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Some problems related to polarization tensors

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1 Introduction

There are a few geometric or physical quantities associated with domains such as eigenvalues, capacities, and moments. The generalized polarization tensors (GPT) are one of those quantities. They appear naturally in the multi-polar asymptotic expansion of the perturbation of fields due to presence of inclusions (domains). The notion of GPTs has been applied to diverse fields of research, theory of composites, inverse problems to recover diametrically small anomalies, bio-medical imaging, dictionary matching (pattern recognition), and electro sensing, to name a few.

The purpose of this article is to introduce the notion of GPTs and briefly review its recent applications, and then discuss two mathematical problems related to GPTs, spectral properties of the Neumann-Poincaré operator and bounds on polarization tensor. This article is a summary of a talk delivered at the workshop "Geometry of Solutions of Partial Differential Equations" at RIMS.

2 Generalized polarization tensors

In this section we introduce the notion of GPTs and review some important properties. Details and proofs of the results described in this section can be found in [8].

Consider a configuration where an inclusion embedded in the free space $\mathbb{R}^d$, $d=2,3$, see Fig. 2.1. The inclusion, which may have a single or multiple connected components, has the conductivity (or dielectric constant) σ_c and the medium $\mathbb{R}^d\setminus\overline{\Omega}$ has σ_m, and $\sigma_c\neq\sigma_m$. Let σ be the conductivity distribution of the configuration so that

$$\sigma=\sigma_m\chi(\mathbb{R}^d\setminus\overline{\Omega})+\sigma_c\chi(\Omega).$$

We then consider the conductivity equation

$$\begin{cases} \nabla\cdot\sigma\nabla u=0 & \text{in } \mathbb{R}^d, \\ u(x)-h(x)=O(|x|^{1-d}) & \text{as } |x|\to\infty \end{cases} \tag{2.1}$$

where h is a given harmonic function in $\mathbb{R}^d$. So, $-\nabla h$ represents the background electrical field and $u-h$ is the perturbation due to presence of the inclusion. In Fig. 2.1 $h(x)$ is given by $a\cdot x$ for some constant vector a, so the field is uniform.

The solution u to (2.1) admits the dipolar asymptotic expansion at infinity when $h(x)=a\cdot x$:

$$u(x)=a\cdot x-\frac{1}{\omega_d}\frac{\langle a,Mx\rangle}{|x|^d}+O(|x|^{-d}), \quad \text{as } |x|\to\infty, \tag{2.2}$$

where w_d is the surface area of the unit sphere in $\mathbb{R}^d$. The $d\times d$ matrix M appeared in the asymptotic expansion is called the *Polarization Tensor* (PT) associated with Ω (or more precisely σ). More generally, u admits the multipolar expansion:

$$u(x)=h(x)+\sum_\alpha\sum_\beta\frac{(-1)^{|\beta|}}{\alpha!\beta!}\partial^\alpha h(0)\partial^\beta\Gamma(x)m_{\alpha\beta}, \quad |x|\to\infty. \tag{2.3}$$

The quantities $m_{\alpha\beta}$ are called *Generalized Polarization Tensors* (GPT). We emphasize that PT and GPTs are not dependent on the given function h, they depend only on the inclusion Ω and the conductivity ratio

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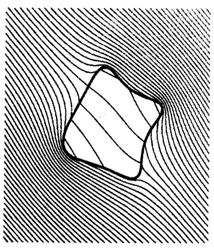

Figure 2.1: An inclusion embedded in the free space with a uniform field

σ_c/σ_m. So, GPTs are the building blocks of asymptotic expansion of the perturbation of the potential (or field) due to presence of the inclusion, and the PT is that of the leading order term. It is worth mentioning that there is ambiguity in the asymptotic expansion (2.3): the point 0 appearing in $h(0)$. We may use other points for expansion of h. This ambiguity can be removed by using spherical harmonics for the expansion[1].

