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Spreading speed and profile for nonlinear diffusion problems with free boundaries

松澤 寛 (Hiroshi Matsuzawa)
Numazu National College of Technology

1 Introduction and Main Results

This article is based on a recent joint research with Professor Yihong Du(University of New England, Australia) and Mr. Maolin Zhou(University of Tokyo, Japan) [7]. In this article we are interested in the following free boundary problem:

$$\begin{cases} u_t - u_{xx} = f(u), & t>0,\ g(t)<x<h(t),\\ u(t,g(t)) = u(t,h(t)) = 0, & t>0,\\ g'(t) = -\mu u_x(t,g(t)) & t>0,\\ h'(t) = -\mu u_x(t,h(t)), & t>0,\\ -g(0)=h(0)=h_0, u(0,x)=u_0(x), & -h_0\le x\le h_0, \end{cases} \tag{1}$$

where $x=h(t)$ and $x=g(t)$ are the moving boundaries to be determined together with $u(t,x)$, μ is a given positive constant, $f:[0,\infty)\to\mathbb{R}$ is C^1, $f(0)=0$ and is of monotstable, or bistable, or of combustion type. The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0>0$, where

$$\mathscr{X}(h_0) := \{\phi\in C^2[-h_0,h_0]:\ \phi(-h_0)=\phi(h_0)=0,\ \phi'(-h_0)>0,\ \phi'(h_0)<0, \phi(x)>0 \text{ in } (-h_0,h_0)\}.$$

For any $h_0>0$ and $u_0\in\mathscr{X}(h_0)$, a triple $(u(t,x),g(t),h(t))$ is a (classical) solution to (1) for $0<t\le T$ if it belongs to $C^{1,2}(G_T)\times C^1[0,T]\times C^1[0,T]$ and all the identities in (1) are satisfied pointwisely, where

$$G_T := \{(t,x)|t\in(0,T],x\in[g(t),h(t)]\}.$$

This problems with $f(u)=au-bu^2$ was introduced by Du and Lin [5] to describe the spreading of a biological or chemical species, with the free boundaries representing the expanding fronts. A deduction of the free boundary condition based on ecological assumption can be found in [4]. The results in [5] were extended to monostable, bistable and combustion types of nonlinearities in Du and Lou [6]. They showed that (1) has a unique solution which is defined for all $t>0$, and as $t\to\infty$, the interval $(g(t),h(t))$ converges either to a finite interval (g_∞,h_∞) , or to $\mathbb{R}$. Moreover, in the former case, $u(t,x)\to 0$ uniformly in x, while in the latter case, $u(t,x)\to 1$ locally uniformly in $x\in\mathbb{R}$ (except for a non-generic transition case when f is of bistable or combustion type). The situation that

$$u\to 0 \text{ and } (g,h)\to(g_\infty,h_\infty)$$

is called the **vanishing** case, and

$$u\to 1 \text{ and } (g,h)\to\mathbb{R}$$

is called the **spreading** case.

When spreading happens, it is shown in [6] that there exists $c^* > 0$ such that

$$\lim_{t\to\infty} \frac{-g(t)}{t} = \lim_{t\to\infty} \frac{h(t)}{t} = c^*. \tag{2}$$

The number c^* is called the asymptotic spreading speed determined by (1).

The main purpose of this article is to obtain a much better estimate for $g(t)$ and $h(t)$ for large t than (2), and describe how the solution approaches the semi-wave when spreading happens.

Before describing our main results, let us define three types of nonlinearities of f mentioned above:

$$(\mathrm{f_M}) \text{ monostable case, } (\mathrm{f_B}) \text{ bistable case, } (\mathrm{f_C}) \text{ combustion case.}$$

We call f is monostable or $(\mathrm{f_M})$ when f is C^1 and it satisfies

$$f(0) = f(1) = 0,\ f'(0) > 0,\ f'(1) < 0,\ (1-u)f(u) > 0 \text{ for } u > 0, u \neq 1.$$

A typical example is $f(u) = u(1-u)$.

We call f is bistable or $(\mathrm{f_B})$, when f is C^1 and it satisfies

$$\begin{cases} f(0) = f(\theta) = f(1) = 0, \\ f(u) < 0 \text{ in } (0,\theta),\ f(u) > 0 \text{ in } (\theta,1),\ f(u) < 0 \text{ in } (1,\infty), \end{cases}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0.$$

A typical example is $f(u) = u(u-\theta)(1-u)$ with $\theta \in \left(0, \frac{1}{2}\right)$.

We call f is combustion type or $(\mathrm{f_C})$, when f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta],\ \ f(u) > 0 \text{ in } (\theta,1),\ \ f'(1) < 0,\ \ f(u) < 0 \text{ in } [1,\infty)$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta+\delta_0).$$

The asymptotic spreading speed c^* mentioned above is determined by the following problem,

$$\begin{cases} q'' - cq' + f(q) = 0 \ \text{ in } (0,\infty), \\ q(0) = 0, q(\infty) = 1, q(z) > 0 \ \text{ in } \ (0,\infty). \end{cases} \tag{3}$$

Proposition 1.1 (Proposition 1.8 and Theorem 6.2 of [6]). *Suppose that f is of $(\mathrm{f_M})$, or $(\mathrm{f_B})$, or $(\mathrm{f_C})$ type. Then for any $\mu > 0$ there exists a unique $c^* = c^*_\mu > 0$ and a unique solution q_{c^*} to (3) with $c = c^*$ such that $q'_{c^*}(0) = \frac{c^*}{\mu}$.*

We remark that this function q_{c^*} is shown in [6] to satisfy $q'_{c^*}(z) > 0$ for $z \geq 0$. We call q_{c^*} a *semi-wave with speed c^**, since the function $w(t,x) := q_{c^*}(c^*t - x)$ satisfies

$$\begin{cases} w_t = w_{xx} + f(w) \ \text{ for } \ t \in \mathbb{R}^1, x < c^*t, \\ w(t, c^*t) = 0,\ -\mu w_x(t, c^*t) = c^*,\ w(t,-\infty) = 1. \end{cases}$$

Our main result is the following theorem.

