数理解析研究所講究録
第 1898 巻 2014 年 156-173

ASYMPTOTIC EXPANSIONS FOR CERTAIN q-SERIES, q-INTEGRALS, q-DIFFERENTIALS AND A FORMULA OF RAMANUJAN FOR SPECIFIC VALUES OF $\zeta(s)$

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ABSTRACT. This is a summarized version of the author's papers [22][24] on asymptotic aspects of the q-series of Lambert type, q-hypergeometric function, q-integrals and q-differentials. Major portions of the results in these papers are rearranged to state in Parts I and II respectively; the first part is devoted to showing intrinsic linkage between asymptotics of certain q-series and a formula of Ramanujan for specific values of the Riemann zeta-function $\zeta(s)$, while several complete asymptotic expansions for multiple q-integrals and q-differentials of Thomae-Jackson type are presented in the second part.

Part I: Asymptotics for q-series and Ranmanujan's formula for $\zeta(s)$

1.1. **Introduction (I).** Throughout the present article, let q be a complex parameter with $|q| < 1$, and the substitution $q = e^{-t}$ will be made as it is needed, where the half-plane $\mathrm{Re}\, t > 0$ is transformed to the unit disk $|q| < 1$. It is the main aim of Part I to present intrinsic linkage between asymptotic expansions of certain q-series (see (1.1.6)–(1.1.8) below) and a formula of Ramanujan for specific values of the Riemann zeta-function at odd integers (see (1.1.9) below). This linkage is in fact hidden in Ramanujan's original work; however, the introduction of the q-series (1.1.2) or (1.1.3) and its treatment based on a Mellin transform technique give us an insight for connecting these two aspects together.

Let z and s be complex variables, and let α and μ be real parameters with $\alpha > 0$. For our later purposes it is convenient to introduce the generalized Lerch zeta-function $\Phi(s, \alpha, z)$ defined by

$$\Phi(s,\alpha,z) = \sum_{n=0}^{\infty}(\alpha+n)^{-s}z^n \tag{1.1.1}$$

for all s if $|z| < 1$, for $\mathrm{Re}\, s > 0$ if $|z| = 1$ and $z \neq 1$, and for $\mathrm{Re}\, s > 1$ if $z = 1$, respectively; this continues to a meromorphic function over the whole s-plane and is one-valued in the complex z-plane cut along the real axis from 1 to $+\infty$ (cf. [13]). We use the notation $e(\mu) = e^{2\pi i\mu}$ hereafter. Then $\Phi(s, \alpha, z)$ reduces to the ordinary Lerch zeta-function $\phi(s, \alpha, \mu)$ when $z = e(\mu)$, so that $\Phi(s, \alpha, 1) = \zeta(s, \alpha)$ is the Hurwitz zeta-function, $e(\mu)\Phi(s, 1, e(\mu)) = \zeta_\mu(s)$ the exponential zeta-function, and so $\Phi(s, 1, 1) = \zeta(s)$ the Riemann zeta-function. We remark that the order of the variables in Φ and ϕ above differs from the usual notation, in order to retain notational consistency with other terminology.

2010 *Mathematics Subject Classification.* Primary 11P82; Secondary 11M35.

Key words and phrases. multiple q-integral, multiple q-differential, Mellin-transform, asymptotic expansion, Lerch zeta-function.

A portion of the present investigation was initiated during the author's academic stay at Mathematisches Institut, Westfälische Wilhelms-Universität Münster. He would like to express his sincere gratitude to Professor Christopher Deninger and the institution for their warm hospitality and constant support. The author was also indebted to Grant-in-Aid for Scientific Research (No. 16540038) from JSPS.

Let β and ν be real parameters with $\beta > 0$. The main object of the present paper is the q-series of the form

$$S_s\left({\alpha,\beta \atop \mu,\nu};q\right) = e(\beta\nu)\sum_{m=0}^{\infty} e((\alpha+m)\mu)q^{(\alpha+m)\beta}\Phi(s,\beta,e(\nu)q^{\alpha+m}), \tag{1.1.2}$$

which is rewritten, by changing the order of summations, as a Lambert series form

$$S_s\left({\alpha,\beta \atop \mu,\nu};q\right) = e(\alpha\mu)\sum_{n=0}^{\infty}(\beta+n)^{-s}\frac{e((\beta+n)\nu)q^{\alpha(\beta+n)}}{1-e(\mu)q^{\beta+n}}. \tag{1.1.3}$$

We shall prove complete asymptotic expansions of $S_s\left({\alpha,\beta \atop \mu,\nu};q\right)$ as $t \to 0$ in the sectorial region $|\arg t| < \pi/2$ (see Theorem 0 below). Let as usual

$$(z;q)_\infty = \prod_{m=0}^{\infty}(1-zq^m), \qquad (z;q)_n = (z;q)_\infty/(zq^n;q)_\infty$$

for any integer n denote q-shifted factorials. Our main formula (1.2.3) in particular implies a complete asymptotic expansion of $\log(q^\alpha;q)_\infty$ as $q \to 1^-$, and it further allows us to treat the q-series

$$F(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}, \tag{1.1.4}$$

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) = \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}. \tag{1.1.5}$$

These are typical examples of the theta series (in the transformed Eulerian form) whose asymptotic behaviours near the singularities at the points $q^k = 1$ ($k = 1, 2, \ldots$) were first considered by Ramanujan in his last letter to Hardy (see [38]). Ramanujan showed

$$F(q) = \left(\frac{t}{2\pi}\right)^{1/2}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+o(1), \tag{1.1.6}$$

$$G(q) = \left(\frac{2}{5-\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+o(1), \tag{1.1.7}$$

$$H(q) = \left(\frac{2}{5+\sqrt{5}}\right)^{1/2}\exp\left(\frac{\pi^2}{15t}+\frac{11t}{60}\right)+o(1), \tag{1.1.8}$$

as $t \to +0$, and similar asymptotic formulae for certain other q-series. In conjunction with this result, (complete) Stirling's formula for the q-gamma function was first established by Moak [31], while Ueno and Nishizawa [37] developed their theory on a q-analogue of the Hurwitz zeta-function and applied it to rederive the same formula, together with asymptotic expansions of $G(q)$ and $H(q)$, similar to (1.1.7) and (1.1.8). The study on asymptotic aspects for more general q-series of the type $\sum_{n=0}^{\infty}a^n q^{bn^2+cn}/(q;q)_n$ was initiated by Ramanujan [35, p. 366] [36, p.359], and was further proceeded by Berndt [7] [8, Chap. 27]. This direction has recently been systematically explored by McIntosh [25][26][27] and Gordon-McIntosh [17][18], in conjunction with transformation properties of the q-series. It is to be remarked that the basic tool applied by these authors is the Euler-Maclaurin summation device. The Mellin transform technique, on the other hand, was applied by Meinardus [29][30] to derive certain asymptotic formulae for fairly general class of partition-type functions. We refer the reader to [2, Chap. 6] for various related works.

Let B_k $(k=0,1,2,\ldots)$ denote the Bernoulli numbers (cf. [13]). Our main theorem also yields Ramanujan's famous formula for specific values of the Riemann zeta-function at odd integers (cf. [5][6]), which asserts, for any integer $n \neq 0$,

$$\xi^{-n}\Big\{\frac{1}{2}\zeta(2n+1)+\sum_{l=1}^{\infty}\frac{l^{-2n-1}}{e^{2l\xi}-1}\Big\}+2^{2n}\sum_{k=0}^{n+1}\frac{B_{2n+2-2k}B_{2k}}{(2n+2-2k)!(2k)!}\xi^{n+1-k}(-\eta)^k$$
$$=(-\eta)^{-n}\Big\{\frac{1}{2}\zeta(2n+1)+\sum_{l=1}^{\infty}\frac{l^{-2n-1}}{e^{2l\eta}-1}\Big\}, \tag{1.1.9}$$

where ξ and η are positive real numbers satisfying $\xi\eta=\pi^2$ and the finite sum on the left-hand side is to be regarded as null if $n<-1$ (see Theorem 2 in Section 1.4). It will later turn out that the excluded case $n=0$ of this formula emerges (in a sense) as asymptotic expansions of $F(q)$, $G(q)$ and $H(q)$ (see Corollary 1.4 in Section 1.3).

