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Diophantine approximation related to polylogarithms

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Abstract

In this article, we show a linear independence criterion for the $s+1$ numbers: 1 and s polylogarithms over an algebraic number field, both in the complex and in the p-adic cases. Our method relies on a Diophantine approximation so-called Padé approximation.

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1 Introduction

For $s=1,2,\cdots$, consider the polylogarithmic function $\mathrm{Li}_s(z)$ defined by

$$\mathrm{Li}_s(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^s}, z\in\mathbb{C}, |z|\leqslant 1\ (z\neq 1 \text{ if } s=1).$$

The function satisfies $\mathrm{Li}_1(z)=-\log(1-z)=\int_0^z\frac{dt}{1-t},\ \mathrm{Li}_{s+1}(z)=\int_0^z\frac{\mathrm{Li}_s(t)}{t}dt$.
In the case $s=1$, it corresponds to the power series expansion of $-\log(1-z)$.

In 1979, E. M. Nikišin [8] investigated sufficient conditions such that for a rational number α, the values of polylogarithmic functions $\mathrm{Li}_1(\alpha),\mathrm{Li}_2(\alpha),\cdots,\mathrm{Li}_s(\alpha)$ and 1 are linearly independent over $\mathbb{Q}$. M. Hata [3] gave in 1990 a general linear independence criterion by creating a new method.

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and K be a number field of finite degree d over $\mathbb{Q}$. Fix a prime $p\in\mathbb{Q}$. For an Archimedean $v|\infty$, denote $|\cdot|_v=|\cdot|$ and for $v\nmid\infty$ of K over p, denote by $|\cdot|_v$ normalized valuation s.t. $|x|_v=p^{-ord_p(x)}$ for $x\in\mathbb{Q}$. Write $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ by p and K_v the completion of K by v, and we put $n_v=[K_v:\mathbb{Q}_v]$ the local degree for v ($v|p$ or $v|\infty$). The completion of the algebraic closure of K_v for $v|p$ is denoted by $\mathbb{C}_p$, which is an algebraically closed field.

We also define for $v|p$:

$$\mathrm{L}i_s^{(p)}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \ z \in \mathbb{C}_p, \ |z|_v < 1.$$

We call polylogarithms, values of the polylogarithmic function $\mathrm{L}i_s$ for $z \in \mathbb{C}$ in the domain of convergence $0 < |z| < 1$ in the complex case, and we say p-adic polylogarithms as values of the polylogarithmic function $\mathrm{L}i_s^{(p)}$ for $z \in \mathbb{C}_p$ with $0 < |z|_v < 1$ in the p-adic case, respectively.

In 2003, T. Rivoal [10] showed a linear independence result of values of polylogarithmic function, by means of the linear independence criterion due to Yu. V. Nesterenko [6].

Theorem A [Rivoal] *Let s be an integer $\geqslant 2$. Let $\alpha = p/q \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$ and $0 < |\alpha| < 1$. For any $\varepsilon > 0$, there exists an integer $A(\varepsilon, p, q) \geqslant 1$ satisfying the following property. If $s \geqslant A(\varepsilon, p, q)$, we have*

$$\dim_{\mathbb{Q}} \{\mathbb{Q} + \mathbb{Q}\mathrm{L}i_1(\alpha) + \cdots + \mathbb{Q}\mathrm{L}i_s(\alpha)\} \geqslant \frac{1-\varepsilon}{1+\log 2} \log(s).$$

Hence it is followed:

Corollary B [Rivoal] *For any $\alpha \in \mathbb{Q}$ with $0 < |\alpha| < 1$, the set $\{\mathrm{L}i_s(\alpha) : s \geqslant 1\}$ contains infinitely many irrational numbers.*

R. Marcovecchio [5] generalized Rivoal's proof for algebraic number fields of higher degree, by means of simultaneous Padé approximation.

However, these results due to Rivoal and Marcovecchio do not imply the irrationality of any chosen polylogarithm. Our motivation is now to obtain examples of irrational or linear independent polylogarithms over $\mathbb{Q}$ or an algebraic number field. We basically refer the argument used in Nikišin in the complex case and that in P. Bel [1] in the p-adic case.

