu_{i,n}$ and the sets.

Lemma 3.1. *Let $N \le 3$. If $x_0 \in \mathcal{N}_{i,n}$, then there exists a neighborhood W of x_0 such that $\mu_{i,n}(W) = 0$. Furthermore we have*

$$\operatorname{supp}\mu_{i,n} \subseteq \mathcal{C}_{i,n}. \tag{3.15}$$

Proof. Let $N \le 3$ and fix $x^0 \in \mathcal{N}_{i,n}$ arbitrarily. Since $\mathcal{N}_{i,n}$ is an open set, there exist a constant $\delta > 0$ and a neighborhood W of x^0 such that

$$u_{i,n}(x) - f(x) > \delta \quad \text{for all} \quad x \in W.$$

Notice that $u_{i,n}$ satisfies

$$\int_\Omega \Delta u_{i,n}\Delta(u_{i,n} - \varphi)\,dx \le -\int_\Omega V_{i,n}(u_{i,n} - \varphi)\,dx \tag{3.16}$$

for any $\varphi \in K$, for $u_{i,n}$ is a solution of $(M_{i,n})$. Then for any $\zeta \in C_0^\infty(W)$ with $0 \le \zeta \le \delta/2$, the function

$$\psi = u_{i,n} - \zeta$$

belongs to K. Taking this ψ as φ in (3.16), we have

$$\int_\Omega [\Delta u_{i,n} \Delta \zeta + V_{i,n} \zeta] \, dx \le 0.$$

Since $\mu_{i,n} \ge 0$, this asserts that

$$\int_\Omega [\Delta u_{i,n} \Delta \zeta + V_{i,n} \zeta] \, dx = 0,$$

i.e., $\mu_{i,n} = 0$ in W. □

Regarding the finiteness of $\mu_{i,n}$, we have the following:

Theorem 3.3. ([13]) *Let $u_{i,n}$ be the solution of $(M_{i,n})$ obtained by Theorem* 3.1. *Then $\mu_{i,n}$ defined in* (3.12) *is a measure in Ω for each $i = 1, \cdots, n$. Moreover there exists a positive constant C being independent of n such that*

$$\tau_n \sum_{i=1}^{n} \mu_{i,n}(\Omega)^2 < C. \tag{3.17}$$

Regarding the regularity of $u_{i,n}$, we obtain the following result:

Theorem 3.4. ([13]) *Let $N \le 3$. It holds that*

$$u_{i,n} \in W^{2,\infty}(\Omega) \tag{3.18}$$

for each $n \in \mathbb{N}$ and $i = 1, \cdots, n$. Moreover, for any $R > 0$ with $\overline{B}_R \subset \Omega$, there exist positive constants C_1 and C_2 being independent of n such that

$$\tau_n \sum_{i=1}^{n} \left\| D^2 u_{i,n} \right\|_{L^\infty(\Omega)}^2 \le C_1 + C_2 \left\| \Delta f \right\|_{L^\infty(\Omega)}^2 . \tag{3.19}$$

4 Existence and regularity of solutions to (P)

In this section, we start with a convergence result of the piecewise linear interpolation u_n. We state several results without its proof. For the precise proof, see [13]. We first show a convergence result which holds in any dimension $N \ge 1$.

Theorem 4.1. *Let u_n be the piecewise linear interpolation of $\{u_{i,n}\}$. Then there exists a function*

$$u \in L^\infty([0, +\infty); H_0^2(\Omega)) \cap H^1_{loc}(0, +\infty; L^2(\Omega))$$

such that

$$(4.1) \qquad u_n \rightharpoonup u \quad in \quad L^2(0,T;H_0^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) \quad as \quad n \to +\infty\,,$$

up to a subsequence, for any $0 < T < +\infty$. *Moreover*

$$\int_0^T \int_\Omega u_t^2 \, dx \, dt \le 2E(u_0)\,,$$

$u(x,t) \ge f(x)$ *for a.e.* $x \in \Omega$ *and for every* $t \in [0,+\infty)$, *and for each* $\alpha \in (0,\frac{1}{2})$ *it holds*

$$(4.2) \qquad u_n \to u \quad in \quad C^{0,\alpha}([0,T];L^2(\Omega)) \quad as \quad n \to +\infty\,.$$