1.2. **The main theorem (I).** Let x and y be complex variables. Apostol [3] introduced the sequence of rational functions $\mathcal{B}_k(x,y)$ $(k\geq 0)$ defined by the Taylor series expansion

$$\frac{ze^{xz}}{ye^z-1}=\sum_{k=0}^{\infty}\frac{\mathcal{B}_k(x,y)}{k!}z^k \tag{1.2.1}$$

with $|\arg y|<\pi$ near $z=0$. The function $\mathcal{B}_k(x,y)$, which coincides with the usual Bernoulli polynomial $B_k(x)$ if $y=1$, is a polynomial in x of degree at most k with coefficients in $\mathbb{Q}(y)$. Next let $\Gamma(s)$ be the gamma function, and $U(a;c;z)$ denote the confluent hypergeometric function defined by

$$U(a;c;z)=\frac{1}{\Gamma(a)}\int_0^{\infty e^{i\varphi}}e^{-zw}w^{a-1}(1+w)^{c-a-1}dw \tag{1.2.2}$$

for $\operatorname{Re}a>0$ and $|\arg z+\varphi|<\pi/2$ with any fixed angle $\varphi\in(-\pi,\pi)$, where the path of integration is taken as a half-line from the origin to $\infty e^{i\varphi}$ (cf. [13]); the domain of z is extended to the whole sector $|\arg z|<3\pi/2$ by rotating suitably the path of integration in (1.2.2).

We now state our main result in Part I.

Theorem 0. *Let α, β, μ and ν be real parameters with $\alpha>0$ and $\beta>0$, $q=e^{-t}$, and let $S_s(\alpha,\beta;\mu,\nu;q)$ be defined by (1.1.2) or (1.1.3). Then for any integer $K\geq 0$ and any complex t in the sector $|\arg t|<\pi/2$ the formula*

$$\begin{aligned}S_s\big({\textstyle\frac{\alpha,\beta}{\mu,\nu}};q\big)&=e(\alpha\mu+\beta\nu)\mathcal{B}_0(\beta,e(\nu))\Gamma(1-s)\phi(1-s,\alpha,\mu)t^{s-1}\\&\quad+e(\alpha\mu+\beta\nu)\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}\mathcal{B}_{k+1}(\alpha,e(\mu))}{(k+1)!}\phi(s-k,\beta,\nu)t^k\\&\quad+R_{s,K}\big({\textstyle\frac{\alpha,\beta}{\mu,\nu}};q\big)\end{aligned} \tag{1.2.3}$$

holds in the region $\operatorname{Re}s<K+1$ except the points $s=k$ $(k=0,1,\ldots,K)$, where $\mathcal{B}_k(x,y)$ is defined by (1.2.1), and the empty sum is to be regarded as null. Here $R_{s,K}\big({\textstyle\frac{\alpha,\beta}{\mu,\nu}};q\big)$ is the remainder term satisfying the estimate

$$R_{s,K}\big({\textstyle\frac{\alpha,\beta}{\mu,\nu}};q\big)=O(|t|^K) \tag{1.2.4}$$

as $t\to 0$ through $|\arg t|\leq\pi/2-\delta$ with any small $\delta>0$, in the region $\operatorname{Re}s<K+1$, where the implied O-constant depends at most on s, K, α, β, μ, ν and δ. In particular

when $K\ge 1$, $0<\alpha\le 1$, $0<\beta\le 1$, $0\le\mu\le 1$ *and* $0\le\nu\le 1$ *the explicit expression* (1.2.5)

$$\begin{aligned}R_{s,K}\left(\begin{smallmatrix}\alpha,\beta\\ \mu,\nu\end{smallmatrix};q\right) &= (-1)^K(2\pi)^{-s}t^{s-1}\Gamma(K+1-s)\\
&\times\Big\{e^{\pi is/2}\sum_{m,n=0}^{\infty}{}' e(-\alpha m-\beta n)(\mu+m)^{-s}f_{s,K}(4\pi^2e^{-\pi i}(\mu+m)(\nu+n)/t)\\
&+e^{-\pi is/2}\sum_{m,n=0}^{\infty}{}' e(\alpha(1+m)+\beta(1+n))(1-\mu+m)^{-s}\\
&\times f_{s,K}(4\pi^2e^{\pi i}(1-\mu+m)(1-\nu+n)/t)\\
&+e^{\pi is/2}\sum_{m,n=0}^{\infty}{}' e(-\alpha m+\beta(1+n))(\mu+m)^{-s}f_{s,K}(4\pi^2(\mu+m)(1-\nu+n)/t)\\
&+e^{-\pi is/2}\sum_{m,n=0}^{\infty}{}' e(\alpha(1+m)-\beta n)(1-\mu+m)^{-s}f_{s,K}(4\pi^2(1-\mu+m)(\nu+n)/t)\Big\}\end{aligned}$$

holds for $|\arg t|<\pi/2$, *in the region* $\mathrm{Re}\, s<K$, *where*

(1.2.6) $$f_{s,K}(z)=U(K+1-s;K+1-s;z)$$

with the confluent hypergeometric function defined by (1.2.2), *and the primed summation symbols indicate that the terms including* $\mu+m=0$ *or* $1-\mu+m=0$, *and* $\nu+n=0$ *or* $1-\nu+n=0$ *are to be omitted in they occur.*

Remark. Asymptotic expansions similar to (1.2.3) follow also for the exceptional points $s=k$ $(k=0,1,2,\ldots)$ as limiting cases of Theorem 0, whose important applications are included in these exceptional cases (see Theorems 1–5 below).

1.3. **Applications to q-factorials and allied functions.** It is seen from the relation $z\Phi(1,1,z)=-\log(1-z)$ for $|z|<1$ and (1.1.2) that

(1.3.1) $$S_1(\alpha,1;0,\nu;q)=-\log(e(\nu)q^\alpha;q)_\infty,$$

and hence Theorem 0 yields

Theorem 1. *Let* α *and* ν *be real with* $\alpha>0$ *and* $0<\nu<1$. *Then the following asymptotic expansions hold for any integer* $K\ge 1$ *and any complex* t *in* $|\arg t|<\pi/2$:

(1.3.2) $$\log(q^\alpha;q)_\infty=-\frac{\pi^2}{6t}-B_1(\alpha)\log t-\log\frac{\Gamma(\alpha)}{\sqrt{2\pi}}+\frac{1}{4}B_2(\alpha)t-\sum_{k=2}^{K-1}\frac{(-1)^kB_kB_{k+1}(\alpha)}{k(k+1)!}t^k-R_{1,K}\left(\begin{smallmatrix}\alpha,1\\ 0,0\end{smallmatrix};q\right);$$

(1.3.3) $$\log(e(\nu)q^\alpha;q)_\infty=-\zeta_\nu(2)t^{-1}-B_1(\alpha)\{\log(2\sin\pi\nu)+\pi iB_1(\nu)\}+\frac{1}{4}B_2(\alpha)(1-i\cot\pi\nu)t-\sum_{k=2}^{K-1}\frac{(-1)^k\mathcal{B}_k(0,e(\nu))B_{k+1}(\alpha)}{k(k+1)!}t^k-R_{1,K}\left(\begin{smallmatrix}\alpha,1\\ 0,\nu\end{smallmatrix};q\right),$$

where the remainder terms $R_{1,K}\left(\begin{smallmatrix}\alpha,1\\ 0,0\end{smallmatrix};q\right)$ *and* $R_{1,K}\left(\begin{smallmatrix}\alpha,1\\ 0,\nu\end{smallmatrix};q\right)$ *satisfy the same estimate as* (1.2.4) *when* $t\to 0$ *through the sector* $|\arg t|\le\pi/2-\delta$ *with any small* $\delta>0$. *In* *particular if $K \geq 2$ and $0 < \alpha \leq 1$, the explicit expressions as in (1.2.5) follow for the remainder terms.*

Remark A complete asymptotic expansion of $(q^\alpha; q)_\infty$ as $q \to 1^-$ was first established by Moak [31] and later rederived by Ueno-Nishizawa [37] in a slightly different form from that of (1.3.2). McIntosh [25][27] proved (1.3.2) for real $t > 0$ with the error term $R_{1,K} = O(t^K)$ in a more general situation.