Explicit formula for PT and GPTs can be found for shapes like disks, balls, ellipses, and ellipsoids. For example, if Ω is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, then

$$M(k, \Omega) = (\sigma_c - \sigma_m)|\Omega| \begin{bmatrix} \dfrac{a+b}{\sigma_m a + \sigma_c b} & 0 \\ 0 & \dfrac{a+b}{\sigma_m b + \sigma_c a} \end{bmatrix}. \tag{2.4}$$

where $k = \sigma_c/\sigma_m$. It is worth mentioning that GPTs for two disks were found by Lim [25].

GPTs carry important geometric information on the shape of the inclusion. For example, the full set of GPTs determines the shape of the inclusion uniquely as proved in [7]. More precisely, if

$$\sum a_\alpha b_\beta m_{\alpha\beta}(k_1, \Omega_1) = \sum a_\alpha b_\beta m_{\alpha\beta}(k_2, \Omega_2)$$

for all harmonic coefficients[2] a_α and b_β, then

$$k_1 = k_2 \quad \text{and} \quad \Omega_1 = \Omega_2.$$

Here k_j is the conductivity ratio. We emphasize that this result holds for inclusions with multiple components. It is worthwhile to compare this result, which may be called the inverse GPT problem, with the inverse spectral problem. It was proved in [18] that the full set of eigenvalues does not determine the shape uniquely. It means that GPTs carry richer information than eigenvalues.

Since the full set of GPTs determines the shape, a natural question which follows would be what about a finite number of GPTs. In this regard, it was found in [12] that there is canonical one-to-one correspondence between the class of PTs and that of ellipses (ellipsoids). So one can recover the equivalent ellipse (ellipsoid) of the inclusion using its PT. The equivalent ellipse of a given domain is an ellipse whose PT is the same as that of the domain. It represents a kind of the averaged shape of the inclusion. The left figure in Fig. 2.2 shows an equivalent ellipse (black curve) of the kite shaped domain (grey curve). It is difficult to see in an analytic way what kind of shape information the higher order GPTs contain. So, a numerical method based on optimization was used to reconstruct the shape using finite number of GPTs [11]. The middle figure in Fig. 2.2 shows the result of reconstruction using GPTs $m_{\alpha\beta}$, $|\alpha| + |\beta| \le 6$. In [6] a level set method is incorporated into the optimization algorithm to recover even the topology, the result of which is shown in the right figure of Fig. 2.2. In relation to shape information of GPTs, it is shown in [21] that the coefficients of the Riemann mapping from outside the unit disk to outside of an simply connected domain can be computed explicitly using GPTs.

[1] The GPTs defined using spherical harmonics are called contracted GPTs. See [20].

[2] a_α is called harmonic coefficients if $\sum_\alpha x^\alpha$ is a harmonic polynomial.

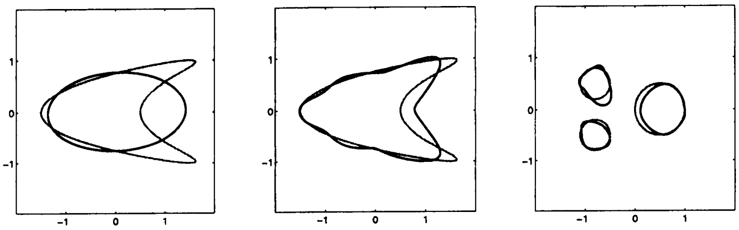

Figure 2.2: Left: the equivalent ellipse (black) of an inclusion (grey). Middle and Right: reconstruction of a single and multiple inclusions using $m_{\alpha\beta}$, $|\alpha| + |\beta| \leq 6$.

The PT M, which is $d \times d$ matrix, enjoys several important properties: it is symmetric and positive definite if $\sigma_c > \sigma_m$ (negative definite if $\sigma_c < \sigma_m$). Generalizing these properties to GPTs, the following properties have been proved in [7].