Proof. Recalling that $u_n(x,\cdot)$ is absolutely continuous on $[0,T]$, for all t_1, $t_2 \in [0,T]$ with $t_1 < t_2$, Hölder's inequality and Fubini's Theorem give us

$$\|u_n(\cdot,t_2) - u_n(\cdot,t_1)\|_{L^2(\Omega)} = \left(\int_0^L \left(\int_{t_1}^{t_2} \frac{\partial u_n}{\partial t}(x,t)\,dt\right)^2 dx\right)^{\frac{1}{2}}$$
$$\le \left(\int_{t_1}^{t_2} \left\|\frac{\partial u_n}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)}^2 dt\right)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}}.$$

Then it follows from (3.7) that

$$(4.3) \qquad \int_{t_1}^{t_2} \int_\Omega u_t^2 \, dx \, dt \le 2E(u_0)$$

and

$$(4.4) \qquad \|u_n(\cdot,t_2) - u_n(\cdot,t_1)\|_{L^2(\Omega)} \le \sqrt{2E(u_0)}(t_2 - t_1)^{\frac{1}{2}}.$$

Since (3.8) yields that

$$(4.5) \qquad \sup_{t\in[0,T]} \|\Delta u_n(\cdot,t)\|_{L^2(\Omega)} \le \sup_{1\le i\le n} \|\Delta u_{i,n}\|_{L^2(\Omega)} \le \sqrt{2E(u_0)},$$

there exists a function $u \in L^2(0,T;H_0^2(\Omega))$ such that $u_n \rightharpoonup u$ in $L^2(0,T;H_0^2(\Omega))$ up to a subsequence. On the other hand, the estimate (3.7) implies that

$$(4.6) \qquad V_n = \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

This means that $\partial u/\partial t \in L^2(0,T;L^2(\Omega))$, i.e., $u \in H^1(0,T;L^2(\Omega))$. Combining (4.4) with Ascoli-Arzelà's Theorem (see e.g. [3, Proposition 3.3.1]), we conclude (4.2).

Since (4.5) means that $\{u_n(t)\}$ is uniformly bounded in $H_0^2(\Omega)$ with respect to $t \in [0,T]$ and $n \in \mathbb{N}$, we deduce from (4.2) that, for each $t \in [0,T]$

$$(4.7) \qquad u_n(t) \rightharpoonup u(t) \quad \text{in} \quad H_0^2(\Omega)$$

up to a subsequence. This asserts that $u \in L^\infty([0,T]; H_0^2(\Omega))$. Moreover, Proposition 3.1 implies that for each $t \in [0,T]$

$$(4.8) \qquad u_n(t) \to u(t) \quad \text{in} \begin{cases} C^{1,\gamma}(\Omega) & \text{for} \quad 0<\gamma<\frac{1}{2} & \text{if} \quad N=1, \\ C^{0,\gamma}(\Omega) & \text{for} \quad 0<\gamma<2-\frac{N}{2} & \text{if} \quad N=2,3, \\ L^q(\Omega) & \text{for} \quad 0<q<+\infty & \text{if} \quad N=4, \\ L^q(\Omega) & \text{for} \quad 0<q<\frac{2N}{N-4} & \text{if} \quad N\geq 5. \end{cases}$$

In particular, if $N \geq 4$,

$$(4.9) \qquad u_n(t) \to u(t) \quad \text{a.e. in} \quad \Omega$$

up to a subsequence. Since $u_n(t) \geq f$ a.e. in Ω for each $n \in \mathbb{N}$ and $t \in [0,T]$, the fact (4.8)-(4.9) yields that $u(t) \geq f$ a.e. in Ω for each $t \in [0,T]$. This completes the proof. □