Corollary 1.1. *For any real $\alpha > 0$ and any integer $K \geq 1$ the formula*

$$\begin{aligned}\log(-q^\alpha; q)_\infty = &\frac{\pi^2}{12t} - B_1(\alpha)\log 2 + \frac{1}{4}B_2(\alpha)t \\ &- \sum_{k=2}^{K-1} \frac{(-1)^k(2^k-1)B_kB_{k+1}(\alpha)}{k(k+1)!}t^k - R_{1,K}\left({\alpha,1 \atop 0,1/2}; q\right)\end{aligned} \tag{1.3.4}$$

holds in $|\arg t| < \pi/2$, where the remainder term $R_{1,K}\left({\alpha,1 \atop 0,1/2}; q\right)$ satisfies the same estimate as (1.2.4). In particular if $0 < \alpha \leq 1$ and $K \geq 2$ the explicit expression as in (1.2.5) follows for the remainder term.

To describe the subsequent results, the change of the base

$$q = e^{-t} \longmapsto e^{-4\pi^2/t} = \widehat{q} \tag{1.3.5}$$

is frequently applied. Noting the facts

$$B_{2h+1} = 0, \qquad h = 1, 2, \ldots, \tag{1.3.6}$$

$$B_k(1-\alpha) = (-1)^k B_k(\alpha), \qquad k = 0, 1, 2 \ldots \tag{1.3.7}$$

(cf. [13]), we find that every term (with $k \geq 2$) of the series in (1.3.2) and (1.3.4) vanishes when $\alpha = 1$, and hence Theorem 0 further reduces to

Corollary 1.2. *The following formulae hold:*

$$\log(q; q)_\infty = -\frac{\pi^2}{6t} - \frac{1}{2}\log\frac{t}{2\pi} + \frac{t}{24} - \sum_{l=1}^\infty l^{-1}\frac{\widehat{q}^l}{1-\widehat{q}^l}, \tag{1.3.8}$$

or in exponential form

$$(q; q)_\infty = \sqrt{\frac{2\pi}{t}}\exp\left(-\frac{\pi^2}{6t} + \frac{t}{24}\right)(\widehat{q}; \widehat{q})_\infty;$$

$$\log(-q; q)_\infty = \frac{\pi^2}{12t} - \frac{1}{2}\log 2 + \frac{t}{24} - \sum_{l=1}^\infty l^{-1}\frac{\widehat{q}^{l/2}}{1-\widehat{q}^l}, \tag{1.3.9}$$

or in exponential form

$$(-q; q)_\infty = \frac{1}{\sqrt{2}}\exp\left(\frac{\pi^2}{12t} + \frac{t}{24}\right)(\widehat{q}^{1/2}; \widehat{q})_\infty.$$

Remark Formulae (1.3.8) and (1.3.9) are classic; these can be found for e.g., in [4, Chap. 3].

Remark. Formulae (1.3.8) and (1.3.9) both give complete (convergent) asymptotic expansions, since for instance the l-th term of the last infinite series in (1.3.8) is of order $\widehat{q}^l/l + O(\widehat{q}^{2l})$ as $l \to \infty$.

It can be observed that the explicit expression (1.2.5) for the remainder term, in certain specific cases (as in the preceding corollary), further reduces to complete (convergent) asymptotic expansions as $t \to 0$ in $|\arg t| < \pi/2$ (see Corollaries 1.3–1.5 below). If one considers, for instance, the logarithm of the pairing $(q^\alpha;q)_\infty(q^{1-\alpha};q)_\infty$ with $0<\alpha<1$, each term (with $k \geq 2$) in its asymptotic series vanishes again by (1.3.6) and (1.3.7). From (1.2.5) and Theorem 1 we can in fact prove:

Corollary 1.3. *The following formula hold for any real α and μ with $0<\alpha<1$ and $0<\mu<1$:*

$$\log\{(q^\alpha;q)_\infty(q^{1-\alpha};q)_\infty\} = -\frac{\pi^2}{3t} + \log(2\sin\pi\alpha) + \frac{1}{2}B_2(\alpha)t - \sum_{l=1}^{\infty} l^{-1}\frac{e((1-\alpha)l)\widehat{q}^l}{1-\widehat{q}^l} - \sum_{l=1}^{\infty} l^{-1}\frac{e(\alpha l)\widehat{q}^l}{1-\widehat{q}^l}, \tag{1.3.10}$$

or in exponential form

$$(q^\alpha;q)_\infty(q^{1-\alpha};q)_\infty = 2(\sin\pi\alpha)\exp\Big\{-\frac{\pi^2}{3t}+\frac{1}{2}B_2(\alpha)t\Big\} \times (e(1-\alpha)\widehat{q};\widehat{q})_\infty(e(\alpha)\widehat{q};\widehat{q})_\infty;$$

(1.3.11)

$$\log\{(e(\mu)q^\alpha;q)_\infty(e(1-\mu)q^{1-\alpha};q)_\infty\} = -\{\zeta_\mu(2)+\zeta_{1-\mu}(2)\}t^{-1} - 2\pi i B_1(\alpha)B_1(\mu) + \frac{1}{2}B_2(\alpha)t - \sum_{l=1}^{\infty} l^{-1}\frac{e((1-\alpha)l)\widehat{q}^{\mu l}}{1-\widehat{q}^l} - \sum_{l=1}^{\infty} l^{-1}\frac{e(\alpha l)\widehat{q}^{(1-\mu)l}}{1-\widehat{q}^l},$$

or in exponential form

$$(e(\mu)q^\alpha;q)_\infty(e(1-\mu)q^{1-\alpha};q)_\infty = \exp\Big[\{\zeta_\mu(2)+\zeta_{1-\mu}(2)\}t^{-1} - 2\pi i B_1(\alpha)B_1(\mu) + \frac{1}{2}B_2(\alpha)t\Big](e(1-\alpha)\widehat{q}^\mu;\widehat{q})_\infty(e(\alpha)\widehat{q}^{1-\mu};\widehat{q})_\infty.$$

We can now restate Ramanujan's asymptotic formula (1.1.6)–(1.1.8) with explicit error terms. It is known that $F(q) = 1/(q;q)_\infty$ (cf. [38, pp.57–58], and the famous Rogers-Ramanujan identities assert that

$$G(q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \quad\text{and}\quad H(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

(cf. [2, (7.1.6) and (7.1.7)]). Formulae (1.3.8) and (1.3.10) therefore imply

Corollary 1.4. *The following formulae hold for $F(q)$, $G(q)$ and $H(q)$ defined by* (1.1.4) *and* (1.1.5):

$$F(q) = \Big(\frac{t}{2\pi}\Big)^{1/2}\exp\Big(\frac{\pi^2}{6t}-\frac{t}{24}\Big)\frac{1}{(\widehat{q};\widehat{q})_\infty}, \tag{1.3.12}$$

or in logarithmic form

$$\log F(q) = \frac{\pi^2}{6t} + \frac{1}{2}\log\frac{t}{2\pi} - \frac{t}{24} + \sum_{l=1}^{\infty} l^{-1}\frac{\hat{q}^l}{1-\hat{q}^l};$$