Here, we do not use Y. Nesterenko's p-adic linear independence criterion [7], instead, we follow a p-adic analogy of the proof of Nikišin with a modified remainder function. This is because we want to avoid in estimating " an integral " in the p-adic case.
The main advantage in the p-adic case is indeed that the valuation of a power series can be calculated in a formal way. Since the least common multiple costs much lower than in the complex case, we could show a better linear independence criterion for p-adic polylogarithms.

Padé approximation is a tool to approximate a transcendental function by rational functions. It is often used in a proof of the irrationality. We recall the standard proof that e is irrational. Suppose $e \in \mathbb{Q}$, then for a sufficiently large positive integer n, the number $n!e$ is an integer. Since we know $n!e = S_n + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots$ with $S_n = \left(2n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + 1\right) \in \mathbb{Z}$, we see the integer $n!e - S_n$ verifies $0 < n!e - S_n < 1$ which leads us to a contradiction. This way is summarized as follows: *let $\beta \in \mathbb{R}$. Suppose that there exist sequences of integers S_n, $T_n \to \infty$ with $T_n\beta - S_n \to 0$ as $n \to \infty$ but $T_n\beta - S_n \neq 0$ for n infinitely often. Then we conclude that β is irrational.* The construction of such integer sequences is in fact realized by putting integers in polynomials with integer coefficients. These polynomials are found as numerators and denominators of the rational functions, searched by Padé approximation. The most difficult part is to prove $T_n\beta - S_n \neq 0$.

2 New results

For $\alpha \in \overline{\mathbb{Q}}$, we write $K = \mathbb{Q}(\alpha), [K:\mathbb{Q}] = d = r_1 + 2r_2$. We put $\alpha^{(i)} (i = 1, \cdots, d)$ the conjugates of α over $\mathbb{Q}$.

Theorem 1 (with Y. Washio) *Let $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < 1$. Let b be the denominator of α^{-1}.*
Suppose

$$|\alpha| \times \prod_{i \neq \mathrm{Id}} \max\{1, \frac{1}{|\alpha^{(i)}|^s}\} < \frac{1}{b^{ds}} \exp\{-s(ds-1)(\log s + 2\log 2 + 1)\}.$$

Then the numbers $1, \mathrm{Li}_1(\alpha), \mathrm{Li}_2(\alpha), \cdots, \mathrm{Li}_s(\alpha)$ are linearly independent over $K = \mathbb{Q}(\alpha)$.

Theorem 2 (with S. David) *Let $v|p$. Consider $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha|_v < 1$. By b, we denote the denominator of α^{-1}.*
Suppose

$$|\alpha|_v^{n_v} \times \prod_{i=1}^{d} \max\{1, \frac{1}{|\alpha^{(i)}|}\} < \frac{1}{b^d} \exp\{-ds(\log s + 2\log 2 + 1)\}.$$

Then the numbers $1, \mathrm{Li}_1^{(p)}(\alpha), \mathrm{Li}_2^{(p)}(\alpha), \cdots, \mathrm{Li}_s^{(p)}(\alpha)$ are linearly independent over $K = \mathbb{Q}(\alpha)$.

2.1 Construction of suitable sequences

In this article, we show the proof of Theorem 1. Let $0 \le q \le s$, $q, s \in \mathbb{Z}$, $1 \le n \in \mathbb{Z}$. Fix a $z \in \mathbb{C}$, $|z| > 1$. For each q, we construct polynomials $A_{iq}(z) \in \mathbb{Q}[z]$ $(i = 1, 2, \cdots, s)$ and $P_q(z) \in \mathbb{Q}[z]$ such that $A_{iq}(z)$ $(i = 1, 2, \cdots, s)$ are not all identically zero, with suitable estimates for coefficients and

$$A_{1q}(z)\mathrm{Li}_1(1/z) + A_{2q}(z)\mathrm{Li}_2(1/z) + \cdots + A_{sq}(z)\mathrm{Li}_s(z) - P_q(z) = \frac{c_{0(q)}}{z^\sigma} + \frac{c_{1(q)}}{z^{\sigma+1}} + \cdots \quad (1)$$

with

$$\deg A_{jq}(z) \le n \ (j = 1, \cdots, q), \quad \deg A_{jq}(z) \le n-1 \ (j = q+1, \cdots, s)$$

where $\sigma = (n+1)q + n(s-q) = ns + q$.
For this, we use Siegel's lemma.