When $N = 1$, we can improve the convergence result obtained in Theorem 4.1:

Theorem 4.2. ([13]) *Let $N = 1$. Let u be the function obtained by Theorem 4.1. Then it holds that $u \in L^2(0,T; W^{2,\infty}(\Omega)) \cap C^{0,\beta}([0,T]; C^{1,\alpha}(\Omega))$ and*

$$(4.10) \qquad u_n \to u \quad \textit{weakly* in} \quad L^2(0,T; W^{2,\infty}(\Omega)) \quad \textit{as} \quad n \to \infty,$$

$$(4.11) \qquad u_n \to u \quad \textit{in} \quad C^{0,\beta}([0,T]; C^{1,\alpha}(\Omega)) \quad \textit{as} \quad n \to \infty$$

for every $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{8})$. Furthermore $u(\cdot, t) \to u_0$ in $C^{1,\alpha}(\Omega)$ as $t \downarrow 0$.

When $N = 2$, 3, we can also improve the result obtained in Theorem 4.1:

Theorem 4.3. ([13]) *Let $N = 2$, 3. Let u be the function obtained by Theorem 4.1. Then it holds that $u \in L^2(0,T; W^{2,\infty}(\Omega)) \cap C^{0,\beta}([0,T]; C^{0,\gamma}(\Omega))$ and*

$$(4.12) \qquad u_n \to u \quad \textit{weakly* in} \quad L^2(0,T; W^{2,\infty}(\Omega)) \quad \textit{as} \quad n \to +\infty,$$

$$(4.13) \qquad u_n \to u \quad \textit{in} \quad C^{0,\beta}([0,T]; C^{0,\gamma}(\Omega)) \quad \textit{as} \quad n \to +\infty$$

for every

$$0 < \beta < \left(\frac{1}{2} - \frac{N}{8}\right)\left(1 - \frac{\gamma}{2 - N/2}\right), \quad 0 < \gamma < 2 - \frac{N}{2}.$$

Furthermore $u(\cdot, t) \to u_0$ in $C^{0,\gamma}(\Omega)$ as $t \downarrow 0$.

Regarding the piecewise constant interpolation for $\{u_{i,n}\}$, i.e., $\tilde{u}_n$ defined in Definition 2.2, we can verify the following:

Lemma 4.1. *Let $\tilde{u}_n$ be the piecewise constant interpolation of $\{u_{i,n}\}$. If $N = 1$, then*

$$(4.14) \qquad \tilde{u}_n \to u \quad \textit{in} \quad L^\infty([0,T]; C^{1,\gamma}(\Omega)) \quad \textit{as} \quad n \to +\infty$$

for every $\gamma \in (0, 1/2)$, *where* u *is the function obtained in Theorem* 4.1. *If* $N = 2,\ 3$, *then*

$$\tilde{u}_n \to u \quad in \quad L^\infty([0,T]; C^{0,\gamma}(\Omega)) \quad as \quad n \to +\infty \tag{4.15}$$

for every $\gamma \in (0, 2 - N/2)$. *Furthermore, for any* $N \geq 1$, *it holds that*

$$\Delta\tilde{u}_n \rightharpoonup \Delta u \quad in \quad L^2(0,T; L^2(\Omega)) \quad as \quad n \to +\infty. \tag{4.16}$$

Let us define μ_n as

$$\mu_n(t) = \mu_{i,n} \quad \text{if} \quad t \in [(i-1)\tau_n, i\tau_n). \tag{4.17}$$

We close the paper with an outline of proof of Theorem 1.1:

4.1 Proof of Theorem 1.1

Let u be the function in Theorem 4.1. Fix $T > 0$ and $\varphi \in C_c^\infty(\Omega \times (0,T))$ with $\varphi \geq 0$ arbitrary. For each $\varepsilon > 0$, let w_ε be a unique minimizer of the functional $G_{i,n}^\varepsilon$ defined by

$$G_{i,n}^\varepsilon(v) := \int_\Omega \left[\frac{1}{2}(\Delta v)^2 + \frac{1}{2\tau_n}(v - u_{i-1,n})^2 + \gamma_\varepsilon(v - f)\right] dx, \tag{4.18}$$

where

$$\gamma_\varepsilon(\lambda) = \begin{cases} \dfrac{\lambda^2}{\varepsilon} & \text{if} \quad \lambda < 0, \\ 0 & \text{if} \quad \lambda > 0, \end{cases} \tag{4.19}$$

$$\beta_\varepsilon(\lambda) = \gamma_\varepsilon'(\lambda). \tag{4.20}$$

Since w_ε satisfies

$$\int_\Omega \left[\Delta w_\varepsilon \Delta\varphi + \frac{1}{\tau_n}(w_\varepsilon - u_{i-1,n})\varphi + \beta_\varepsilon(w_\varepsilon - f)\varphi\right] dx = 0,$$

we observe from the definition of β_ε that

$$\int_\Omega \left[\Delta w_\varepsilon \Delta\varphi + \frac{1}{\tau_n}(w_\varepsilon - u_{i-1,n})\varphi\right] dx = -\int_\Omega \beta_\varepsilon(w_\varepsilon - f)\varphi\, dx \geq 0.$$

Letting $\varepsilon \to 0$, the proof of Theorem 3.3 implies that

$$\int_\Omega \left[\Delta u_{i,n} \Delta\varphi + \frac{1}{\tau_n}(u_{i,n} - u_{i-1,n})\varphi\right] dx \geq 0.$$

Integrating it over $[0,T]$ and using Definitions 2.1 and 2.2, we deduce that

$$\int_0^T \int_\Omega \left[\Delta\tilde{u}_n(x,t)\Delta\varphi(x,t) + V_n(x,t)\varphi(x,t)\right] dxdt \geq 0. \tag{4.21}$$

It follows from (4.16) that

$$\int_0^T\int_\Omega \Delta\tilde{u}_n(x,t)\Delta\varphi(x,t)\,dxdt \to \int_0^T\int_\Omega \Delta u(x,t)\Delta\varphi(x,t)\,dxdt \quad \text{as} \quad n\to+\infty,$$

and while (4.6) gives us

$$\int_0^T\int_\Omega V_n(x,t)\varphi(x,t)\,dxdt \to \int_0^T\int_\Omega u_t(x,t)\varphi(x,t)\,dxdt \quad \text{as} \quad n\to+\infty.$$

Thus, letting $n\to+\infty$ in (4.21), we observe that

$$\int_0^T\int_\Omega [\Delta u(x,t)\Delta\varphi(x,t) + u_t(x,t)\varphi(x,t)]\,dxdt \geq 0. \tag{4.22}$$

Since φ is arbitrary, (4.22) implies that

$$\Delta^2 u(x,t) + u_t(x,t) \geq 0 \quad \text{a.e. in} \quad \Omega\times(0,T), \tag{4.23}$$

where $\Delta^2 u$ is written in the distribution sense. Moreover, the regularity of u follows from Theorems 4.1–4.3.

We now prove (1.7). By (4.17) and Theorem 3.3, we observe that

$$\begin{aligned} \|\mu_n\|_{L^2([0,T];\mathcal{M}(\Omega))} &:= \int_0^T\left(\int_\Omega d\mu_n\right)^2 dt \\ &= \sum_{i=1}^n \int_{(i-1)\tau_n}^{i\tau_n}\left(\int_\Omega d\mu_{i,n}\right)^2 dt = \tau_n\sum_{i=1}^n \mu_{i,n}(\Omega)^2 < C. \end{aligned} \tag{4.24}$$