(1.3.13) $$G(q) = \left(\frac{2}{5-\sqrt{5}}\right)^{1/2} \exp\left(\frac{\pi^2}{15t} - \frac{t}{60}\right) \frac{1}{(e(1/5)\hat{q}^{1/5}; \hat{q}^{1/5})_\infty (e(4/5)\hat{q}^{1/5}; \hat{q}^{1/5})_\infty},$$

or in logarithmic form

$$\log G(q) = \frac{\pi^2}{15t} + \frac{1}{2}\log\left(\frac{2}{5-\sqrt{5}}\right) - \frac{t}{60} + \sum_{l=1}^{\infty} l^{-1}\frac{e(l/5)\hat{q}^{l/5}}{1-\hat{q}^{l/5}} + \sum_{l=1}^{\infty} l^{-1}\frac{e(4l/5)\hat{q}^{l/5}}{1-\hat{q}^{l/5}};$$

(1.3.14) $$H(q) = \left(\frac{2}{5+\sqrt{5}}\right)^{1/2} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) \frac{1}{(e(2/5)\hat{q}^{1/5}; \hat{q}^{1/5})_\infty (e(3/5)\hat{q}^{1/5}; \hat{q}^{1/5})_\infty},$$

or in logarithmic form

$$\log H(q) = \frac{\pi^2}{15t} + \frac{1}{2}\log\left(\frac{2}{5+\sqrt{5}}\right) + \frac{11t}{60} + \sum_{l=1}^{\infty} l^{-1}\frac{e(2l/5)\hat{q}^{l/5}}{1-\hat{q}^{l/5}} + \sum_{l=1}^{\infty} l^{-1}\frac{e(3l/5)\hat{q}^{l/5}}{1-\hat{q}^{l/5}}.$$

We next mention slightly different type of implications from Theorem 1. To this aim several necessary terminologies are prepared. The q-gamma and q-beta functions are defined respectively by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha} \qquad \text{and} \qquad B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

whose limits as $q \to 1^-$ are known to be the ordinary gamma function and the beta function $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, respectively (cf. [16]). Whilst the basic hypergeometric function ${}_2\phi_1(a,b;c;q,z)$ is defined by

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} z^n, \qquad |z| < 1,$$

for any complex a, b and c with $c \neq q^{-n}$ $(n = 0,1,2,\ldots)$, whose particular case $a = q^\alpha$, $b = q^\beta$ and $c = q^\gamma$ gives a q-analogue of Gauss' hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ (cf. [16, 1.2]). It is known that the classical Gauss' and Kummer's summation formulae

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

where $\mathrm{Re}(\gamma-\alpha-\beta) > 0$, $\gamma \neq -n$ $(n = 0,1,2,\ldots)$, and

$${}_2F_1(\alpha,\beta;1+\alpha-\beta;-1) = \frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha/2)}{\Gamma(1+\alpha)\Gamma(1+\alpha/2-\beta)},$$

where $1+\alpha-\beta \neq -n$ $(n = 0,1,2,\ldots)$, have q-analogues of the form

$${}_2\phi_1(q^\alpha,q^\beta;q^\gamma;q,q^{\gamma-\alpha-\beta}) = \frac{(q^{\gamma-\alpha};q)_\infty(q^{\gamma-\beta};q)_\infty}{(q^\gamma;q)_\infty(q^{\gamma-\alpha-\beta};q)_\infty},$$

$${}_2\phi_1(q^\alpha,q^\beta;q^{1+\alpha-\beta};q,-q^{1-\beta}) = \frac{(-q;q)_\infty(q^{1+\alpha};q^2)_\infty(q^{2+\alpha-2\beta};q^2)_\infty}{(q^{1+\alpha-\beta};q)_\infty(-q^{1-\beta};q)_\infty}$$

respectively (cf. [16, 1.5; 1.8]). Combining formulae (1.3.2) and (1.3.4) with appropriate exponents (in place of α) we can prove

Corollary 1.5. *Let α, β, γ be positive real numbers. Then the following formulae hold for any integer $K \geq 1$ when $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$:*

$$\log \Gamma_q(\alpha) = \log \Gamma(\alpha) - \frac{1}{4}(\alpha-1)(\alpha-2)t + \sum_{k=2}^{K-1} \frac{B_k}{kk!}\left\{\frac{(-1)^k B_{k+1}(\alpha)}{k+1} + 1 - \alpha\right\}t^k + O(|t|^K)$$

for $\alpha > 0$;

$$\log B_q(\alpha,\beta) = \log B(\alpha,\beta) + \frac{1}{2}(\alpha\beta - 1)t + \sum_{k=2}^{K-1} \frac{B_k}{kk!}\left\{\frac{(-1)^k C_{k+1}(\alpha,\beta)}{k+1} + 1\right\}t^k + O(|t|^K)$$

for $\alpha > 0$ and $\beta > 0$, where

$$C_k(\alpha,\beta) = B_k(\alpha) + B_k(\beta) - B_k(\alpha+\beta);$$

$$\log {}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q, q^{\gamma-\alpha-\beta}) = \log {}_2F_1(\alpha,\beta;\gamma;1) - \frac{1}{2}\alpha\beta t - \sum_{k=2}^{K-1} \frac{(-1)^k B_k D_{k+1}(\alpha,\beta,\gamma)}{k(k+1)!}t^k + O(|t|^K)$$

for $\gamma - \alpha > 0$, $\gamma - \beta > 0$, $\gamma > 0$ and $\gamma - \alpha - \beta > 0$, where

$$D_k(\alpha,\beta,\gamma) = B_k(\gamma-\alpha) + B_k(\gamma-\beta) - B_k(\gamma) - B_k(\gamma-\alpha-\beta);$$

$$\log {}_2\phi_1(q^\alpha, q^\beta; q^{1+\alpha-\beta}; q, -q^{1-\beta}) = \log {}_2F_1(\alpha,\beta;1+\alpha-\beta;-1) - \sum_{k=2}^{K-1} \frac{(-1)^k B_k E_{k+1}(\alpha,\beta)}{k(k+1)!}t^k + O(|t|^K)$$

for $1+\alpha > 0$, $2+\alpha-2\beta > 0$, $1+\alpha-\beta > 0$ and $1-\beta > 0$, where

$$E_k(\alpha,\beta) = 2^{k-1}B_k(\alpha/2+1/2) + 2^{k-1}B_k(1+\alpha/2-\beta) - B_k(1+\alpha-\beta) - (2^{k-1}-1)B_k(1-\beta).$$

Here the implied O-constants depend at most on K, α, β, γ and δ.

1.4. **Connections with Ramanujan's formula for $\zeta(2n+1)$.** We next describe that our main theorem implies Ramanujan's formula for $\zeta(2n+1)$ and its several variants. In order to clarify symmetricity of the following results we introduce the new parameter $\tau = t/2\pi$. Then the case $\alpha = \beta = 1$, $\lambda = \mu = 0$ and $s = 2n+1$ ($n = \pm1, \pm2, \ldots$) of Theorem 0 reduces to the following equivalent form of (1.1.9).