Thanks to

$$\int_0^1 x^{M-1} \left(\log \frac{1}{x}\right)^{k-1} dx = \frac{\Gamma(k)}{M^k} \qquad (M \in \mathbb{N}), \tag{2}$$

we obtain

$$\begin{aligned} \mathrm{Li}_k(1/z) &= \sum_{m \ge 1} \frac{z^{-m}}{m^k} = \sum_{m \ge 1} \frac{z^{-m}}{\Gamma(k)} \int_0^1 x^{m-1} \left(\log \frac{1}{x}\right)^{k-1} dx \\ &= \frac{1}{\Gamma(k)} \int_0^1 \left(\log \frac{1}{x}\right)^{k-1} \left(\sum_{m \ge 1} \frac{x^{m-1}}{z^m}\right) dx \\ &= \frac{1}{\Gamma(k)} \int_0^1 \frac{\left(\log \frac{1}{x}\right)^{k-1}}{z-x} dx, \end{aligned}$$

hence

$$\begin{aligned} A_{kq}(z)\mathrm{Li}_k(1/z) &= \frac{1}{\Gamma(k)} \int_0^1 \frac{A_{kq}(z)}{z-x} \left(\log \frac{1}{x}\right)^{k-1} dx \\ &= \frac{1}{\Gamma(k)} \int_0^1 \frac{A_{kq}(z) - A_{kq}(x)}{z-x} \left(\log \frac{1}{x}\right)^{k-1} dx + \frac{1}{\Gamma(k)} \int_0^1 \frac{A_{kq}(x)}{z-x} \left(\log \frac{1}{x}\right)^{k-1} dx. \end{aligned}$$

We then have

$$\sum_{k=1}^{s} A_{kq}(z)\mathrm{Li}_k(1/z) = \sum_{k=1}^{s} I_1^{(k,q)}(z) + \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log \frac{1}{x}\right)^{k-1} \frac{dx}{z-x}$$

where $I_1^{(k,q)}(z)$ is the first term of the right-hand side of the second line of the above identity. Setting $P_q(z) = \sum_{k=1}^{s} I_1^{(k,q)}(z)$, we get

$$\sum_{k=1}^{s} A_{kq}(z)\mathrm{Li}_k(1/z) - P_q(z) = \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log \frac{1}{x}\right)^{k-1} \frac{dx}{z-x} = \frac{c_{0(q)}}{z^{\sigma}} + \frac{c_{1(q)}}{z^{\sigma+1}} + \cdots.$$

The identity that we have made

$$\int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} \frac{dx}{z-x} = \frac{c_{0(q)}}{z^\sigma} + \frac{c_{1(q)}}{z^{\sigma+1}} + \cdots \tag{3}$$

has the form

$$\frac{1}{z}\int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} x^{1-1}dx + \cdots + \frac{1}{z^{\sigma-1}}\int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} x^{\sigma-1-1}dx$$
$$+\frac{1}{z^\sigma}\int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} x^{\sigma-1}dx + \cdots$$

concerning with the left-hand side of (3), thanks to the uniform convergence.

Thus we obtain for $\nu = 1, 2, \cdots, \sigma - 1$:

$$\int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} x^{\nu-1}dx = 0 \tag{4}$$

For $t \geq 1$ we define the function

$$R(t) = \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} x^{t-1}dx.$$

We now see that $R(t)$ is a rational function; indeed, putting

$$A_{kq}(x) = \sum_{j=0}^{n-\varepsilon_k} c_{kj}^{(q)} x^j$$

with

$$\varepsilon_k = \begin{cases} 0 & \text{for} \quad 1 \leq k \leq q, \\ 1 & \text{for} \quad q+1 \leq k \leq s. \end{cases}$$

