115

TOPOLOGICAL STRUCTURES IN STATIONARY EULER FLOWS

ALBERTO ENCISO AND DANIEL PERALTA-SALAS

ABSTRACT. In this paper we review recent research on certain topological aspects of the vortex lines of stationary ideal fluids. We will mainly focus on the study of knotted and linked vortex lines and vortex tubes, which is a topic that can be traced back to Lord Kelvin and was popularized by the works of Arnold and Moffatt on topological hydrodynamics in the 1960s. In this context, we will provide a leisurely introduction to some recent results concerning the existence of steady solutions to the Euler equation in Euclidean space with a prescribed set of vortex lines and thin vortex tubes of arbitrarily complicated topology.

1. INTRODUCTION

Our goal in this paper is to review some problems in fluid mechanics whose common denominator is that the main object of interest are the integral curves of the velocity and vorticity of the fluid, which are usually called stream and vortex lines, respectively. Mathematically, these problems are extremely appealing because they give rise to remarkable connections between different areas of mathematics, such as PDEs, dynamical systems and differential geometry. From a physical point of view, these questions are often considered in some approaches to turbulence and hydrodynamical instability.

Regarding the study of the topological structure of stream and vortex lines, one aspect that has attracted considerable attention is the existence of knotted and linked lines. The interest in these questions goes back to Helmholtz, who discovered the phenomenon of the transport of vorticity, and to Lord Kelvin, who developed an atomic theory in which atoms were understood as thin knotted vortex tubes in an ideal fluid: the ether. Although this atomic theory was abandoned after some years, it was a major boon for the development of knot theory.

In modern times, the main figures in the study of knotted stream and vortex lines are Vladimir Arnold, who proved the celebrated structure theorem for steady flows and introduced the asymptotic linking number, and Keith Moffatt, to whom we owe the introduction of the helicity in fluid mechanics and its connection with the entangledness of the fluid. An excellent reference for these and other questions, which are still a very active area of research known as topological hydrodynamics [14], is the monograph [3].

The paper is organized as follows. In Section 2 we recall some basic concepts related to the Euler equation for ideal fluids. In Section 3 we review some heuristic arguments suggesting the existence of stream and vortex lines of any knot type in steady Euler flows and state Arnold's structure theorem. In Section 4 we introduce

ALBERTO ENCISO AND DANIEL PERALTA-SALAS

Beltrami fields, which are used in Section 5 to prove a realization theorem for linked stream and vortex lines [7]. A readable detailed sketch of the proof is also given in this section. To conclude, in Section 6 we state a deeper theorem that ensures the existence of thin vortex tubes of any link type in steady Euler flows.

2. The Euler equation

In this paper we will consider the Euler equation for ideal fluids in $\mathbb{R}^3$:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P, \qquad \operatorname{div} u = 0.$$

The unknowns are the velocity field $u(x,t)$ and the pressure function $P(x,t)$. The integral curves of the velocity field (that is, the solutions to the non-autonomous ODE

$$\dot{x}(t) = u(x(t), t)$$

for some initial condition $x(t_0) = x_0$) are called *particle paths* and describe the motion of the particles in the fluid. The trajectories of $u(x,t)$ at fixed time t are called *streamlines*, and thus the streamline pattern changes with time if the flow is unsteady. If the flow is steady, it is obvious that the particle paths coincide with the streamlines.

Another time-dependent vector field that plays a crucial role in fluid mechanics is the *vorticity*, defined by

$$\omega := \operatorname{curl} u.$$

The integral curves of the vorticity $\omega(x,t)$ at fixed time t (that is to say, the solutions to the autonomous ODE

$$\dot{x}(\tau) = \omega(x(\tau), t)$$

for some initial condition $x(0) = x_0$) are the *vortex lines* of the fluid at time t.