- Symmetry: If $\{a_\alpha\}$ and $\{b_\beta\}$ are such that $\sum a_\alpha x^\alpha$ and $\sum b_\beta x^\beta$ are harmonic polynomials, then

$$\sum a_\alpha b_\beta m_{\alpha\beta} = \sum a_\alpha b_\beta m_{\beta\alpha}. \tag{2.5}$$

- Positivity: If $\sigma_c > (<)\sigma_m$, then

$$\sum a_\alpha a_\beta m_{\alpha\beta} > (<)0. \tag{2.6}$$

- Bounds: If $f(x) = \sum_{\alpha \in I} a_\alpha x^\alpha$ is a harmonic polynomial, then

$$\int_\Omega |\nabla f|^2 \, dx \leq \frac{\sigma_c + \sigma_m}{|\sigma_c - \sigma_m|} \left| \sum_{\alpha,\beta \in I} a_\alpha a_\beta m_{\alpha\beta} \right| \leq C \int_\Omega |\nabla f|^2 \, dx \,. \tag{2.7}$$

The bounds (2.7) hold on the PT. But there are tighter bounds on the PT:

$$\frac{\sigma_m}{\sigma_c - \sigma_m} \mathrm{Tr}(M) \leq |\Omega|(d - 1 + \frac{\sigma_m}{\sigma_c}), \tag{2.8}$$

and

$$\frac{\sigma_c - \sigma_m}{\sigma_m} \mathrm{Tr}(M^{-1}) \leq \frac{d - 1 + \frac{\sigma_c}{\sigma_m}}{|\Omega|}, \tag{2.9}$$

where Tr denotes the trace and $|\Omega|$ the volume of Ω. The bounds are called the Hashin-Shtrikman bounds after names of the scientists who first found the optimal bounds on the effective conductivity of isotropic two-phase composites [19]. These bounds have been obtained by Lipton [26], and later by Capdeboscq-Vogelius [14]. They are optimal in the sense that every matrix satisfying bounds is realized as the PT of an inclusion [2, 13].

It is interesting to observe that the PT of the ellipse given by (2.4) satisfies the equality in (2.9) (it is true for ellipsoids as well). In [28] Pólya and Szegö raised the conjecture that the inclusion whose polarization tensor has the minimal trace among inclusions of the same volume takes the shape of a disk or a ball. In terms of the Hashin-Shtrikman bounds what they asked amounts to that if the equality in (2.9) holds and all eigenvalues are the same then the domain must be a disk or a ball. At the time they wrote the book the bounds (2.8) and (2.9) were not known. In the sense of isoperimetric inequalities the correct conjecture is that if the equality holds in (2.9), then the domain must be an ellipse or an ellipsoid, and the original conjecture Pólya and Szegö is a part of the extended version. This conjecture was proved by Kang and Milton [23]. More interestingly this conjecture was related to the conjecture of Eshelby. Eshelby found that if the inclusion is elliptical or ellipsoidal, then the solution to (2.1) for $h(x) = a \cdot x$ is linear inside the inclusion (see Fig. 2.3), and then conjectured that ellipses and ellipsoids are the only shape with this property. It was proved in [22, 23] that these two conjectures are equivalent and true.

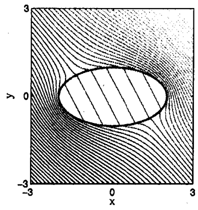

Figure 2.3: Field is uniform inside the elliptic inclusions.

3 Applications of GPTs

The notion of polarization tensor (polarizability tensor) has been widely used in the theory of composites to compute the effective properties of two phase composites, for which we refer to the comprehensive book of Milton [27] (see also [8]). Other classical usage of the PT includes the study of potential flow [28] and low frequency asymptotic of wave [15].

The notion of PT (or GPT) has been attracting much attention lately due to its usage in the inverse problem to detect diametrically small (point-like) inclusions. The related electrical impedance tomography problem is described as follows: Let D be a diametrically small inclusion buried in a domain Ω (at some distance from $\partial\Omega$) and let

$$\sigma = \chi(\Omega \setminus D) + \sigma_c \chi(D)$$

For a given $g \in L^2_0(\partial\Omega)$ (current) let u be the solution to the Neumann boundary value problem

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \sigma \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g \\ \displaystyle\int_{\partial\Omega} u \, d\sigma = 0. \end{cases} \tag{3.1}$$

The problem is to detect the location and/or some geometric information (and unknown σ_c) of the inclusion using a finite number of data $(u|_{\partial\Omega}, g)$ measured on $\partial\Omega$.