Theorem A. *Assume that f is of* (f_M), (f_B), *or* (f_C) *type and (u, g, h) is the unique solution to* (1) *for which spreading happens. Let (c^*, q_{c^*}) be given by Proposition 1.1. Then there exist $\hat{H}, \hat{G} \in \mathbb{R}$ such that*

$$\begin{aligned} &\lim_{t\to\infty}(h(t) - c^*t - \hat{H}) = 0, \ \lim_{t\to\infty} h'(t) = c^*, \\ &\lim_{t\to\infty}(g(t) + c^*t - \hat{G}) = 0, \ \lim_{t\to\infty} g'(t) = -c^*, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\lim_{t\to\infty} \sup_{x\in[0,\,h(t)]} |u(t,x) - q_{c^*}(h(t) - x)| = 0, \\ &\lim_{t\to\infty} \sup_{x\in[g(t),\,0]} |u(t,x) - q_{c^*}(x - g(t))| = 0. \end{aligned} \tag{5}$$

Here we remark that estimates (4) are much sharper than (2) obtained by [6].

Now we recall that the corresponding Cauchy problem:

$$\begin{cases} U_t = U_{xx} + f(U), \ t > 0, \ x \in \mathbb{R}, \\ U(0,x) = U_0(x), \ x \in \mathbb{R}. \end{cases} \tag{6}$$

This problem have been extensively studied. For example, the classical paper of Aronson and Weinberger [1] contains a systematic investigation of this problem (and [2] contains its higher-dimensional extension). They have obtained various sufficient conditions for $\lim_{t\to\infty} U(t,x) = 1$ ("spreading" or "propagation") and for $\lim_{t\to\infty} U(t,x) = 0$ ("vanishing" or "extinction") when U_0 is nonnegative and has compact support. [1, 2] show that when $\lim_{t\to\infty} U(t,x) = 1$ as $t \to \infty$ locally uniformly in $\mathbb{R}$, there exists $c_0 > 0$ such that, for any small $\varepsilon > 0$,

$$\lim_{t\to\infty} \max_{|x|\geq(c_0+\varepsilon)t} U(t,x) = 0, \quad \lim_{t\to\infty} \max_{|x|\leq(c_0-\varepsilon)t} |U(t,x) - 1| = 0.$$

In this sense, the number c_0 is usually called the spreading speed determined by (6) and is determined by the well-known problem of travelling wave

$$\begin{cases} Q'' - cQ' + f(Q) = 0, \ Q > 0, \ \text{ in } \mathbb{R}, \\ Q(-\infty) = 0, \ Q(+\infty) = 1, \ Q(0) = 1/2. \end{cases} \tag{7}$$

The relationship between the spreading speed c^* which is determined by (1) and that c_0 determined by (6) is given in Theorem 6.2 of [6].

As we will explain below, fundamental differences arise between the free boundary problem and the Cauchy problem. First, for monostable case, (7) has multiple solutions while for bistable or combustion case, (7) has a unique solution. More precisely, when f is (f_M), (7) has a solution Q_c if and only if $c \geq c_0$ and when f is (f_B) or (f_C), $c_0 > 0$ is the unique value of c such that (7) has a solution Q_c. Moreover, Q_c is unique when it exists for a given c. Second, there is an essential difference in how the solution of (6) approaches the traveling waves. A classical result of Fife and McLeod [8] shows that for f of type (f_B), and for appropriate initial function U_0, the solution U to (6) satisfies

$$\begin{aligned} &|U(t,x) - Q_{c_0}(c_0t - x + C_+)| \leq Ke^{-\omega t} \text{ for } x > 0, \\ &|U(t,x) - Q_{c_0}(c_0t + x + C_-)| \leq Ke^{-\omega t} \text{ for } x < 0 \end{aligned}$$

for some $C_\pm \in \mathbb{R}$. On the other hand, when (f_M) holds and furthermore $f(u) \le f'(0)u$ for $u \in (0,1)$, there exist constants $C_\pm$ such that

$$\lim_{t\to\infty} \max_{x\ge 0} \left| U(t,x) - Q_{c_0}\left(c_0 t - \frac{3}{c_0}\log t - x + C_+\right)\right| = 0,$$
$$\lim_{t\to\infty} \max_{x\le 0} \left| U(t,x) - Q_{c_0}\left(c_0 t - \frac{3}{c_0}\log t + x + C_-\right)\right| = 0$$

The term $\frac{3}{c_0}\ln t$ is known as the logarithmic Bramson correction; see [3, 9, 11, 13] for more details. Our main theorem claims that there is not such a difference among (f_M), (f_B) and (f_C) in our free boundary problem.