The study of vortex lines is a classic topic in fluid mechanics that can be traced back to Helmholtz [13] and Lord Kelvin [22] in the XIX century. In particular, the analysis of these objects is central in topological fluid mechanics, an area that has attracted considerable attention after the foundational works of Arnold [1, 2] and Moffatt [18] and lies somewhere between the theory of partial differential equations, dynamical systems and differential geometry.

This paper is devoted to the study of stream and vortex lines. More precisely, the kind of questions we will consider in this paper refer to the topological structure of these lines of a fluid: as we will see, our basic goal is to ascertain whether these lines can be of arbitrary knot (or link) type.

In this direction, it should be noted that the most interesting situation is that of *steady* fluids. In this case, the velocity field does not depend on time and the Euler equation can be written as

(2.1) $$u \wedge \operatorname{curl} u = \nabla B, \qquad \operatorname{div} u = 0,$$

where $B := P + \frac{1}{2}|u|^2$ is the Bernoulli function. The reason why stream and vortex lines have been throughly studied for steady fluids is that, on the one hand, they are somehow connected with the important phenomenon of Lagrangian turbulence and that, on the other hand, there are physical arguments, known for decades, that suggest the existence of stationary solutions with stream and vortex lines of arbitrarily complicated topology. Let us start by briefly reviewing these arguments.

3. TRANSPORT OF VORTICITY, MAGNETIC RELAXATION AND KNOTTED VORTEX LINES

The argument suggesting the existence of vortex lines with complex topology, which is essentially due to Helmholtz [13], is based on the transport of vorticity. The basic idea is the following. Suppose that $u(x, t)$ is a time-dependent solution of the Euler equation. Then its vorticity satisfies the equation

$$\frac{\partial \omega}{\partial t} = [\omega, u],$$

with $[\cdot, \cdot]$ the commutator of vector fields. Therefore, the vorticity at time t can be expressed in terms of the vorticity $\omega_0(x)$ at time t_0 as

$$\omega(x, t) = (\phi_{t,t_0})_* \omega_0(x),$$

where $(\phi_{t,t_0})_*$ denotes the push-forward of the non-autonomous flow of the velocity field between the times t_0 and t.

From this expression for the vorticity it stems that the vortex lines at time t are diffeomorphic to those at time t_0. Therefore, one can attempt to construct the initial vorticity ω_0 with a prescribed set of vortex lines. This can be done as follows. Let L be any finite link in $\mathbb{R}^3$. As it has trivial normal bundle, we can ensure that there are two smooth functions f, g of compact support in $\mathbb{R}^3$ such that L is the union of connected components of $f^{-1}(1) \cap g^{-1}(1)$, and that at these components the intersection is transverse. Using these functions, we can prescribe the initial vorticity as the divergence-free vector field

$$\omega_0 := \nabla f \times \nabla g.$$

Through the Biot–Savart operator, this initial vorticity corresponds to the initial velocity

$$u_0(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y) \wedge \omega_0(y)}{|x - y|^3} dy,$$

which falls off at infinity as $|u(x)| < C/|x|^2$ and lies in the Sobolev space $H^k(\mathbb{R}^3)$ for all k. By construction, the field ω_0 is tangent to the level sets of the functions f and g, and the gradients of f and g are not collinear at any point of L. Therefore, the link L is a union of periodic trajectories of the initial vorticity ω_0, so *if* there is a global solution to the Euler equation with initial datum u_0, the solution u has a set of vortex lines diffeomorphic to the link L at all times. In particular, *if* the fluid $u(x, t)$ evolves, for large times, into an equilibrium state, characterized by a steady solution to Euler $u_\infty(x)$, it is conceivable (although certainly not at all obvious) that this steady solution should also have a set of periodic vortex lines diffeomorphic to L. Of course, these hypotheses prevent us from promoting this heuristic argument to a rigorous result.