The connection of the PT (and GPT) to the inverse problem lies in the asymptotic expansion of u on $\partial\Omega$, namely,

$$u(x) = u_0(x) - M(D)\nabla u_0(z) \cdot \nabla_z N(x,z) + O(\delta^{d+1}), \quad x \in \partial\Omega, \tag{3.2}$$

where u_0 is the solution in absence of D, *i.e.*, ,

$$\begin{cases} \Delta u_0 = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = g \\ \displaystyle\int_{\partial\Omega} u_0 \, d\sigma = 0, \end{cases}$$

z represents the location of D, $N(x,z)$ is the Neumann function (the Green function for the Neumann problem), and δ is the order of small diameter of D. It is worth mentioning that $M(D)$ is of order δ^d. If there are more than one anomalies, say $D_1, \ldots, D_N$ at the locations $z_1, \ldots, z_N$, respectively, then the formula becomes

$$u(x) = u_0(x) - \sum_{j=1}^{N} M(D_j)\nabla u_0(z_j) \cdot \nabla_z N(x,z_j) + O(\delta^{d+1}), \quad x \in \partial\Omega. \tag{3.3}$$

This formula was first found by Friedman and Vogelius [17], and effectively used to detect anomalies [10, 12].

The asymptotic expansion method has been developed in various other contexts such as elasticity, heat conduction and waves, and applied to diverse problems arising from non-destructive evaluation, multi-static and medical imaging. Also various reconstruction algorithms using (3.3) among which are simple pole algorithm, MUSIC algorithm and topological derivative algorithm. Compressed sensing may be applied for reconstruction. We refer interested readers to a survey paper [9] and a recent book [5] and references therein.

More recently the notion of GPT has been applied to the dictionary matching. Since GPTs obey certain rules under rigid motions and scaling, an infinity number of invariants can be constructed using GPTs, and these invariants can be used for dictionary matching. Another important application was found in enhancement of near cloaking. It was shown that by coating a spherical inclusion with multi-layer spherical structure one can make GPTs up to arbitrary order vanish. This GPT vanishing structure enhances the near cloaking effect dramatically. We also refer to [5] and references therein for these development. For the last we mention another recent application. GPTs was used for modelling sensing mechanism of blind fishes living in mud [1], which is an important development in electro-sensing.

4 Neumann-Poincaré operator and GPTs

The Neumann-Poincaré (NP) operator is a classical operator (as the name suggests) arising naturally when solving the Dirchlet or Neumann problem using integral operators such as the single or double layer potentials. In order to introduce it, let $\Gamma(x)$ be the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| \,, & d = 2 \,, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d} \,, & d \geq 3 \,. \end{cases} \tag{4.1}$$

The single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ of a function $\varphi \in L^2(\partial\Omega)$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y) \,, \quad x \in \mathbb{R}^d. \tag{4.2}$$

It enjoys the jump relation

$$\frac{\partial}{\partial\nu} \mathcal{S}_{\partial\Omega}[\varphi]\big|_{\pm}(x) = \Big(\pm\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\Big)[\varphi](x), \quad x \in \partial\Omega \,, \tag{4.3}$$

where the operator $\mathcal{K}^*_{\partial\Omega}$ is defined by

$$\mathcal{K}^*_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \varphi(y)\, d\sigma(y) \,. \tag{4.4}$$

Here the subscripts + and − denote the limits from outside and inside of Ω to $\partial\Omega$, respectively, and ν denotes the outward unit normal to $\partial\Omega$. The operator $\mathcal{K}^*_{\partial\Omega}$ (or $\mathcal{K}_{\partial\Omega}$) is called the Neumann-Poincaré operator associated with the domain Ω.