2 Basic and Known Results

In this section we give some basic and known results which will be frequently used later. The first two results are for $f(u)$ more general than the three types of nonlinearities in Theorem 1.2. They only require

$$f \text{ is } C^1 \text{ and } f(0) = 0. \tag{8}$$

Lemma 2.1 (Lemma 2.2 of [6]). *Suppose that* (8) *holds,* $T \in (0,\infty)$, $\overline{g}$, $\overline{h} \in C^1[0,T]$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T,\ \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \ge \overline{u}_{xx} + f(\overline{u}), & 0 < t \le T, \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} \ge u, & 0 < t \le T, x = \overline{g}(t), \\ \overline{u} = 0,\ \overline{h}'(t) \ge -\mu \overline{u}_x, & 0 < t \le T, x = \overline{h}(t). \end{cases}$$

If

$$\overline{g}(t) \ge g(t) \text{ in } [0,T], h_0 \le \overline{h}(0),\ u_0(x) \le \overline{u}(0,x) \text{ in } [\overline{g}(0), h_0],$$

where (u,g,h) *is a solution to* (1), *then*

$$h(t) \le \overline{h}(t) \text{ in } (0,T],$$
$$u(t,x) \le \overline{u}(t,x) \text{ for } t \in (0,T] \text{ and } \overline{g}(t) < x < h(t).$$

The function $\overline{u}$, or the triple $(\overline{u}, \overline{g}, \overline{h})$ in Lemma 2.1 is usually called an upper solution of (1). We can define a lower solution by reversing the inequalities in the obvious places. There is a symmetric version of Lemma 2.1, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.

Lemma 2.2 (Lemma 2.6 of [6]). *Suppose that* (8) *holds,* (u,g,h) *is a solution to* (1) *defined for* $t \in [0,T_0)$ *for some* $T_0 \in (0,\infty)$, *and there exists* $C_1 > 0$ *such that*

$$u(t,x) \le C_1 \text{ for } t \in [0,T_0) \text{ and } x \in [g(t), h(t)].$$

Then there exists C_2 *depending on* C_1 *but independent of* T_0 *such that*

$$-g'(t),\ h'(t) \in (0, C_2] \text{ for } t \in (0,T_0).$$

Moreover, the solution can be extended to some interval $(0,T)$ *with* $T > T_0$.

Lemma 2.3 (Lemma 6.5 of [6]). *Suppose that f is of* (f_M), (f_B), *or* (f_C) *type. Let* (u, g, h) *be the unique solution of* (1) *for which spreading happens. For any* $c \in (0, c^*)$ *there exist* $\delta \in (0, -f'(1))$, $T_0 > 0$ *and* $M > 0$ *such that for* $t \geq T_0$,

$$[g(t), h(t)] \supset [-ct, ct], \tag{9}$$

$$u(t, x) \geq 1 - Me^{-\delta t} \text{ for } x \in [-ct, ct], \tag{10}$$

$$u(t, x) \leq 1 + Me^{-\delta t} \text{ for } x \in [g(t), h(t)]. \tag{11}$$

3 Outline of Proof of Theorem A

In this section we will give proof of Theorem A. Please see [7] for detailed proof. Throughout this section we assume that f is of type (f_M), (f_B), or f_C and (u, g, h) is a solution to (1) for which spreading happens. Our proof is divided into three parts:

- Part 1: Boundedness of $|g(t) + c^*t|$ and $|h(t) - c^*t|$.
- Part 2: We will prove that for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{\tilde{t}_n\}$ and $\hat{H} \in \mathbb{R}$ such that

$$h(\tilde{t}_n + \cdot) - c^*(\tilde{t}_n + \cdot) \to \hat{H} \text{ in } C^1_{\text{loc}}(\mathbb{R})$$
$$u(\tilde{t}_n, z + c^*\tilde{t}_n) \to q_{c^*}(\hat{H} \to z)$$

  as $n \to \infty$.

- Part 3: We will prove Theorem A by constructing finer upper and lower solutions based on the result in part 2.

In this article we will focus on Part 2. For other parts, please see [7].

3.1 Part1: Boundedness of $|g(t) + c^*t|$ and $|h(t) - c^*t|$

Proposition 3.1. *There exists* $C > 0$ *such that*

$$|g(t) + c^*t|, |h(t) - c^*t| \leq C \text{ for all } t > 0. \tag{12}$$

This proposition is proved by constructing suitable upper and lower solutions.

Fix $c \in (0, c^*)$. From Lemma 2.3, there exist $\delta \in (0, -f'(1))$, $M > 0$ and $T_0 > 0$ such that for $t \geq T_0$, (9), (10) and (11) hold. Since $0 < \delta < -f'(1)$ we can find some $\eta > 0$ such that

$$\begin{cases} \delta \leq -f'(u) & \text{for } 1 - \eta \leq u \leq 1 + \eta, \\ f(u) \geq 0 & \text{for } 1 - \eta \leq u \leq 1. \end{cases}$$

By enlarging T_0 we may assume that

$$Me^{-\delta T_0} < \eta/2.$$

We take $M' > M$ satisfying

$$M'e^{-\delta T_0} \leq \eta.$$

Since $q_{c^*}(z) \to 0$ as $z \to \infty$, we can find $X_0 > 0$ such that

$$(1 + M'e^{-\delta T_0})q_{c^*}(X_0) \geq 1 + Me{-}\delta T_0.$$

We now construct an upper solution $(\overline{u}, \overline{g}, \overline{h})$ to (1) as follows:

$$\begin{aligned}
&\overline{g}(t) := g(t),\\
&\overline{h}(t) := c^*(t - T_0) + \sigma M'(e^{-\delta T_0} - e^{-\delta t}) + h(T_0) + X_0,\\
&\overline{u}(t,x) := (1 + M'e^{-\delta t}q_{c^*}(\overline{h}(t) - x),
\end{aligned}$$

where $\sigma > 0$ is a positive constant to be determined. We have following lemma.