The heuristic argument in support of the existence of knotted stream lines is based on the phenomenon of magnetic relaxation. To explain this argument [19], let us consider the following magnetohydrodynamic system with viscosity μ:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla P + \mu \Delta v + H \times \operatorname{curl} H,$$

$$\frac{\partial H}{\partial t} = [H, v], \quad \operatorname{div} v = \operatorname{div} H = 0.$$

In this equation, $v(x,t)$ represents the velocity field of a plasma, $H(x,t)$ is the associated magnetic field and $P(x,t)$ is the pressure of the plasma.

Just as in the case of vortex lines, the idea is to take initial conditions (H_0, v_0) such that H_0 has a set of periodic trajectories given by a link L. This can be done as in the case of vortex lines. Then one can argue that, *if* there is a global solution with this choice of initial conditions, it is reasonable that the viscous term $\mu \Delta v$ forces the velocity to become negligible as $t \to \infty$. If the magnetic field also has some definite limit $H_\infty(x)$ as $t \to \infty$, then this limit field satisfies

$$H_\infty \times \operatorname{curl} H_\infty = \nabla P_\infty\,, \qquad \operatorname{div} H_\infty = 0\,.$$

By Eq. (2.1), H_∞ is then a steady solution to the Euler equation. Since the magnetic field is transported by the flow of the velocity field, the same argument as above suggests that one can hope that H_∞ should have a set of periodic trajectories (i.e., stream lines) diffeomorphic to the link L. The problems that appear when one tries to make this argument rigorous are similar to those appearing in the case of vortex lines.

In spite of the fact that it is very challenging to make them rigorous, these arguments are the main theoretical basis for the well known conjecture that there are steady solutions to the Euler equation having stream and vortex lines of any link topology. A priori, this conjecture is quite striking in view of Arnold's celebrated structure theorem [3], which asserts that, under mild technical assumptions, the stream and vortex lines of a steady solution to Euler whose velocity field is not everywhere collinear with its vorticity are nicely stacked in a rigid structure akin to those which appear in the study of integrable Hamiltonian systems:

Theorem 3.1 (Arnold's structure theorem). *Let u be a solution to the steady Euler equation in a bounded domain $\Omega \subset \mathbb{R}^3$ with analytic boundary. Suppose that u is tangent to the boundary and analytic in the closure of the domain. If u and its vorticity are not everywhere collinear, then there is an analytic set C, of codimension at least 1, so that $\Omega \backslash C$ consists of a finite number of subdomains in which the dynamics of u is of one of the following two types:*

- *The subdomain is trivially fibered by tori invariant under u. On each torus, the flow of u is conjugate to a linear flow (rational or irrational).*
- *The subdomain is trivially fibered by cylinders invariant under u whose boundaries sit on $\partial\Omega$. All the trajectories of u on each cylinder are periodic.*

Heuristically, this structure should somehow restrict the way the vortex lines are arranged. Partial results in this direction have been shown in [10], where it is proved that under appropriate hypotheses the stream or vortex lines of steady solutions with non-collinear velocity and vorticity cannot be of certain knot types.

4. BELTRAMI FIELDS

In his structure theorem, Arnold emphasized that the key hypothesis is that the velocity and the vorticity should not be everywhere collinear [2], and actually conjectured that when this condition is not satisfied, i.e. when the velocity and vorticity are everywhere parallel, then one should be able to construct steady solutions to the Euler equation with stream and vortex lines of arbitrary topological complexity.

Therefore, if one tries to construct steady solutions to the Euler equation with stream or vortex lines of any link type, it is natural to consider solutions of the form

$$\text{(4.1)} \qquad \operatorname{curl} u = f\,u\,, \qquad \operatorname{div} u = 0\,,$$

with f a smooth function on $\mathbb{R}^3$. Taking the divergence in this equation we infer that $\nabla f \cdot u = 0$, i.e., that f is a first integral of the velocity. As a consequence of this, the trajectories of u must lie on the level sets of the function f. We have proved recently [9] that there are no nontrivial solutions to Eq. (4.1) for an open and dense set of factors f in the C^k topology, $k \geqslant 7$. In particular, there are no nontrivial solutions whenever f has a regular level set diffeomorphic to the sphere. This result is reminiscent of (and somehow complementary to) Arnold's structure theorem, cf. Theorem 3.1, for steady solutions with nonconstant Bernoulli function (that is, for solutions where u and curl u are not collinear).