Theorem 2 (Ramanujan). *Let $q = e^{-2\pi\tau}$ and $\widehat{q} = e^{-2\pi/\tau}$ with $\operatorname{Re}\tau > 0$. Then for any integer $n \neq 0$ the formula*

$$(1.4.1)\qquad S_{2n+1}\left(\begin{smallmatrix}1,1\\0,0\end{smallmatrix};q\right) + \frac{1}{2}\zeta(2n+1) + \frac{1}{2}(2\pi)^{2n+1}\sum_{k=0}^{n+1}\frac{(-1)^k B_{2n+2-2k}B_{2k}}{(2n+2-2k)!(2k)!}\tau^{2n+1-2k}$$
$$= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}1,1\\0,0\end{smallmatrix};\widehat{q}\right) + \frac{1}{2}\zeta(2n+1)\right\}$$

holds.

Theorem 0 further yields the following several variants of (1.1.9).

Theorem 3. *Let q and $\widehat{q}$ be as in Theorem 2. Then the following formulae hold for any integer n and any real α and μ with $0 < \alpha < 1$ and $0 < \mu < 1$:*

$$(1.4.2)\quad S_{2n+1}\left(\begin{smallmatrix}\alpha,1\\0,\mu\end{smallmatrix};q\right) + S_{2n+1}\left(\begin{smallmatrix}1-\alpha,1\\0,1-\mu\end{smallmatrix};q\right) + (2\pi)^{2n+1}\sum_{k=0}^{2n+2}\frac{(-i)^k B_{2n+2-k}(\alpha)B_k(\mu)}{(2n+2-k)!k!}\tau^{2n+1-k}$$
$$= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}\mu,1\\0,1-\alpha\end{smallmatrix};\widehat{q}\right) + S_{2n+1}\left(\begin{smallmatrix}1-\mu,1\\0,\alpha\end{smallmatrix};\widehat{q}\right)\right\};$$

$$(1.4.3)\quad S_{2n}\left(\begin{smallmatrix}\alpha,1\\0,\mu\end{smallmatrix};q\right) - S_{2n}\left(\begin{smallmatrix}1-\alpha,1\\0,1-\mu\end{smallmatrix};q\right) - (2\pi)^{2n}\sum_{k=0}^{2n+1}\frac{(-i)^k B_{2n+1-k}(\alpha)B_k(\mu)}{(2n+1-k)!k!}\tau^{2n-k}$$
$$= i(-1)^n\tau^{2n-1}\left\{S_{2n}\left(\begin{smallmatrix}\mu,1\\0,1-\alpha\end{smallmatrix};\widehat{q}\right) - S_{2n}\left(\begin{smallmatrix}1-\mu,1\\0,\alpha\end{smallmatrix};\widehat{q}\right)\right\},$$

where $B_k(x)$ denotes the k-th Bernoulli polynomial.

Remark. Eie and Chen [12] recently obtained the same formula as (1.4.2) in a quite different manner, basing on their theorems for multiple zeta functions associated with polynomials.

Theorem 4. *Let q and $\widehat{q}$ be as in Theorem 2. Then the following formulae hold for any integer n and any real β and λ with $0 < \beta < 1$ and $0 < \lambda < 1$:*

$$(1.4.4)\qquad S_{2n+1}\left(\begin{smallmatrix}1,\beta\\\lambda,0\end{smallmatrix};q\right) + S_{2n+1}\left(\begin{smallmatrix}1,1-\beta\\1-\lambda,0\end{smallmatrix};q\right) + \zeta(2n+1,\beta)$$
$$+ (2\pi)^{2n+1}\sum_{k=0}^{2n+2}\frac{i^k\mathcal{B}_{2n+2-k}(0,e(\lambda))\mathcal{B}_k(0,e(\beta))}{(2n+2-k)!k!}\tau^{2n+1-k}$$
$$= (-1)^n\tau^{2n}\left\{S_{2n+1}\left(\begin{smallmatrix}1,\lambda\\1-\beta,0\end{smallmatrix};\widehat{q}\right) + S_{2n+1}\left(\begin{smallmatrix}1,1-\lambda\\\beta,0\end{smallmatrix};\widehat{q}\right) + \zeta(2n+1,1-\lambda)\right\}$$

except when $n = 0$;

$$(1.4.5)\qquad S_{2n}\left(\begin{smallmatrix}1,\beta\\\lambda,0\end{smallmatrix};q\right) - S_{2n}\left(\begin{smallmatrix}1,1-\beta\\1-\lambda,0\end{smallmatrix};q\right) + \zeta(2n,\beta)$$
$$- (2\pi)^{2n}\sum_{k=0}^{2n+1}\frac{i^k\mathcal{B}_{2n+1-k}(0,e(\lambda))\mathcal{B}_k(0,e(\beta))}{(2n+1-k)!k!}\tau^{2n-k}$$
$$= i(-1)^n\tau^{2n-1}\left\{S_{2n}\left(\begin{smallmatrix}1,\lambda\\1-\beta,0\end{smallmatrix};\widehat{q}\right) - S_{2n}\left(\begin{smallmatrix}1,1-\lambda\\\beta,0\end{smallmatrix};\widehat{q}\right) - \zeta(2n,1-\lambda)\right\},$$

where $\mathcal{B}_k(x,y)$ is defined by (1.2.1).

Part II: Asymptotics for multiple q-integrals and q-differentials

2.1. **Introduction (II).** Suppose temporarily that q is a real parameter with $0<q<1$. Let $\varphi(u)$ be a function integrable on the interval $[0,x]$. A q-analogue of the ordinary integral $\int_0^x \varphi(u)du$, in the form

$$\int_0^x \varphi(u)d_qu=(1-q)x\sum_{n=0}^{\infty}\varphi(q^nx)q^n, \tag{2.1.1}$$

was introduced by Thomae [34] in 1869 and studied by Jackson [19] during 1910–1951 (see also [16, p.23, Chap.1, 1.11]). The formulation in (2.1.1) is motivated from the fact that

$$\lim_{q\to1^-}\int_0^x \varphi(u)d_qu=\int_0^x\varphi(u)du \tag{2.1.2}$$

holds for all $\varphi(u)$ continuous on $[0,x]$. On the other hand, a q-analogue of the ordinary differentiation is formulated as

$$\partial_{q,z}\psi(z)=\frac{\psi(z)-\psi(qz)}{(1-q)z} \tag{2.1.3}$$

(cf. [16, p.27, 1.12]), which asserts that

$$\lim_{q\to1^-}\partial_{q,z}\psi(z)=\psi'(z)=\partial_z\psi(z), \tag{2.1.4}$$

say, for all $\psi(z)$ complex differentiable at z.

Throughout the following, q is a complex parameter with $0<|q|<1$, and the substitution $q=e^{-t}$ will be made if necessary, upon transforming the half-plane $\mathrm{Re}\,t>0$ to the unit disk $|q|<1$. A complex domain $D\subset\mathbb{C}$ is called *star-shaped* if $0\in D$ and for any $z\in D$ the line segment $\overline{0,z}$ is included in D. We suppose throughout that $f(z)$ is a function holomorphic in a star-shaped domain D, and ρ_f denotes the distance between 0 and the singularity of $f(z)$ being closest to 0.

We introduce the q-integral and q-differential operators $\mathcal{I}^x_{q,z}$ and $\mathcal{D}^y_{q,z}$ defined for any real $x>0$ and $y\ge0$ by

$$\mathcal{I}^x_{q,z}f(z)=\int_0^1 u^{x-1}f(uz)d_qu=z^{-x}\int_0^z w^{x-1}f(w)d_qw, \tag{2.1.5}$$

$$\mathcal{D}^y_{q,z}f(z)=\frac{f(z)-q^yf(qz)}{1-q}=z^{-y}(z\partial_{q,z})\{z^yf(z)\} \tag{2.1.6}$$

for any z in $|z|<\rho_f$, where the latter equalities follow from (2.1.1) and (2.1.3) respectively.