Lemma 3.2. *For sufficiently large $\sigma > 0$, $u(t,x)$ and $h(t)$ satisfy*

$$\begin{aligned}
&u(t,x) \leq \overline{u}(t,x) \text{ for } t > T_0,\ x \in [g(t), h(t)],\\
&h(t) \leq \overline{h}(t) \text{ for } t \geq T_0.
\end{aligned}$$

To prove this lemma, we check that $(\overline{u}, \overline{g}, \overline{h})$ satisfies

$$\begin{aligned}
&\overline{u}_t - \overline{u}_{xx} \geq f(\overline{u}) \quad \text{for} \quad t > T_0,\ \overline{g}(t) < x < \overline{h}(t),\\
&\overline{u}(t, \overline{g}(t)) \geq g(t, \overline{g}(t)) \quad \text{for} \quad t \geq T_0,\\
&\overline{u}(t, \overline{h}(t)) = 0,\ \overline{h}'(t) \geq -\mu \overline{u}_x(t, \overline{h}(t)) \quad \text{for} \quad t \geq T_0,\\
&h(T_0) \leq \overline{h}(T_0), u(T_0, x) \leq \overline{u}(T_0, x) \quad \text{for} \quad x \in [\overline{g}(T_0), \overline{h}(T_0)]
\end{aligned}$$

for sufficiently large $\sigma > 0$. Please see [7] for detail.

Next we construct a lower solution $(\underline{u}, \underline{g}, \underline{h})$ to bound u and h from below. Let c, M and δ be as before. We now define $\underline{g}(t)$, $\underline{h}(t)$ and $\underline{u}(t,x)$ as follows:

$$\begin{aligned}
&\underline{g}(t) = -ct,\\
&\underline{h}(t) = c^*(t - T_0) + cT_0 - \sigma M(e^{-\delta T_0} - e^{-\delta t}).\\
&\underline{u}(t,x) = (1 - Me^{-\delta t})q_{c^*}(\underline{h}(t) - x).
\end{aligned}$$

Lemma 3.3. *For sufficiently large $\sigma > 0$, $u(t,x)$ and $h(t)$ satisfy*

$$\begin{aligned}
&\underline{u}(t,x) \leq u(t,x) \text{ for } t > T_0,\ x \in [\underline{g}(t), h(t)],\\
&\underline{h}(t) \leq h(t) \text{ for } t \geq T_0.
\end{aligned}$$

To prove this lemma, we check that $(\underline{u}, \underline{g}, \underline{h})$ satisfies

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} \leq f(\underline{u}) \quad \text{for} \quad t > T_0,\ \underline{g}(t) < x < \underline{h}(t),\\
&\underline{u}(t, \underline{g}(t)) \leq g(t, \underline{g}(t)) \quad \text{for} \quad t \geq T_0,\\
&\underline{u}(t, \underline{h}(t)) = 0,\ \underline{h}'(t) \leq -\mu \underline{u}_x(t, \underline{h}(t)) \quad \text{for} \quad t \geq T_0,\\
&\underline{h}(T_0) \leq h(T_0), \underline{u}(T_0, x) \leq u(T_0, x) \quad \text{for} \quad x \in [\overline{g}(T_0), \overline{h}(T_0)]
\end{aligned}$$

for sufficiently large $\sigma > 0$. Please see [7] for detail.

Proof of Proposition 3.1. From Lemmas 3.2 and 3.3 we have

$$\begin{aligned}(c-c^*)T_0 - \sigma M(e^{-\delta T_0} - e^{-\delta t}) \le h(t) - c^*t \\ - c^*T_0 + \sigma M'(e^{-\delta T_0} - e^{-\delta t}) + h(T_0) + X_0,\end{aligned}$$

for $t \ge T_0$. Hence if we define

$$C := \max\{-c^*T_0 + \sigma M' e^{-\delta T_0} + h(T_0) + X_0, (c^*-c)T_0 + \sigma M e^{-\delta T_0}, \max_{t\in[0,T_0]} |h(t) - c^*t|\}$$

then

$$|h(t) - c^*t| \le C \ \text{ for all } \ t > 0.$$

This complete the proof of Proposition 3.1. □

3.2 Part 2: Convergence along a subsequence of $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$.

Set

$$v(t,x) := u(t, x + c^*t), H(t) = h(t) - c^*t.$$

We note that H and H' are bounded function by Proposition 3.1 and Lemma 2.2. By Lemmas 3.2 and 3.3 we have

$$(1 - Me^{-\delta t})q_{c^*}(\underline{h}(t) - x) \le u(t,x) \le (1 + M'e^{-\delta t})q_{c^*}(\overline{h}(t) - x)$$

for $t \ge T_0$ and $x \in [-ct, h(t)]$, where we have assumed that $q_{c^*}(z) = 0$ for $z \le 0$. Since $f'(1) < 0$ we have

$$|1 - q_{c^*}(z)| \le Ce^{-\gamma z} \ \text{ for some } C > 0 \text{ and } \gamma > 0 \tag{13}$$

(see [8] and [7]). Using this and the boundedness of the functions $\underline{h}(t) - c^*t$ and $\overline{h}(t) - c^*t$, we easily see that there exists some $C' > 0$ such that

$$|1 - v(t,z)| \le C'(e^{\gamma z} + e^{-\delta t}) \tag{14}$$

for $t \ge T_0$ and $z \in [-(c+c^*)t, H(t)]$.

Following proposition is key proposition to prove our main theorem.