Accordingly, in order to construct solutions with complex orbit structures we will focus our attention on *Beltrami fields*, which satisfy the equation

$$\operatorname{curl} u = \lambda u$$

for some nonzero constant λ. Obviously the streamlines of a Beltrami field are the same as its vortex lines, so henceforth we will only refer to the latter.

There is abundant numerical evidence and some analytical results that suggest that the dynamics of a Beltrami field can be extremely complex. The most thoroughly studied examples of Beltrami field are the ABC fields, introduced by Arnold himself and discussed in detail, e.g., in [6]:

$$u(x) = \left(A\sin x_3 + C\cos x_2,\; B\sin x_1 + A\cos x_3,\; C\sin x_2 + B\cos x_1 \right).$$

Here A, B, C are real parameters. It is remarkable that all our intuition about Beltrami fields comes from the analysis of a few exact solutions, which basically consist of fields with Euclidean symmetries and the ABC family.

An interesting approach to the conjecture on the existence of linked vortex lines in steady solutions to Euler, due to Etnyre & Ghrist (1999), hinges on the connection of Beltrami fields with contact geometry [11]. The main observation is the following. Let u be a Beltrami field and α its dual 1-form, so that the Beltrami equation can be written using the Hodge $*$-operator as

$$\text{(4.2)} \qquad *d\alpha = \lambda \alpha\,.$$

Therefore, if the Beltrami field does not vanish anywhere, we have that

$$\alpha \wedge d\alpha = \lambda |u|^2 \, dx_1 \wedge dx_2 \wedge dx_3$$

does not vanish either, so that by definition α defines a contact 1-form. Conversely, if α is a contact 1-form in $\mathbb{R}^3$, there is a smooth Riemannian metric g adapted to the form α so that this 1-form satisfies Eq. (4.2) with the Hodge $*$-operator corresponding to the metric g. The vector field associated with α is a Beltrami field with respect to the metric g.

The reason why this observation is useful is that the machinery of contact geometry is very well suited for the construction of contact forms whose associated vector fields (which are called Reeb fields) have a prescribed set of periodic trajectories. Therefore, one finds that there is a metric in $\mathbb{R}^3$, which in general is neither flat nor complete, such that the Euler equation in this metric admits a steady solution

of Beltrami type having a set of vortex lines of any link type. This strategy does not work when we consider the Euler equation for a fixed (e.g. Euclidean) metric.

5. REALIZATION THEOREM FOR VORTEX LINES

In this section we shall review a recent result that shows how Beltrami fields can be used to prove that there are steady solutions to the Euler equation with a set of periodic vortex lines diffeomorphic to any given link [7]. The statement applies to Beltrami fields with any nonzero constant λ; obviously for $\lambda = 0$ the claim does not hold true, as u would be a gradient field and, as such, could not have any periodic trajectories.

Theorem 5.1. *Let $L \subset \mathbb{R}^3$ be a finite link and let λ be any nonzero real number. Then one can deform the link L by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in any C^m norm, such that $\Phi(L)$ is a set of vortex lines of a Beltrami field u, which satisfies the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$. Moreover, u falls off at infinity as $|D^j u(x)| < C_j / |x|$.*

We have only considered the case of finite links, but the case of locally finite links can be tackled similarly at the expense of losing the decay condition of the velocity field. In particular, taking into account the fact that the knot types modulo diffeomorphism are countable, this yields a positive answer to a question of Williams [23] and Etnyre & Ghrist [11]: is there a steady solution to the Euler equation whose streamlines realize all knot types at the same time?