We have

$$R(t) = \int_0^1 \left\{ \sum_{k=1}^{s} \frac{\sum_{j=0}^{n-\varepsilon_k} c_{kj}^{(q)} x^j}{\Gamma(k)} \left(\log \frac{1}{x}\right)^{k-1} \right\} x^{t-1} dx$$

$$= \sum_{k=1}^{s} \left\{ \frac{1}{\Gamma(k)} \sum_{j=0}^{n-\varepsilon_k} c_{kj}^{(q)} \frac{\Gamma(k)}{(t+j)^k} \right\}$$

$$= \sum_{k=1}^{s} \left\{ \sum_{j=0}^{n-\varepsilon_k} \frac{c_{kj}^{(q)}}{(t+j)^k} \right\}$$

$$= \sum_{k=1}^{q} \left\{ \sum_{j=0}^{n} \frac{c_{kj}^{(q)}}{(t+j)^k} \right\} + \sum_{k=q+1}^{s} \left\{ \sum_{j=0}^{n-1} \frac{c_{kj}^{(q)}}{(t+j)^k} \right\}$$

$$= \sum_{k=1}^{q} \left\{ \sum_{j=0}^{n} \frac{c_{kj}^{(q)}}{(t+j)^k} \right\} + \sum_{j=0}^{n-1} \left\{ \sum_{k=1}^{s} \frac{c_{kj}^{(q)}}{(t+j)^k} - \sum_{k=1}^{q} \frac{c_{kj}^{(q)}}{(t+j)^k} \right\}$$

$$= \sum_{k=1}^{q} \frac{c_{kn}^{(q)}}{(t+n)^k} + \sum_{j=0}^{n-1} \left(\sum_{k=1}^{s} \frac{c_{kj}^{(q)}}{(t+j)^k} \right).$$

Therefore the function

$$R(t) t^s (t+1)^s (t+2)^s \cdots (t+n-1)^s (t+n)^q$$

is a polynomial of degree not exceeding $ns+q-1 = \sigma - 1$ with $R(1) = R(2) = \cdots = R(\sigma - 1) = 0$. Thus we have

$$R(t) = \gamma \frac{(t-1)(t-2)\cdots(t-\sigma+1)}{t^s(t+1)^s \cdots (t+n-1)^s (t+n)^q},$$

with $\gamma \neq 0$. By normalizing the polynomial we may take $\gamma = 1$. Finally we get

$$R(t) = \sum_{j=0}^{n-1} \left(\sum_{k=1}^{s} \frac{c_{kj}^{(q)}}{(t+j)^k} \right) + \sum_{k=1}^{q} \frac{c_{kn}^{(q)}}{(t+n)^k} = \frac{(t-1)(t-2)\cdots(t-\sigma+1)}{t^s(t+1)^s \cdots (t+n-1)^s (t+n)^q}.$$

Lemma 1 *Put*

$$H_{nq}(z) = A_{1q}(z) E_1(z) + \cdots A_{sq}(z) E_s(z) - P_q(z).$$

There exists a constant $c > 0$ *such that for* $^\forall z \in \mathbb{C},\ |z| > 1$*:*

$$|H_{nq}(z)| \leq n^c \left(\frac{1}{|z|}\right)^{\sigma} \left(1 + \frac{1}{s}\right)^{-ns(s+1)}$$

Proof) We have

$$H_{nq}(z) = \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} \frac{dx}{z-x}.$$

Now let us be in the case $\left|\frac{x}{z}\right| < 1$ with $x \neq 0, x \in \mathbb{R}$. Then we have

$$\frac{1}{x-z} = \frac{1}{2iz}\int_{\mathrm{Re}(t)=\frac{1}{2}} \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin\pi(t-1)} \tag{5}$$

The function $f(t) = \dfrac{1}{\sin\pi(t-1)}$ has poles at $t-1 \in \mathbb{Z}$ of order 1. Then

$$\begin{aligned}
\mathrm{Res}_{t=n+1}\left(-\frac{x}{z}\right)^{t-1}\frac{1}{\sin\pi(t-1)} &= \lim_{t\to n+1}\frac{t-n-1}{\sin\pi(t-1)}\left(-\frac{x}{z}\right)^{t-1}\\
&= \lim_{h\to 0}\frac{h}{\sin\pi(n+h)}\left(-\frac{x}{z}\right)^{n+h}\\
&= \lim_{h\to 0}\frac{\pi h}{(-1)^n\sin\pi h}\cdot\frac{1}{\pi}\cdot\left(-\frac{x}{z}\right)^{n+h}\\
&= \frac{1}{\pi}\left(\frac{x}{z}\right)^n
\end{aligned}$$