Proposition 3.4. *For any sequence $\{t_n\} \subset \mathbb{R}$ satisfying $\lim_{n\to\infty} t_n = \infty$, there exists as subsequence $\tilde{t}_n \subset \{t_n\}$ such that*

$$\lim_{n\to\infty} H(\tilde{t}_n + \cdot) = \hat{H} \ \textit{ in } \ C^1_{\text{loc}}(\mathbb{R})$$

for some constant $\hat{H} \in \mathbb{R}$ and

$$\lim_{n\to\infty} \sup_{z\in[-(c+c^*),\hat{H}]} |v(\tilde{t}_n, z) - q_{c^*}(\hat{H} - z)| = 0.$$

Here we have used the convention that $q_{c^*}(z) = 0$ for $z \le 0$ and $v(t,z) = 0$ for $z \ge H(t)$.

First we can easily check that v satisfies

$$\begin{cases} v_t = v_{zz} + c^* v_z + f(v), & t > 0,\ z \in (g(t) - c^*t, H(t)), \\ v(t, H(t)) = 0, & t > 0, \\ H'(t) = -\mu v_z(t, H(t)), & t > 0. \end{cases} \tag{15}$$

By using (14), standard parabolic L^p estimate, Sobolev imbedding and parabolic Schauder estimates([10] and [12]), we obtain that there exists some $C'' > 0$ such that

$$|v_z(t,z)|, |v_{zz}(t,z)| \le C''(e^{-\gamma z} + e^{-\delta t}) \text{ for } t \ge T_0, \text{ and } z \in (g(t) - c^*t, H(t)). \tag{16}$$

We will need the following energy functional

$$E(t) := \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \left\{ \frac{1}{2} v_z^2 - F(v) \right\} dz,$$

where

$$F(v) = \int_0^v f(s) ds.$$

We have following lemma.

Lemma 3.5. *The functional $E(t)$ is bounded from below and satisfies*

$$\begin{aligned} E'(t) = &-\frac{h(t)^2}{2\mu^2}(h'(t) - c^*)e^{(h(t)-c^*t)} + \frac{g'(t)^2}{2\mu^2}(g'(t) - c^*)e^{c^*(g(t)-c^*t)} \\ &- \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z}\{v_{zz} + c^*v_z + f(v)\}^2 dz. \end{aligned}$$

We can easily show that $E(t)$ is bounded from below by using estimate (14). The identity for $E'(t)$ follows from a direct calculation, integration by parts and (15). Please see [7] for details.

Let us define

$$E_0(t) := \frac{1}{2\mu^2} \int_0^t e^{c^*(h(s)-c^*s)} h'(s)^2 (h'(s) - c^*) ds$$

and

$$\tilde{E}(t) := E(t) + E_0(t).$$

Since

$$h'(s)^2\{h'(s) - c^*\} - (c^*)^2\{h'(s) - c^*\} = \{h'(s) + c^*\}\{h'(s) - c^*\}^2 \ge 0,$$

we have

$$\begin{aligned} E_0(t) &\ge \frac{1}{2\mu^2} \int_0^t e^{c^*(h(s)-c^*s)} (c^*)^2 (h'(s) - c^*) ds \\ &= \frac{c^*}{2\mu^2} \int_0^t \frac{d}{ds}\{e^{c^*(h(s)-c^*s)}\} ds \\ &= \frac{c^*}{2\mu^2}(e^{c^*H(t) - c^*h_0}) \ge -C_0 \end{aligned}$$

for some $C_0 > 0$ independent of t. Thus $\tilde{E}(t)$ is bounded from below and $\tilde{E}'(t) \le 0$. Hence we have

$$\lim_{t\to\infty} \tilde{E}(t) = E_\infty > -\infty.$$

We have following lemma.

Lemma 3.6. $\lim_{t\to\infty} \tilde{E}'(t) = 0$.

Proof. If $\tilde{E}'(t)$ is uniformly continuous, then we necessarily have $\lim_{t\to\infty} \tilde{E}'(t) = 0$. In fact, if $\tilde{E}'(t)$ does not converge to 0 as $t \to \infty$, there exists $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and $\varepsilon > 0$ such that $\tilde{E}'(t_n) \le -\varepsilon$. Since $\tilde{E}(t)$ is uniformly continuous, $\tilde{E}'(t) \le -\varepsilon/2$ for $t \in [t_n, t_n + \delta]$ for some $\delta > 0$ independent of n. Then we have, by passing to a subsequence of $\{t_n\}$ if necessary,

$$\begin{aligned} E_\infty - \tilde{E}(t_1) &= \int_{t_1}^{\infty} \tilde{E}'(t)dt \\ &\le \int_{\cup_{n=1}^{\infty}[t_n,t_n+\delta]} \tilde{E}'(t)dt = -\infty. \end{aligned}$$

This contradicts $E_\infty > -\infty$.

Since

$$\lim_{t\to\infty} \frac{g'(t)^2}{2\mu^2}(g'(t) - c^*)e^{c^*(g(t)-c^*t)} = 0$$

we only have to show that the second term of $\tilde{E}'(t)$

$$-\int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z}\{v_{zz} + c^*v_z + f(v)\}^2 dz$$

is uniformly continuous in t for large t. From (16), this will be done by showing that for any $L > 0$, v, v_z and v_{zz} are uniformly continuous in t for $z \in [-L, H(t)]$.