It should be mentioned that the steady solutions to the Euler equation that we construct in the theorem do not have finite energy: being Beltrami fields, the field satisfies $\Delta u = -\lambda^2 u$, so it cannot be in $L^2(\mathbb{R}^3)$. Nadirashvili has proved recently [20] that the $1/|x|$ decay we have is optimal within the class of Beltrami solutions (not necessarily with constant proportionality factor, see Eq. (4.1)), nonetheless, so in particular our solutions are real analytic and belong to the space $L^p(\mathbb{R}^3)$ for all $p > 3$ (which is optimal as well according to Nadirashvili's result). Notice that the $1/|x|$ decay was not proved in Ref. [7] (indeed, in this paper the Beltrami field was not shown to satisfy any conditions at infinity), but follows from the more refined global approximation theorem that we have proved in [8].

We shall next sketch the proof of Theorem 5.1. The heart of the problem is that one needs to extract topological information from a PDE. Generally speaking, topological techniques (such as those used in [11]) are too 'soft' to capture what happens in a PDE, while analytical techniques (see e.g. [16]) have not been very successful in these kinds of problems either. We will resort to an intermediate approach. The basic philosophy is to use the methods of differential topology and dynamical systems to control auxiliary constructions and those of PDEs to relate these auxiliary constructions to the Euler equation.

To simplify the exposition, we will divide the presentation in three steps. In Steps 1 and 2 we will construct a local Beltrami field, defined in a neighborhood of the link L, for which L is a set of robust vortex lines. In Step 3 we will approximate this local Beltrami field by a global Beltrami field that has a set of vortex lines diffeomorphic to L.

Step 1. Let us take a connected component L_1 of the link L. It is well know that, perturbing the knot a little through a small diffeomorphism, we can assume that the knot L_1 is analytic. Since the normal bundle of a knot is trivial, we can take an analytic strip (or ribbon) Σ around L_1. More precisely, there is an analytic embedding h of the cylinder $\mathbb{S}^1 \times (-\delta, \delta)$ into $\mathbb{R}^3$ whose image is Σ and such that $h(\mathbb{S}^1 \times \{0\}) = L_1$.

In a small tubular neighborhood N_1 of the knot L_1 we can take an analytic coordinate system

$$(\theta, z, \rho) : N_1 \to \mathbb{S}^1 \times (-\delta, \delta) \times (-\delta, \delta)$$

adapted to the strip Σ. Basically, θ and z are suitable extensions of the angular variable on the knot and of the signed distance to L_1 as measured along the strip Σ, while ρ is the signed distance to Σ.

The reason why this coordinate system is useful is that it allows us to define a vector field w_1 in the neighborhood N_1 that is key in the proof: simply, w_1 is the field dual to the closed 1-form

$$d\theta - z\, dz\,.$$

From this expression and the definition of the coordinates it stems that w_1 is an analytic vector field tangent to the strip Σ and that L_1 is a stable hyperbolic periodic trajectory of the pullback of w_1 to the strip Σ.

Step 2. The field w_1 we constructed in Step 1 will now be used to define a local Beltrami field v_1. To this end we will consider the Cauchy problem

$$(5.1) \qquad\qquad \operatorname{curl} v_1 = \lambda v_1\,, \qquad v_1|_\Sigma = w_1\,.$$

One cannot apply the Cauchy–Kowalewski theorem directly because the curl operator does not have any non-characteristic surfaces as its symbol is an skew-symmetric matrix. In fact, a direct computation shows that there are some analytic Cauchy data w_1, tangent to the surface Σ, for which this Cauchy problem does not have any solutions: a necessary condition for the existence of a solution, when the field w_1 is tangent to Σ, is that the pullback to the strip of the 1-form dual to the Cauchy datum must be a closed form.