Remark. If the base q is restricted to the range $0<q<1$, then the domain of z in which the definitions in (2.1.5) and (2.1.6) are valid is extended to the whole D by its star-shapedness.

Proposition 1. *The operator relations*

$$\mathcal{I}^x_{q,z}\mathcal{D}^x_{q,z}=1 \qquad \textit{and} \qquad \mathcal{D}^x_{q,z}\mathcal{I}^x_{q,z}=1$$

hold for any $x>0$, where 1 denotes the identity operation.

It is the main aim of Part II to pursue the directions in (2.1.2) and (2.1.4) further; this leads us to show that complete asymptotic expansions as $t\to0$ through the sector

$|\arg t| < \pi/2$ exist for the multiple q-integrals $(\mathcal{I}_{q,z}^x)^r f(q^y z)$ (Theorem 5) and the multiple q-differentials $(\mathcal{D}_{q,z}^x)^r f(q^y z)$ (Theorem 6) with any integer $r \geq 1$, under fairly generic situations. A full extension of the domain of z in which Theorems 5 and 6 are valid is possible if $0 < q < 1$ (Theorem 7). Several applications of our main formulae (2.2.4) and (2.2.9) will further be given for the Hurzitz-Lerch zeta-function (Theorems 8 and 9), q-factorials (Corollary 8.1), and q-analogues of the exponential functions (Corollary 8.2), of the binomial functions (Corollary 8.3), and of the poly-logarithmic functions (Corollaries 8.4 and 9.1). As for methodology, it is fundamental to apply a Mellin transform technique in the proofs of Theorems 5 and 6.

2.2. **The main theorems (II).** Let r be any integer, and w a complex variable. To describe our results we introduce the functions $A_{f,k}(x,z)$ and Nörlund's generalized Bernoulli polynomials $B_k^{(r)}(y)$ of rank r (cf. [32]) defined respectively for $k = 0, 1, \ldots$ by the Taylor series expansions

$$e^{xw} f(e^w z) = \sum_{k=0}^{\infty} \frac{A_{f,k}(x,z)}{k!} w^k, \tag{2.2.1}$$

$$e^{yw} \left(\frac{w}{e^w - 1}\right)^r = \sum_{k=0}^{\infty} \frac{B_k^{(r)}(y)}{k!} w^k \tag{2.2.2}$$

near $w = 0$. Note that $B_k^{(1)}(y) = B_k(y)$ is the usual Bernoulli polynomial, and so $B_k(0) = B_k$ is the usual Bernoulli number. We write $B_k^{(r)}(0) = B_k^{(r)}$, and use Euler's differential operator $\vartheta_z = z\partial_z$.

We state our first main result in Part II.

Theorem 5. *Let x and y be real parameters with $x > 0$ and $y \geq 0$, $q = e^{-t}$, and $r \geq 1$ an arbitrary fixed integer. Further let $(\mathcal{I}_{q,z}^x)^r f(z)$ denote the r-times iterated operation of* (2.1.5) *to any function $f(z)$ holomorphic in a star-shaped domain D, and define the coefficients $A_{f,-j}(x,z)$ $(j = 1, 2, \ldots)$ by*

$$A_{f,-j}(x,z) = \int_0^1 u_j^{x-1} \int_0^1 u_{j-1}^{x-1} \cdots \int_0^1 u_1^{x-1} f(u_1 \cdots u_j z) du_1 \cdots du_j. \tag{2.2.3}$$

Then for any integer $K \geq 0$ the formula

$$\begin{aligned} \frac{q^{xy}}{(1-q)^r} (\mathcal{I}_{q,z}^x)^r f(q^y z) &= \sum_{j=1}^{r} \frac{(-1)^{r-j} A_{f,-j}(x,z) B_{r-j}^{(r)}(y)}{(r-j)!} t^{-j} \\ &\quad + \sum_{k=0}^{K-1} \frac{(-1)^{r+k} A_{f,k}(x,z) B_{r+k}^{(r)}(y)}{(r+k)!} t^k + R_{f,K}^{(r)}(x,y;q,z) \end{aligned} \tag{2.2.4}$$

holds in the sector $|\arg t| < \pi/2$ and on the disk $|z| < \rho_f$. Here $R_{f,K}^{(r)}$ is the remainder term expressed by a certain inverse Mellin transform, and satisfies the estimate

$$R_{f,K}^{(r)}(x,y;q,z) = O(|t|^K) \tag{2.2.5}$$

as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$, where the implied O-constant depends at most on r, x, y, z, K and δ. In particular if $0 \leq y \leq r$ and $K \geq 1$ the *representation*

$$(2.2.6)\qquad R^{(r)}_{f,K}(x,y;q,z)=(-t)^K\sum_{l=0}^{r-1}\frac{(-1)^{r-1-l}B^{(r)}_{r-1-l}(y)}{l!(r-1-l)!}\sum_{n=-\infty}^{\infty}{}'\frac{e(ny)}{(2\pi in)^{K+l}}\times\left(\frac{\partial}{\partial u}\right)^l u^{K+l}\int_0^1\xi^{xtu+2\pi in-1}(x+\vartheta_z)^K f(\xi^{tu}z)d\xi\Big|_{u=1}$$

follows, where the primed summation symbol indicates that the term with $n=0$ is to be omitted. with $n=0$.

Remark 3. The explicit expression (2.2.6) will be used to extend the domain of z where (2.2.4) with (2.2.5) is valid (see Theorem 7).

From a point of view of applications it is necessary to establish the asymptotic expansions for $(\mathcal{I}^x_{q,z})^r f(z)$ both with and without the associated q-multiples (see (2.3.5), (2.3.11) and (2.3.12) below). The case $y=0$ of Theorem 5 in fact yields, in view of the latter equality in (2.1.5), the following corollary.

Corollary 5.1. *Let r and x be as in Theorem 5. Then for any integer $K\geq 0$ the asymptotic formula*

$$(2.2.7)\qquad \int_0^z w_r^{-1}\int_0^{w_r} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)d_qw_1\cdots d_qw_r=\sum_{k=0}^{K-1}\frac{(-1)^kC_{f,k}(x,z)}{k!}t^k+O(|t|^K)$$

holds as $t\to 0$ through $|\arg t|\leq\pi/2-\delta$ for any small $\delta>0$, on the disk $|z|<\rho_f$ with $|\arg z|<\pi$, where the implied O-constant depends at most on x, z, K and δ. Here the coefficients $C^{(r)}_{f,k}$ $(k=0,1,\dots)$ are given by

$$(2.2.8)\qquad C^{(r)}_{f,k}(x,z)=\sum_{j=\max(1,r-k)}^{r}\binom{k}{r-j}B^{(-r)}_{k-r+j}B^{(r)}_{r-j}\times\int_0^z w_j^{-1}\int_0^{w_j}w_{j-1}^{-1}\cdots w_2^{-1}\int_0^{w_2}w_1^{x-1}f(w_1)dw_1\cdots dw_j+\sum_{j=0}^{k-r}\binom{k}{r+j}B^{(-r)}_{k-r-j}B^{(r)}_{r+j}\vartheta_z^j\{z^xf(z)\},$$

which is reduced if $r=1$ to

$$C^{(1)}_{f,k}(x,z)=\frac{1}{k+1}\left[\int_0^z w^{x-1}f(w)dw+\sum_{j=0}^{k-1}\binom{k+1}{j+1}B_{j+1}\vartheta_z^j\{z^xf(z)\}\right],$$

where the empty sums are to be regarded as null.

The case $K=1$ of Corollary 5.1 implies the following.