By the residue formula, we have $\displaystyle\int_{L+C}\left(-\frac{x}{z}\right)^{t-1}\frac{dt}{\sin\pi(t-1)} = 2\pi i\cdot\frac{1}{\pi}\sum_{n=0}^{N-1}\left(\frac{x}{z}\right)^n$;

$$\begin{aligned}
&C: t = Ne^{i\theta} - \frac{1}{2} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)\\
&L: \mathrm{Re}(t) = \frac{1}{2} \quad (|\mathrm{Im}(t)| \le N).
\end{aligned}$$

For the half circle C, we have

$$\begin{aligned}
\left|\sin\pi\left(Ne^{i\theta}+\frac{1}{2}-1\right)\right| &= \left|\sin\pi\left(Ne^{i\theta}+\frac{1}{2}\right)\right| = |\cos(\pi Ne^{i\theta})|\\
&= |\cos(\pi N\cos\theta + i\pi N\sin\theta)|\\
&= \frac{1}{2}\left|e^{i\pi N\cos\theta}e^{-\pi N\sin\theta} + e^{-i\pi N\cos\theta}e^{\pi N\sin\theta}\right|\\
&\ge \frac{1}{2}\left|e^{-\pi N\sin\theta} - e^{\pi N\sin\theta}\right|.
\end{aligned}$$

Then

$$\left|\int_C \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin\pi(t-1)}\right| = \left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\frac{x}{z}\right)^{Ne^{i\theta}+\frac{1}{2}-1} \frac{1}{\sin\pi(Ne^{i\theta}+\frac{1}{2}-1)} \cdot Nie^{i\theta} d\theta\right|$$

$$= \left|\frac{x}{z}\right|^{N-\frac{1}{2}} \cdot 4N \int_0^{\frac{\pi}{2}} \frac{d\theta}{e^{\pi N \sin\theta} - e^{-\pi N \sin\theta}} \longrightarrow 0 \quad (N \to \infty).$$

Thus

$$-\int_{\mathrm{Re}(t)=\frac{1}{2}} \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin\pi(t-1)} = 2i \cdot \frac{1}{1-\frac{x}{z}} = 2iz \cdot \frac{1}{z-x}.$$

Since $0 < x \leq 1$ and $|z| > 1$,

$$H_{nq}(z) = \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1} \left\{-\frac{1}{2iz}\int_{\mathrm{Re}(t)=\frac{1}{2}} \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin\pi(t-1)}\right\} dx$$

$$= \frac{1}{2iz}\int_{\mathrm{Re}(t)=\frac{1}{2}} \left\{\int_0^1 \left(\sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log\frac{1}{x}\right)^{k-1}\right) \left(-\frac{x}{z}\right)^{t-1} dx\right\} \frac{dt}{\sin\pi t}$$

$$= \frac{1}{2iz}\int_{\mathrm{Re}(t)=\frac{1}{2}} R(t) \left(-\frac{1}{z}\right)^{t-1} \frac{dt}{\sin\pi t}.$$

Shifting the line of the integration, we get

$$H_{nq}(z) = \frac{1}{2iz}\int_{\mathrm{Re}t=\sigma-\frac{1}{2}} R(t) \left(-\frac{1}{z}\right)^{t-1} \frac{1}{\sin\pi t} dt.$$

For $t = \sigma - \dfrac{1}{2} + iw$ we then have

$$\left|R(t)\left(-\frac{1}{z}\right)^{t-1} \frac{1}{\sin\pi t}\right| = \left|\frac{z^{\frac{1}{2}-iw}}{z^{\sigma}}\right| \left|\frac{\Gamma(t)}{\Gamma(t-\sigma+1)\sin\pi t}\right| \left|\frac{1}{t^s(t+1)^s\cdots(t+n-1)^s(t+n)^q}\right|$$

$$\leq \frac{O(n^c)}{|z^{\sigma}|} \left|\frac{\Gamma(t)}{\Gamma(t-\sigma+1)\sin\pi t}\right| \left|\frac{1}{t^s(t+1)^s\cdots(t+n-1)^s}\right|.$$