Through a more elaborate argument that involves a Dirac-type operator, one can prove that this condition is not only necessary but also sufficient. Therefore, the properties of the field w_1 constructed in Step 1 allow us to ensure that there is a unique analytic field v_1 in a neighborhood of the knot L_1 which solves the Cauchy problem (5.1). Taking now the neighborhood N_1 small enough, we can assume that v_1 is defined in its closure $\overline{N_1}$.

It is obvious that the knot L_1 is a periodic trajectory of the local Beltrami field v_1. As a matter of fact, it is easy to check that this trajectory is hyperbolic (and therefore stable under small perturbations). The idea is that, by construction, the strip Σ is an invariant manifold under the flow of v_1 that contracts into L_1 exponentially. As the flow of v_1 preserves volume because v_1 is divergence-free, there must exist an invariant manifold that is exponentially expanding and intersects Σ transversally on L_1, which guarantees the hyperbolicity of the periodic trajectory L_1.

As a consequence of this hyperbolicity, L_1 is a robust periodic trajectory. More concretely, by the hyperbolic permanence theorem any field u_1 that is close enough

to v_1 in the $C^m(N_1)$ norm has a periodic trajectory diffeomorphic to L_1, and this diffeomorphism can be chosen C^m-close to the identity (and different from the identity only in N_1). Here m is any positive integer.

Step 3. Applying the previous argument to each component L_i of the link L we obtain (pairwise disjoint) tubular neighborhoods N_i around each knot L_i and local Beltrami fields v_i defined in $\overline{N_i}$. This defines a Beltrami field v in the closed set

$$S := \bigcup_i \overline{N_i}.$$

The global Beltrami field u is obtained through a Runge-type theorem for the operator $\mathrm{curl} - \lambda$. This result, whose proof makes use of functional-analytic methods and Green's functions estimates [8], allows us to approximate the local Beltrami field v by a global Beltrami field u in the $C^m(S)$ norm. More precisely, for any positive ϵ and any positive integer m there is a global Beltrami field u, satisfying the fall-off condition $|D^j u(x)| < C_j/|x|$ for all $j \geqslant 0$, such that

$$\sum_{j=0}^m \left| D^j u(x) - D^j v(x) \right| < \epsilon$$

for all $x \in S$. (The case of locally finite links requires an analog of this result in which the field u does not satisfy the fall-off condition but the positive constant ϵ can be replaced by any positive function $\epsilon(x)$, which can be allowed to tend to zero at infinity arbitrarily fast).

To conclude the proof of the theorem it is enough to take ϵ small enough so that the hyperbolic permanence theorem ensures that if $\|u - v_i\|_{C^m(N_i)} < \epsilon$ then there is a diffeomorphism Φ_i of $\mathbb{R}^3$ such that $\Phi_i(L_i)$ is a periodic trajectory of u and $\Phi_i - \mathrm{id}$ is supported in N_i and such that $\|\Phi_i - \mathrm{id}\|_{C^m(\mathbb{R}^3)}$ is as small as one wishes. Therefore, the diffeomorphism Φ defined as

$$\Phi(x) := \begin{cases} \Phi_i(x) & \text{if } x \in N_i \text{ for some } i, \\ x & \text{otherwise} \end{cases}$$

maps the link L into a set of vortex lines of the Beltrami field u and is arbitrarily close to the identity in the C^m norm.

6. Realization theorem for thin vortex tubes

In Theorem 5.1 we have established the existence of steady solutions to the Euler equation in $\mathbb{R}^3$ with vortex (or stream) lines of any link type. However, there are still many open questions about the structure of vortex lines in steady incompressible fluids that are of great interest, both physically and mathematically.

A long-standing problem in this direction is Lord Kelvin's conjecture [22] that knotted and linked thin vortex tubes can arise in steady solutions to the Euler equation. This conjecture was motivated by results due to Helmholtz and Maxwell's observations of what they called 'water twists'. Another recent related experiment, where nontrivially knotted vortex tubes are produced in laboratory for the first time, is described in [15]. We recall (see e.g. [17]) that a (closed) *vortex tube* is defined as a domain in $\mathbb{R}^3$ that is the union of vortex lines and whose boundary is an embedded torus.