Let $M = (m_{ij})$ be the polarization tensor of Ω. It is connected to the NP operator by

$$m_{ij} = \int_{\partial\Omega} y^j \left(\lambda I - \mathcal{K}^*_{\partial\Omega}\right)^{-1} [\nu_i](y)\, d\sigma(y), \tag{4.5}$$

where

$$\lambda = \frac{\sigma_c - \sigma_m}{2(\sigma_c - \sigma_m)}.$$

(See [8].) So, we may write $m_{ij} = m_{ij}(\lambda) = m_{ij}(\lambda, \Omega)$. GPTs also can be expressed using the NP operator, but here we restrict the discussion only to the PT.

It is known that

$$-\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle = -\int_{\partial\Omega} \varphi \mathcal{S}_{\partial\Omega}[\psi] d\sigma$$

is an inner product on $L^2_0(\partial\Omega)$ (square integrable functions with the mean zero). If we let $\mathcal{H}$ be the Hilbert space $L^2_0(\partial\Omega)$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle,$$

then $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}$. (See [20] for example.) Since the spectrum of $\mathcal{K}^*_{\partial\Omega}$ lies in $(-1/2,1/2)$ (see [24]), $\mathcal{K}^*_{\partial\Omega}$ admits the spectral resolution: there is a family of projection operators $\mathcal{E}(t)$ on $\mathcal{H}$ (called a resolution of identity) such that

$$\mathcal{K}^*_{\partial\Omega} = \int_{-1/2}^{1/2} t \, d\mathcal{E}(t). \tag{4.6}$$

It then follows from (4.5) that

$$m_{ij} = \langle y_j, (\lambda I - \mathcal{K}^*_{\partial\Omega})^{-1}[\nu_i] \rangle = \int_{-1/2}^{1/2} \frac{1}{\lambda - t} d\langle \mathcal{E}(t)[\nu_i], y_j \rangle.$$

So we have

$$m_{ij}(\lambda, \Omega) = \int_{-1/2}^{1/2} \frac{d\mu_{ij}(t)}{\lambda - t}, \quad \lambda \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]. \tag{4.7}$$

where

$$d\mu_{ij} := d\langle \mathcal{E}(t)[\nu_i], y_j \rangle.$$

If $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{K}^*_{\partial\Omega}$ is compact on $\mathcal{H}$ and the spectral resolution becomes

$$\mathcal{K}^*_{\partial\Omega} = \sum_{j=1}^{\infty} \lambda_j \varphi_j \otimes \varphi_j, \tag{4.8}$$

where $\lambda_1, \lambda_2, \ldots$ ($|\lambda_1| \geq |\lambda_2| \geq \ldots$) are eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}$ counting multiplicities, and $\varphi_1, \varphi_2, \ldots$ are the corresponding (normalized) eigenfunctions. So we have

$$d\mu_{ij} = \sum_{k=1}^{\infty} \langle \varphi_k, \nu_i \rangle_{\mathcal{H}} \langle \varphi_k, y_j \rangle \delta(t - \lambda_k) dt. \tag{4.9}$$

Here δ is the Dirac mass.

5 Discussions

Let us now discuss on some problems related to the polarization tensors.

If $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, then the spectral measure for the PT $d\mu_{ij}$ is a singular measure supported at the eignevalues of the corresponding NP operator. It is quite interesting to see how this measure looks like when $\partial\Omega$ is merely Lipschitz. It is particularly interesting to see whether the measure has non-vanishing absolutely continuous part. If domain is a touching disks, then the measure is absolutely continuous (we will discuss on it in the forthcoming paper). But the domain is not Lipschitz, it has cusps.

In relation to the question above, it is an outstanding problem to develop the spectral theory of the NP operator. Even though the NP operator has been studied for long time, not much is known about its spectrum. Moreover, there is growing interest in the spectrum of the NP operator in relation to the plasmonics. For this we refer to [3, 4] and references therein.

Another challenging problem would be derivation of bounds like (2.8) and (2.9) on PT $M(\lambda)$ when λ is complex. We emphasize that $M(\lambda)$ is an analytic function of λ in $\mathbb{C} \setminus [-1/2, 1/2]$.

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