We first consider problem (15) over the domain $[t_0 - 1, t_0 + 1] \times [-L - 1, H(t_0) - \eta/3] \subset \mathbb{R}^2$ for $t_0 \in \mathbb{R}$, $L > 0$ and $\eta > 0$. Since $\|v\|_\infty$ and $\|f(v)\|_\infty$ are bounded, we can apply the parabolic L^p estimate(see, for example [10] or [12]) to obtain

$$\|v\|_{W_p^{1,2}([t_0-1/2,t_0+1]\times[-L-1/2,H(t_0)-\eta/2])} \le C$$

for some $C > 0$ which does note depend on t_0. Here we note that $H(t_0)$ has a bound independent of $t_0 > 0$. By Sobolev imbedding (see [10]) we have

$$\|v\|_{C^{\frac{1+\nu}{2},1+\nu}([t_0-1/2,t_0+1]\times[-L-1/2,H(t_0)-\eta/2])} \le C'.$$

Using this and the Schauder estimate (see [12]) we obtain

$$\|v\|_{C^{1+\frac{\alpha}{2},2+\alpha}([t_0,t_0+1]\times[-L,H(t_0)-\eta])} \le C'' \tag{17}$$

for some $C'' > 0$ which does not depend on t_0 (Please see [7] for detail).

Next we consider the domain $\{(t,z)|t \in [t_0 - 1, t_0 + 1], z \in [H(t) - L, H(t)]\}$ for $L > 0$. We first straighten the boundary $z = H(t)$. Let

$$z = y + H(t),\ w(t,y) = v(t, y + H(t)).$$

Then w satisfies

$$\begin{cases} w_t = w_{yy} + (H'(t) + c^*)w_y + f(w), & t > 0, y \in (g(t) - c^*t - H(t), 0), \\ w(t,0) = 0, & t > 0, \\ H'(t) = -\mu w_y(t,0) - c^*, & t > 0. \end{cases} \tag{18}$$

Since $\|w\|_\infty$, $\|f(w)\|_\infty$, and $\|H'\|_\infty$ are bounded we can apply the parabolic estimate [10, 12] to obtain

$$\|w\|_{W_p^{1,2}([t_0-1/2,t_0+1]\times[-3L/2,0])} \le C$$

for some $C > 0$ which does not depend on t_0. By Sobolev imbedding (see [10]) we have

$$\|w\|_{C^{\frac{1+\nu}{2},1+\nu}([t_0-1/2,t_0+1]\times[-3L/2,0])} \le C'$$

for some $\nu \in (0,1)$ and $C' > 0$ which do not depend on t_0. This implies that H' and $f(w)$ are Hölder continuous and by the parabolic Schauder estimate we obtain

$$\|w\|_{C^{1+\frac{\alpha}{2},2+\alpha}([t_0,t_0+1]\times[-L,0])} \le C'' \tag{19}$$

for some $\alpha \in (0,1)$ and $C'' > 0$ which do not depend on t_0.

From (16) and (19) we see that v, v_z and v_{zz} are uniformly continuous in t for $z \in [-L, H(t)]$. This completes the proof. □

Now we prove following lemma.

Lemma 3.7. *For any sequence $\{t_n\}$ satisfying $\lim_{n\to\infty} t_n = \infty$ and any $K > 0$ there exists a subsequence $\{\tilde{t}_n\} \subset \{t_n\}$ such that*

$$\lim_{n\to\infty} H(\tilde{t}_n + \cdot) = \hat{H} \quad in \quad C^1_{\rm loc}(\mathbb{R})$$

for some $\hat{H} \in \mathbb{R}$ and

$$\lim_{n\to\infty} \sup_{z\in[-K,\hat{H}]} |v(\tilde{t}_n, z) - q_{c^*}(\hat{H} - z)| = 0.$$

Proof. Without loss of generality we assume that $\{t_n\}$ is an increasing sequence of positive numbers satisfying $\lim_{n\to\infty} t_n = \infty$. Define

$$\begin{aligned} &v_n(t,z) := v(t + t_n, z),\ w_n(t + t_n, y) := w(t + t_n, y), \\ &H_n(t) := H(t + t_n),\ \overline{G}_n(t) := g(t + t_n) - c^*(t + t_n) - H_n(t). \end{aligned}$$

By (18) we have

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = \dfrac{\partial^2 w_n}{\partial y^2} + (H_n'(t) + c^*)\dfrac{\partial w_n}{\partial y} + f(w_n), & t > -t_n, y \in (\overline{G}_n(t), 0), \\ w_n(t,0) = 0, & t > -t_n, \\ H_n'(t) = -\mu \dfrac{\partial w_n}{\partial y}(t,0) - c^*, & t > -t_n. \end{cases} \tag{20}$$

Since $\|w_n\|_\infty$, $\|f(w_n)\|_\infty$ and $\|H_n'\|_\infty$ are bounded, we can use the parabolic L^p estimate, Sobolev imbedding and Schauder estimate to deduce that $\{w_n\}$ is bounded in $C^{1+\frac{\alpha}{2},2+\alpha}([-R,R]\times[-R,0])$ for any $R>0$, Hence H_n' is uniformly bounded in $C^\alpha(I)$ for any bounded interval $I\subset\mathbb{R}$. Hence there exists a subsequence of $\{t_n\}$, still denoted by $\{t_n\}$ such that

$$H_n' \to \tilde{H} \text{ in } C^{\alpha'}_{\text{loc}}(\mathbb{R})$$

for some $\alpha'\in(0,\alpha/2)$. By using parabolic Schauder estimates again for the equation in (23), we can see that

$$w_n \to \hat{w} \;\text{ in }\; C^{1+\frac{\alpha'}{2},2+\alpha'}_{\text{loc}}(\mathbb{R}\times(-\infty,0]), \tag{21}$$

along a further subsequence, and $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + (\tilde{H}(t)+c^*)\hat{w}_y + f(\hat{w}), & t\in\mathbb{R}, y<0,\\ \hat{w}(t,0)=0, & t\in\mathbb{R},\\ \tilde{H}(t) = -\mu\hat{w}_y(t,0)-c^*, & t\in\mathbb{R}. \end{cases} \tag{22}$$