Kelvin's conjecture is fully consistent with Arnold's views of steady ideal fluids. Indeed, after establishing his structure theorem, Arnold conjectured [2] that, contrary to what happens in the non-collinear case, Beltrami fields could present vortex lines of the same topological complexity as the trajectories of any divergence-free vector field. By KAM theory, typically these trajectories give rise to infinitely many invariant tori and chaotic regions between them.

There is strong numerical evidence of the existence of thin vortex tubes in the Euler equation, both in the case of steady and time-dependent fluid flows. Indeed, thin vortex tubes have long played a key role in the construction and numerical exploration of possible blow-up scenarios for the Euler equation, which in turn has led to rigorous results such as [4, 5]. A particularly influential scenario in this direction is [21], which discusses how an initial condition with a certain set of linked thin vortex tubes might lead to singularity formation in finite time.

Recently we have proved [8] a realization theorem for thin vortex tubes of any link type that is roughly analogous to that of Theorem 5.1. To state this result, let us denote by $\mathcal{T}_\epsilon(L)$ the ϵ-thickening of a given link L in $\mathbb{R}^3$, that is, the set of points that are at distance at most ϵ from L. The realization theorem can then be stated as follows:

Theorem 6.1. *Let L be a finite link in $\mathbb{R}^3$. For small enough ϵ, one can transform the collection of pairwise disjoint thin tubes $\mathcal{T}_\epsilon(L)$ by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in any C^m norm, so that $\Phi[\mathcal{T}_\epsilon(L)]$ is a set of vortex tubes of a Beltrami field u, which satisfies the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ for some nonzero constant λ. Moreover, the field u decays at infinity as $|D^j u(x)| < C_j/|x|$.*

Indeed, the proof of this theorem also yields information on the structure of the vortex lines inside each vortex tube. This structure is extremely rich: there are infinitely many nested invariant tori (which bound vortex tubes) and a set of elliptic periodic trajectories diffeomorphic to the link L near the core of the vortex tubes. It should be emphasized that the vortex tubes we construct are not 'infinitely thin': the construction is valid for all ϵ smaller than some constant $\epsilon_0(L)$ that only depends on the geometry of the link.

The proof of Theorem 6.1 also relies on the combination of a robust local construction and a global approximation result, as in the case of Theorem 5.1. Indeed, this global approximation result was tacitly used in the statement of Theorem 5.1 to ensure that our Beltrami fields fall off at infinity. However, the construction of the robust local solution (which takes most of the paper) is much more sophisticated than in the case of vortex lines and requires entirely different ideas.

Basically, the robustness of the tubes follows from a KAM-theoretic argument with two small parameters: the thinness of the tubes and the constant λ. The local solution must now be defined in the whole tubes, not just on a neighborhood of the boundary. This makes it impossible to construct the local solution using a theorem of Cauchy-Kowalewski type, as we did in Step 2 of Theorem 5.1. Instead, we need to consider a boundary value problem for the curl operator in which the tangential part of the field cannot be prescribed. As a consequence of this, one cannot directly take local Beltrami fields which satisfy the non-degeneracy conditions of the KAM-type theorem: these conditions must be extracted from the equation using fine PDE estimates. This is in great contrast with the prescription of the Cauchy datum that

ALBERTO ENCISO AND DANIEL PERALTA-SALAS

we made in Step 1 of Theorem 5.1, which readily ensures the hyperbolicity of the periodic trajectory, and leads to very subtle problems with a deep interplay of PDE and dynamical systems techniques.