Corollary 5.2. *Under the same assumptions as in Corollary 5.1 we have the limiting relation*

$$\lim_{\substack{q\to 1\\ |q|<1}} \int_0^z w_r^{-1} \int_0^{w_r} w_{r-1}^{-1} \cdots w_2^{-1} \int_0^{w_2} w_1^{x-1} f(w_1) d_q w_1 \cdots d_q w_r$$

$$= C_{f,0}^{(r)}(x,z) = \int_0^z w_r^{-1} \int_0^{w_r} w_{r-1}^{-1} \cdots w_2^{-1} \int_0^{w_2} w_1^{x-1} f(w_1) dw_1 \cdots dw_r.$$

We proceed to state our second main result in Part II. For this, let $\Gamma(s)$ denote the gamma function, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n the rising factorial.

Theorem 6. *Let $x \geq 0$ and $y \geq 0$ be real parameters, $q = e^{-t}$, and $r \geq 1$ an arbitrarily fixed integer. Further let $(\mathcal{D}_{q,z}^x)^r f(z)$ denote the r-times iterated operation of* (2.1.6) *to any function $f(z)$ holomorphic in a star-shaped domain D. Then for any integer $K \geq 0$ the formula*

$$q^{xy}\left(\frac{1-q}{t}\right)^r (\mathcal{D}_{q,z}^x)^r f(q^y z) = \sum_{k=0}^{K-1} \frac{(-1)^k A_{f,r+k}(x,z) B_k^{(-r)}(y)}{k!} t^k + R_{f,K}^{(-r)}(x,y;q,z) \tag{2.2.9}$$

holds in the sector $|\arg t| < \pi/2$ and on the disk $|z| < \rho_f$. Here $R_{f,K}^{(-r)}$ is the remainder term expressed by a certain inverse Mellin transform, and satisfies the estimate

$$R_{f,K}^{(-r)}(x,y;q,z) = O(|t|^K) \tag{2.2.10}$$

as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ with any small $\delta > 0$, where the implied O-constant depends at most on r, x, y, z, K and δ. Furthermore, for any real $x \geq 0$ and $y \geq 0$, and any integer $K \geq 0$,

$$R_{f,K}^{(-r)}(x,y;q,z) = \frac{(-1)^{r+K} t^K}{\Gamma(r+K)} \sum_{n=0}^{r} \frac{(-r)_n}{n!} (y+n)^{r+K} \int_0^1 (1-\xi)^{r+K-1} q^{x(y+n)\xi} \times (x+\vartheta_z)^{r+K} f(q^{(y+n)\xi} z) d\xi. \tag{2.2.11}$$

In view of the latter equality in (2.1.6), the case $y = 0$ of Theorem 6 in fact yields the following corollary.

Corollary 6.1. *Let r and x be as in Theorem 6. Then for any integer $K \geq 0$ the asymptotic formula*

$$(z\partial_{q,z})^r \{z^x f(z)\} = \sum_{k=0}^{K-1} \frac{(-1)^k C_{f,k}^{(-r)}(x,z)}{k!} t^k + O(|t|^K) \tag{2.2.12}$$

holds as $t \to 0$ through $|\arg t| \leq \pi/2 - \delta$ for any small $\delta > 0$, on the disk $|z| < \rho_f$ with $|\arg z| < \pi$, where the implied O-constant depends at most on r, x, z, K and δ. Here the coefficients $C_{f,k}^{(-r)}$ $(k = 0, 1, \ldots)$ are given by

$$C_{f,k}^{(-r)}(x,z) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}^{(r)} B_j^{(-r)} \vartheta_z^{r+j} \{z^x f(z)\}, \tag{2.2.13}$$

which reduces if $r = 1$ to

$$C_{f,k}^{(-1)}(x,z) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{k-j} \vartheta_z^{1+j} \{z^x f(z)\}.$$

The case $K = 1$ of Corollary 6.1 implies the following corollary.

Corollary 6.2. *Under the same assumptions as in Corollary 6.1 we have the limiting relation*

$$\lim_{\substack{q\to 1\\ |q|<1}} (z\partial_{q,z})^r f(z) = C_{f,0}^{(-r)}(x,z) = (z\partial_z)^r\{z^x f(z)\}.$$

We lastly proceed to state the full extension of the domain of z in Theorems 5 and 6 under the restriction that $0 < q < 1$ (see Remark just below of (2.1.6)).

Theorem 7. *Set $q = e^{-t}$ with any real $t > 0$, and let $f(z)$ be any function holomorphic in a star-shaped domain D.*

i) *Let x and y be real with $x > 0$ and $0 \le y \le r$. Then the asymptotic expansion* (2.2.4) *with the estimate* (2.2.5) *when $t \to 0^+$, as well as the explicit expression* (2.2.6), *remain valid throughout the domain D;*
ii) *Let $x \ge 0$ and $y \ge 0$ be real. Then the asymptotic expansion* (2.2.9) *with the estimate* (2.2.10) *when $t \to 0^+$, as well as the explicit expression* (2.11), *remain valid throughout the domain D;*
iii) *The asymptotic expansion* (2.2.7) *with* (2.2.8) *when $t \to 0^+$ for $x > 0$, and also* (2.2.12) *with* (2.2.13) *when $t \to 0^+$ for $x \ge 0$, remain valid both throughout the domain D.*

2.3. **Applications of Theorems 5 and 6.** We suppose throughout this section that $0 < q < 1$. Let $[s]_q = (1-q^s)/(1-q)$ be a q-analogue of s, and $[s]_{q;n} = \prod_{m=0}^{n-1}[s+m]_q$ and $[1]_{q;n} = [n]_q!$ for $n = 0, 1, \dots$ denote q-analogues of the rising factorial and the factorial of n respectively (cf. [16, p.7, Chap.1]), where the empty products are regarded to be 1. Note that the limiting relation $\lim_{q\to 1^-}[s]_q = s$ implies that

$$\lim_{q\to 1^-}[s]_{q;n} = (s)_n \quad \text{and} \quad \lim_{q\to 1^-}[n]_q! = n!. \tag{2.3.1}$$

Recall that the generalized Lerch zeta-function $\Phi(s,x,z)$ is defined by

$$\Phi(s,x,z) = \sum_{m=0}^{\infty}(x+m)^{-s}z^m \tag{2.3.2}$$

for any complex s if $|z| < 1$, and for $\operatorname{Re} s > 1$ if $|z| = 1$ (cf. [13]); this is continued to a holomorphic function of $(s,z) \in \mathbb{C}\times D$, where

$$D = \{z \in \mathbb{C} \mid |\arg(1-z)| < \pi\} = \mathbb{C}\setminus[1,+\infty) \tag{2.3.3}$$

is a complex cut-plane; note here that D is a star-shaped domain. We can therefore apply the part i) of Theorem 7 (upon (2.2.4) with (2.2.5)) to $f(z) = \Phi(s,x,z)$, and obtain the following theorem.

Theorem 8. *Let x and y be real with $x > 0$ and $0 \le y \le r$, and s any complex. Then for any integer $K \ge 0$ the asymptotic expansion*

$$\begin{aligned}\frac{q^{xy}}{(1-q)^r}(\mathcal{I}_{q,z}^x)^r\Phi(s,x,q^y z) &= \sum_{j=1}^{r}\frac{(-1)^{r-j}\Phi(s+j,x,z)B_{r-j}^{(r)}(y)}{(r-j)!}t^{-j}\\ &\quad+\sum_{k=0}^{K-1}\frac{(-1)^{r+k}\Phi(s-k,x,z)B_{r+k}^{(r)}(y)}{(r+k)!}t^k + O(t^K)\end{aligned} \tag{2.3.4}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on r, s, x, y, z, and K.