Since

$$H_n(t) = H_n(0) + \int_0^t H_n'(s)ds$$

and $H_n'\to\tilde{H}$ in $C^{\alpha'}_{\text{loc}}(\mathbb{R})$, we obtain

$$H_n(t)\to\hat{H}(t) := \tilde{H}(0) + \int_0^t \tilde{H}(s)ds \;\text{ as }\; n\to\infty \;\text{ in }\; C^{1+\alpha'}_{\text{loc}}(\mathbb{R}).$$

Thus $\tilde{H}(s)=\hat{H}'(t)$ and $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + (\hat{H}'(t)+c^*)\hat{w}_y + f(\hat{w}), & t\in\mathbb{R}, y<0,\\ \hat{w}(t,0)=0, & t\in\mathbb{R},\\ \hat{H}'(t) = -\mu\hat{w}_y(t,0)-c^*, & t\in\mathbb{R}. \end{cases}$$

Next we examine v_n. From (15) we can see that v_n satisfies

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \dfrac{\partial^2 v_n}{\partial y^2} + c^*\dfrac{\partial v_n}{\partial y} + f(v_n), & t>-t_n, z<H_n(t),\\ v_n(t,H_n(t))=0, & t>-t_n,\\ H_n'(t) = -\mu\dfrac{\partial v}{\partial y}(t,H_n(t)) - c^*, & t>-t_n. \end{cases} \tag{23}$$

For any $\varepsilon>0$ we consider (23) over

$$\Omega_\varepsilon := \{(t,z)|t\in[-\varepsilon^{-1},\varepsilon^{-1}], z\in[-\varepsilon^{-1},\hat{H}-\varepsilon]\}.$$

Applying the parabolic Scauder estimate, we have by passing to a subsequence

$$v_n\to\hat{v} \;\text{ in }\; C^{1+\frac{\alpha'}{2},2+\alpha'}(\Omega_\varepsilon)$$

and $\hat{v}$ satisfies

$$\hat{v} = \hat{v}_{zz} + c^*\hat{v}_z + f(\hat{v}) \text{ in } \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by using the diagonal argument we have along a further subsequence we may assume $v_n \to \hat{v}$ in $C^{1+\frac{\alpha'}{2},2+\alpha'}_{\text{loc}}(\Omega_0)$ with $\Omega_0 = \{(t,z)|t \in \mathbb{R}, z < \hat{H}(t)\}$.

Nest we show $\hat{v} \equiv 0$ and $\hat{v}(t,z) \equiv \hat{v}$. By Lemma 3.6 we have

$$\tilde{E}'(t+t_n) = \frac{g'(t+t_n)^2}{2\mu^2}(g'(t+t_n) - c^*)e^{c^*[g(t+t_n)-c^*(t+t_n)]} - \int_{g(t+t_n)-c^*(t+t_n)}^{H(t+t_n)} e^{c^*z}\{(v_n)_{zz} + c^*(v_n)_z + f(v_n)\}^2 dz \to 0$$

as $n \to \infty$. Since

$$\frac{g'(t+t_n)^2}{2\mu^2}(g'(t+t_n) - c^*)e^{c^*[g(t+t_n)-c^*(t+t_n)]} \to 0$$

as $n \to \infty$, we have for any $K > 0$ and $\varepsilon > 0$,

$$0 \le \int_{-K}^{\hat{H}(t)-\varepsilon} e^{c^*z}\{\hat{v}_{zz} + c^*\hat{v}_z + f(\hat{v})\}^2 dz$$
$$\lim_{n\to\infty} \int_{g(t+t_n)-c^*(t+t_n)}^{H(t+t_n)} e^{c^*z}\{(v_n)_{zz} + c^*(v_n)_z + f(v_n)\}^2 dz = 0.$$

Since $\varepsilon, K > 0$ are arbitrarily, we obtain

$$\hat{v}_{zz} + c^*\hat{v}_z + f(\hat{v}) = 0 \text{ in } \Omega_0.$$

Hence $\hat{v}_t \equiv 0$ and $\hat{v}(t,z) \equiv \hat{v}(z)$.

To determine the boundary condition of $\hat{v}$ at $z = \hat{H}(t)$, we consider $\hat{v}$ on

$$\left\{(t,z)|t \in [-\varepsilon^{-1}, \varepsilon^{-1}], z \in [\hat{H}(t) - \varepsilon, \hat{H}(t)]\right\}.$$

From the relation

$$v_n(t,z) = v(t+t_n, z) = w(t+t_n, z - \hat{H}(t+t_n)) = w_n(t, z - H_n(t))$$

and by (21) we see

$$\lim_{n\to\infty} \sup_{z\in[\hat{H}(t)-\varepsilon,\hat{H}(t)]} |w_n(t, z - H_n(t)) - \hat{w}(t, z - \hat{H}(t))| = 0$$

if we define $w_n(t,y) = 0$ for $y \ge 0$ and $\hat{w}(t,y) = 0$ for $y \ge 0$. It follows that $\hat{v}(t,z) \equiv \hat{w}(t, z - \hat{H}(t))$. Hence $\hat{v}(t, \hat{H}(t)) = 0$ and

$$\hat{H}'(t) = -\mu\hat{v}_z(t, \hat{H}(t)) - c^*, \ t \in \mathbb{R}.$$

From $0 = \hat{v}(t, \hat{H}(t)) = \hat{v}(\hat{H}(t))$ and the fact that $\hat{v}(z) \ge 0$ for $z < \hat{H}(t)$. We obtain by the strong maximum principle $\hat{v}(z) > 0$ for $z < \hat{H}(t)$ and by the Hopf lemma $\hat{v}_z(\hat{H}(t)) < 0$. On the other hand