We shall not give any further details concerning the proof of this result, which is beyond the scope of this review. However, to conclude we would like to mention an important property of the structure of the vortex lines inside the vortex tubes whose existence is established in Theorem 6.1: this structure is *stable* in the following sense. On the one hand, it is robust under small perturbations of the field u, meaning that the trajectories of any field which is close enough to u in the C^k norm have the same structure. On the other hand, the boundary of each vortex tube is Lyapunov-stable under the flow of the Beltrami field u. In particular, from Theorem 6.1 we recover Theorem 5.1 and improve it by ensuring that the set of vortex lines diffeomorphic to the given link is linearly stable, while in Theorem 5.1 is unstable.

ACKNOWLEDGEMENTS

REFERENCES

[1] V.I. Arnold, Sur la topologie des écoulements stationnaires des fluides parfaits. *C R Acad Sci Paris* 261 (1965) 17–20.

[2] V.I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann Inst Fourier* 16 (1966) 319–361.

[3] V.I. Arnold, B. Khesin, *Topological methods in hydrodynamics*. Springer, 1999.

[4] D. Córdoba, C. Fefferman, On the collapse of tubes carried by 3D incompressible flows. *Comm Math Phys* 222 (2001) 293–298.

[5] J. Deng, T.Y. Hou, X. Yu, Geometric properties and nonblowup of 3D incompressible Euler flow. *Comm PDE* 30 (2005) 225–243.

[6] T. Dombre, et al., Chaotic streamlines in the ABC flows. *J Fluid Mech* 167 (1986) 353–391.

[7] A. Enciso, D. Peralta-Salas, Knots and links in steady solutions of the Euler equation. *Ann of Math* 175 (2012) 345–367.

[8] A. Enciso, D. Peralta-Salas, Existence of knotted vortex tubes in steady Euler flows. Preprint (arXiv:1210.6271).

[9] A. Enciso, D. Peralta-Salas, Beltrami fields with a nonconstant proportionality factor are rare. Preprint (arXiv:1402.6825).

[10] J. Etnyre, R. Ghrist, Stratified integrals and unknots in inviscid flows. *Contemp Math* 246 (1999) 99–111.

[11] J. Etnyre, R. Ghrist, Contact topology and hydrodynamics III. Knotted orbits. *Trans Amer Math Soc* 352 (2000) 5781–5794.

[12] M.H. Freedman, Z.X. He, Divergence-free fields: energy and asymptotic crossing number. *Ann of Math* 134 (1991) 189–229.

[13] H. von Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J Reine Angew Math* 55 (1858) 25–55.

[14] B. Khesin, Topological fluid dynamics. *Notices AMS* 52 (2005) 9–19.

[15] D. Kleckner, W.T.M. Irvine, Creation and dynamics of knotted vortices. *Nature Phys.* 9 (2013) 253–258.

[16] P. Laurence, E.W. Stredulinsky, Two-dimensional magnetohydrodynamic equilibria with prescribed topology. *Comm Pure Appl Math* 53 (2000) 1177–1200.

[17] A.J. Majda, A.L. Bertozzi, *Vorticity and incompressible flow*. Cambridge University Press, 2002.

[18] H.K. Moffatt, The degree of knottedness of tangled vortex lines. *J Fluid Mech* 35 (1969) 117–129.

[19] H.K. Moffatt, Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology I. *J Fluid Mech* 159 (1985) 359–378.

[20] N. Nadirashvili, Liouville theorem for Beltrami flow. Preprint (arXiv:1403.1414).

[21] R.B. Pelz, Symmetry and the hydrodynamic blow-up problem. *J Fluid Mech* 444 (2001) 299–320.

[22] W. Thomson (Lord Kelvin), Vortex statics. *Proc. R. Soc. Edinburgh* 9 (1875) 59–73.

[23] R.F. Williams, The universal templates of Ghrist. *Bull. Amer. Math. Soc.* 35 (1998) 145–156.

INSTITUTO DE CIENCIAS MATEMÁTICAS, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, 28049 MADRID, SPAIN

E-mail address: aenciso@icmat.es

INSTITUTO DE CIENCIAS MATEMÁTICAS, CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS, 28049 MADRID, SPAIN

E-mail address: dperalta@icmat.es