This implies that

$$\mu_n \rightharpoonup \overline{\mu} \quad \text{weakly in} \quad L^2(0,T;\mathcal{M}(\Omega))$$

up to a subsequence. Setting

$$\mu_t := \Delta^2 u + u_t,$$

we observe from (4.23) that μ is a measure on $\Omega\times(0,T)$, and it holds that $\overline{\mu} = \mu_t$ by uniqueness of the limit. Since μ_n converges to μ_t weakly in $L^2(0,T;\mathcal{M}(\Omega))$, it follows from (4.24) that

$$\|\mu_t\|_{L^2(0,T;\mathcal{M}(\Omega))} \leq \liminf_{n\to\infty}\|\mu_n\|_{L^2(0,T;\mathcal{M}(\Omega))} \leq C.$$

This is equivalent to (1.7), and implies that μ_t is a positive Radon measure on Ω for a.e. $t\in(0,T)$.

Finally we prove that u is a solution of the problem (P). To prove this assertion, it is sufficient to show that, if $u>f$, then $\Delta^2 u + u_t = 0$ holds. Let us set

$$\mathcal{N} := \{(x,t)\in\Omega\times(0,T):\ u(x,t) > f(x)\}.$$

Since u is continuous in $\Omega \times (0,T)$ by Theorems 4.2 and 4.3, $\mathcal{N}$ is an open set, so that, for any $(x^0,t^0) \in \mathcal{N}$, there exist $\delta > 0$ and a neighborhood $W \times (t_1,t_2)$ of (x^0,t^0) such that

$$u(x,t) - f(x) > \delta \quad \text{in} \quad W \times (t_1,t_2). \tag{4.25}$$

Lemma 4.1 implies that there exists a number $N > 0$ such that

$$\tilde{u}_n(x,t) > u(x,t) - \frac{\delta}{2} \quad \text{in} \quad W \times (t_1,t_2) \quad \text{for any} \quad n > N.$$

Combining this with (4.25), we have, for any $n > N$,

$$\tilde{u}_n(x,t) > f(x) + \frac{\delta}{2} \quad \text{in} \quad W \times (t_1,t_2). \tag{4.26}$$

Let $\zeta \in C_0^\infty(W \times (t_1,t_2))$ with $0 \le \zeta \le \delta/2$. Then (4.26) asserts that

$$\psi(x,t) := \tilde{u}_n(x,t) - \zeta(x,t) \in K \quad \text{for each} \quad t \in [0,T].$$

Taking this ψ as φ in (3.16) and integrating it with respect to t on $(0,T)$, we obtain

$$\int_0^T \int_\Omega \Delta u_{i,n}(x)\zeta(x,t)\,dxdt \le -\int_0^T \int_\Omega V_{i,n}(x)\zeta(x,t)\,dxdt. \tag{4.27}$$

From the definition (4.17), the inequality can be reduced to

$$\sum_{i=1}^{n} \int_{(i-1)\tau_n}^{i\tau_n} \int_\Omega \zeta(x,t) d\mu_n dt \le 0. \tag{4.28}$$

Since $\mu_n \ge 0$, we see that the integral in (4.28) must be equal to 0, i.e.,

$$\mu_n(W \times (t_1,t_2)) = 0\,. \tag{4.29}$$

It follows from (4.24) that

$$\|\mu_n\|_{\mathcal{M}(\Omega\times(0,T))} := \int_0^T \int_\Omega d\mu_n dt < C.$$

Thus we deduce that μ_n converges to μ_t weakly in $\mathcal{M}(\Omega \times (0,T))$, i.e.,

$$\int_0^T\int_\Omega \varphi(x,t) d\mu_n dt \to \int_0^T\int_\Omega \varphi(x,t)\, d\mu_t dt$$

for any $\varphi \in C_0^\infty(\Omega \times (0,T))$. This fact also yields that

$$\|\mu_t\|_{\mathcal{M}(\Omega\times(0,T))} \le \liminf_{n\to+\infty} \|\mu_n\|_{\mathcal{M}(\Omega\times(0,T))}. \tag{4.30}$$

Combining (4.29) with (4.30), we conclude that

$$\mu_t(W \times (t_1,t_2)) = 0\,, \tag{4.31}$$

which completes the proof. □

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