, from $0 = \hat{v}(\hat{H}(t))$ we deduce $0 = \hat{v}_z(\hat{H}(t))\hat{H}'(t)$. Therefore $\hat{H}'(t) \equiv 0$ and $\hat{H}(t) \equiv \hat{H}$. It follows that $c^* = -\mu\hat{v}_z(\hat{H})$. Hence $\hat{v}$ satisfies

$$\begin{cases} \hat{v}_{zz} + c^*\hat{v}_z + f(\hat{v}) = 0, & z < \hat{H}, \\ \hat{v}(\hat{H}) = 0, \hat{v}_z(\hat{H}) = -c^*/\mu. \end{cases}$$

This implies, by the uniqueness of solution to initial value problem, we can conclude that $\hat{v}(z) = q_{c^*}(\hat{H} - z)$. The proof is now complete. □

Now we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. From (14) we have

$$|v(t_n, z) - 1| \le C'(e^{\gamma z} + e^{-\delta t_n}) \text{ for } z \in [-(c + c^*)t_n, H(t_n)].$$

Therefore, from (13) it holds that for any $\varepsilon > 0$, there exists $K > 0$ and $T > 0$ such that

$$\sup_{z \in [-(c+c^*)t_n, -K]} |v(t_n, z) - q_{c^*}(\hat{H} - z)| < \varepsilon$$

for $t_n > T$. On the other hand from Lemma 3.7, for large t_n,

$$\sup_{z \in [-K, \hat{H}]} |v(t_n, z) - q_{c^*}(\hat{H} - z)| < \varepsilon$$

and

$$|h(t_n) - c^*t_n - \hat{H}| < \varepsilon. \tag{24}$$

Hence we have

$$\sup_{z \in [-(c+c^*)t_n, \hat{H}]} |v(t_n, z) - q_{c^*}(\hat{H} - z)| < \varepsilon \tag{25}$$

for all large n. This complete of the proof of the proposition. □

3.3 Part 3: Completion of the proof of Theorem A

In this section we prove Theorem A by constructed finer upper and lower solution as we constructed in part 1. We first present how we construct the upper solution. Take an arbitrary $\varepsilon > 0$ and fix t_n such that (24) and (25) hold and $e^{-\delta t_n} \le \varepsilon$. From (24) and (25) we have

$$\begin{aligned} &v(t_n, z) \le q_{c^*}(\hat{H} - z) + \varepsilon \text{ for } z \in [-(c + c^*), \hat{H}], \\ &H(t_n) = h(t_n) - c^*t_n \le \hat{H} + \varepsilon \end{aligned}$$

Hence we have

$$v(t_n, z) \le q_{c^*}(\hat{H} + \varepsilon - z) + \varepsilon \text{ for } z \in [-(c + c^*)t, \hat{H} + \varepsilon].$$

We note that we can find $N > 1$ independent of $\varepsilon > 0$ such that

$$(1 + N\varepsilon)q_{c^*}(\hat{H} + N\varepsilon - z) \ge q_{c^*}(\hat{H} + \varepsilon - z) + \varepsilon \text{ for } z \le \hat{H} + \varepsilon$$

(Please see [7] for detail).

Now we define an upper solution $(\overline{u}, \overline{g}, \overline{h})$ as follows:

$$\overline{u}(t,x) = (1 + N\varepsilon e^{-\delta(t-t_n)} q_{c^*}(\overline{h}(t) - x),$$
$$\overline{h}(t) = \hat{H} + c^*t + N\varepsilon + N\varepsilon\sigma(1 - e^{-\delta(t-t_n)}),$$
$$\overline{g}(t) = g(t).$$

As in the proof of Lemma 3.2 we can check $(\overline{u}, \overline{g}, \overline{h})$ satisfies the condition in Lemma 2.1 for $t \geq t_n$ (see [7]). Hence we obtain

$$u(t,x) \leq q_{c^*}(\hat{H} + N\varepsilon(1+\sigma) + c^*t - x) + \varepsilon N e^{-\delta(t-t_n)}, \tag{26}$$
$$h(t) - c^*t - \hat{H} \leq N\varepsilon(1+\sigma) \tag{27}$$

for $t \geq t_n$ and $x \in [\overline{g}(t), h(t)] = [g(t), g(t)]$,

Similarly we can obtain by constructing a lower solution that for some $c \in (0, c^*)$ and some $N > 1$

$$q_{c^*}(\hat{H} - N\varepsilon(1+\sigma) + c^*t - x) - \varepsilon N e^{-\delta(t-t_n)} \leq u(t,x), \tag{28}$$
$$- N\varepsilon(1+\sigma) \leq h(t) - c^*t - \hat{H} \tag{29}$$

for $t \geq t_n$ and $x \in [\underline{g}(t), \underline{h}(t)] = [-ct, \underline{h}(t)]$ where

$$\underline{h}(t) = \hat{H} + c^*t - N\varepsilon - N\varepsilon\sigma(1 - e^{-\delta(t-t_n)}).$$

(Please see [7]). From (26) to (29), we can easily conclude that

$$|u(t,x) - q_{c^*}(h(t) - x)| \leq C\varepsilon \quad \text{for} \quad x \in [-ct, h(t)]$$
$$|h(t) - c^*t - \hat{H}| \leq C\varepsilon$$

for $t \geq t_n$. This completes the proof of Theorem A.

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