Let $\mathrm{Li}_l(z)$ for any $l \in \mathbb{Z}$ be the poly-logarithmic function defined by $\mathrm{Li}_l(z) = z\Phi(l,1,z)$ for any $z \in D$. It is seen from (2.1.1), (2.1.5), (2.1.8) and the relation $\log(1-z) = -z\Phi(1,1,z)$, by (2.3.2), that

$$\log(q^y z; q)_\infty = -\frac{q^y z}{1-q}\mathcal{I}^1_{q,z}\Phi(1,1,q^y z) \tag{2.3.5}$$

for any real $y \geq 0$ and in $|\arg(1-z)| < \pi$. Then the case $(r,s,x) = (1,1,1)$ of Theorem 7 yields the following corollary.

Corollary 8.1. *Let y be real with $0 \leq y \leq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\log(q^y z; q)_\infty = -\mathrm{Li}_2(z)t^{-1} - \sum_{k=0}^{K-1} \frac{(-1)^{k+1}\mathrm{Li}_{1-k}(z)B_{k+1}(y)}{(k+1)!}t^k + O(t^K) \tag{2.3.6}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on y, z and K.

Remark The assertion (2.3.6) was first established by McIntosh [25][27] in a more general setting.

We next present the applications to q-analogues of the exponential and binomial functions defined respectively by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \qquad \left(|z| < \frac{1}{1-q}\right),$$

$$f_q(y;z) = \sum_{n=0}^{\infty} \frac{[y]_{q;n}}{[n]_q!}z^n \qquad (|z| < 1),$$

from which with (3.1) the limiting relations $\lim_{q\to 1^-} e_q(z) = e^z$ and $\lim_{q \to 1^-} f_q(y;z) = (1-z)^{-y}$ follow. It is known that the q-binomial theorem (cf. [16, p.8, Chap.1, 1.3]) asserts that

$$e_q(z) = \frac{1}{((1-q)z;q)_\infty} \quad \text{and} \quad f_q(y;z) = \frac{(q^y z;q)_\infty}{(z;q)_\infty} \tag{2.3.7}$$

for any $y \geq 0$; these further provide the meromorphic continuations of $e_q(z)$ and $f_q(y;z)$ respectively over the whole z-plane.

Corollary 8.1 can therefore be applied to the right sides above on yielding the following corollaries.

Corollary 8.2. *For any integer $K \geq 0$ the asymptotic expansion*

$$\log e_q(z) = z + \sum_{k=1}^{K-1} \alpha_k(z)t^k + O(t^K) \tag{2.3.8}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, and this further implies that

$$e_q(z) = e^z\left\{1 + \sum_{k=1}^{K-1} \beta_k(z)t^k + O(t^K)\right\}$$

as $t \to 0^+$, where the coefficients $\alpha_k(z)$ and $\beta_k(z)$ are given by

$$\alpha_k(z) = \sum_{j=0}^{k} \frac{(-1)^{k-j} B_{k-j}}{(k-j)!} \sum_{h=0}^{j} (1+h)^{k-j-2} \frac{B_{j-h}^{(-h-1)} z^{1+h}}{(j-h)!}, \tag{2.3.9}$$

$$\beta_k(z) = \sum_{\substack{\sum_{j=1}^{k} jl_j = k \\ l_j \geq 0 \ (j=1,\dots,k)}} \prod_{j=1}^{k} \frac{\alpha_j(z)^{l_j}}{l_j!}$$

for $k = 0, 1, \dots$, and the implied O-constants depend on z and K.

Corollary 8.3. *Let y be real with $0 \leq y \leq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\log f_q(y; z) = \sum_{k=0}^{K-1} \frac{(-1)^{k+1} \operatorname{Li}_{1-k}(z)}{(k+1)!} \{B_{k+1} - B_{k+1}(y)\} t^k + O(t^K)$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, and this further implies that

$$f_q(y; z) = (1-z)^{-y} \left\{ 1 + \sum_{k=1}^{K-1} \gamma_k(y, z) t^k + O(t^K) \right\}$$

as $t \to 0^+$, where the coefficients $\gamma_k(y, z)$ are given by

$$\gamma_k(y, z) = (-1)^k \sum_{\substack{\sum_{j=1}^{k} jl_j = k \\ l_j \geq 0 \ (j=1,\dots,k)}} \prod_{j=1}^{k} \frac{1}{l_j!} \left[\frac{\operatorname{Li}_{1-j}(z)}{(j+1)!} \{B_{j+1}(y) - B_{j+1}\} \right]^{l_j}$$

for $k = 0, 1, \dots$. Here the implied O-constants depend at most on y, z and K.

We thirdly present applications to a q-analogue of the poly-logarithmic function $\operatorname{Li}_{q,l}(z)$ for any $l \in \mathbb{Z}$ defined by

$$\operatorname{Li}_{q,l}(z) = \sum_{m=0}^{\infty} \frac{z^{1+m}}{[1+m]_q^l} \qquad (|z| < 1), \tag{2.3.10}$$

which with (2.3.1) asserts that $\lim_{q \to 1^-} \operatorname{Li}_{q,l}(z) = \operatorname{Li}_l(z)$. We can in fact show

$$\operatorname{Li}_{q,r}(z) = z(\mathcal{I}_{q,z}^1)^r \Phi(0, 1, z) \tag{2.3.11}$$

for any integer $r \geq 0$; this further provides the meromorphic continuation of $\operatorname{Li}_{q,r}(z)$ for all $z \in D$. Corollary 5.1 can therefore be applied upon taking $f(z) = \Phi(0, 1, z)$ to yield the following corollary.

Corollary 8.4. *Let $r \in \mathbb{Z}$ be arbitrarily fixed with $r \geq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\operatorname{Li}_{q,r}(z) = \sum_{k=0}^{K-1} \frac{(-1)^k C_{f,k}^{(r)}(1, z)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the coefficients $C^{(r)}_{f,k}$ are given by

$$C^{(r)}_{f,k}(1,z) = \sum_{j=\max(1,r-k)}^{r} \binom{k}{r-j} B^{(-r)}_{k-r+j} B^{(r)}_{r-j} \operatorname{Li}_j(z) + \sum_{j=0}^{k-r} \binom{k}{r+j} B^{(-r)}_{k-r-j} B^{(r)}_{r+j} \operatorname{Li}_{-j}(z)$$

for $k = 0, 1, \ldots$. Here the implied O-constant depends at most on r, z and K.

We fourthly discuss the applications of Theorem 6; this at first yields on taking $f(z) = \Phi(s,x,z)$ the following theorem.

Theorem 9. *Let $x \geq 0$ and $y \geq 0$ be real, and s any complex. Then for any integer $K \geq 0$ the asymptotic expansion*

$$q^{xy}\left(\frac{1-q}{t}\right)^r (\mathcal{D}^x_{q,z})^r \Phi(s,x,q^y z) = \sum_{k=0}^{K-1} \frac{(-1)^k \Phi(s-r-k,x,z) B^{(-r)}_k(y)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on r, s, x, y, z and K.

We can in fact show

$$\operatorname{Li}_{q,-r}(z) = z(\mathcal{D}^1_{q,z})^r \Phi(0,1,z) \tag{2.3.12}$$

for any integer $r \geq 0$. Corollary 6.1 can therefore be applied by taking $f(z) = \Phi(0,1,z)$ to yield the following corollary.

Corollary 9.1. *Let $r \in \mathbb{Z}$ be arbitrarily fixed with $r \geq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\operatorname{Li}_{q,-r}(z) = \sum_{k=0}^{K-1} \frac{(-1)^k C^{(-r)}_{f,k}(1,z)}{k!} t^k + O(t^K)$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the coefficients $C^{(-r)}_{f,k}$ are given by

$$C^{(-r)}_{f,k}(1,z) = \sum_{j=0}^{k} \binom{k}{j} B^{(r)}_{k-j} B^{(-r)}_j \operatorname{Li}_{-r-j}(z)$$

for $k = 0, 1, \ldots$. Here the implied O-constant depends at most on r, z and K.

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