Finally we obtain

$$\left|R(t)\left(-\frac{1}{z}\right)^{t-1} \frac{1}{\sin\pi t}\right| \leq \frac{O(n^c)}{|z^{\sigma}|} e^{-\frac{\pi}{2}|w|} \left(\frac{ns+s}{ns+n}\right)^{ns(s+1)}$$

$$\leq \frac{O(n^c)}{|z^{\sigma}|} e^{-\frac{\pi}{2}|w|} \left(1+\frac{1}{s}\right)^{-ns(s+1)}.$$

Therefore

$$|H_{nq}(z)| \le \frac{1}{|2z|}\int_{\mathrm{Re}(t)=\sigma-\frac{1}{2}} \frac{n^c}{|z|^\sigma} e^{-\frac{\pi}{2}|w|}\left(1+\frac{1}{s}\right)^{-ns(s+1)} dt \le \frac{n^c}{|z|^\sigma}\left(1+\frac{1}{s}\right)^{-ns(s+1)}. \quad \square$$

Lemma 2 *Let d_n be the least common multiple of the numbers $1,2,\cdots,n$. Then for any $1 \le k \le s$ the number $d_n^{s-k} c_{kj}^{(q)} \in \mathbb{Z}$ for $j = 0,1,\cdots,n$ (here $c_{kn}^{(q)} = 0$ for $k > q$).*

Proof) Proven as in [8] or [10].

Lemma 3 *The determinant of the following matrix satisfies:*

$$\Delta(z) = \begin{vmatrix} A_{10}(z) & A_{20}(z) & \cdots & A_{s0}(z) & P_0(z) \\ A_{11}(z) & A_{21}(z) & \cdots & A_{s1}(z) & P_1(z) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1s}(z) & A_{2s}(z) & \cdots & A_{ss}(z) & P_s(z) \end{vmatrix} \equiv \text{constant} \neq 0.$$

Proof) For $q = 0,1,\cdots,s$, we have

$$\Delta_q(z) = (-1)^{q+s} \begin{vmatrix} A_{10}(z) & A_{20}(z) & \cdots & A_{s0}(z) \\ A_{11}(z) & A_{21}(z) & \cdots & A_{s1}(z) \\ \vdots & \vdots & & \vdots \\ A_{1,q-1}(z) & A_{2,q-1}(z) & \cdots & A_{s,q-1}(z) \\ A_{1,q+1}(z) & A_{2,q+1}(z) & \cdots & A_{s,q+1}(z) \\ \vdots & \vdots & & \vdots \\ A_{1s}(z) & A_{2s}(z) & \cdots & A_{ss}(z) \end{vmatrix} \tag{6}$$

where $\Delta_q(z)$ is the co-factor for the $(q, s+1)$-th element.
We have $\deg \Delta_q(z) \le (n-1)+n+\cdots+n = ns-1 \quad (q \neq 0)$, and for $q = 0$, $\deg \Delta_0(z) = ns$.
Let β be the product of the leading coefficient of $A_{qq}(z) \quad (q = 0,1,\cdots,s)$. Then

$$\Delta(z) = \sum_{q=0}^{s} P_q(z)\Delta_q(z) = -\left(\beta c_{0(0)} + \frac{h_1}{z} + \frac{h_2}{z^2} + \cdots\right).$$

This is because

$$
\begin{array}{lrcl}
 & A_{10}(z)\mathrm{Li}_1(1/z)\Delta_0(z)+\cdots+A_{s0}(z)\mathrm{Li}_s(1/z)\Delta_0(z)-P_0(z)\Delta_0(z) & = & \Delta_0(z)\left(\dfrac{c_{0(0)}}{z^{ns}}+\cdots\right) \\
 & A_{12}(z)\mathrm{Li}_1(1/z)\Delta_1(z)+\cdots+A_{s1}(z)\mathrm{Li}_s(1/z)\Delta_1(z)-P_1(z)\Delta_1(z) & = & \Delta_1(z)\left(\dfrac{c_{0(1)}}{z^{ns+1}}+\cdots\right) \\
 & & \vdots & \\
+) & A_{1s}(z)\mathrm{Li}_1(1/z)\Delta_s(z)+\cdots+A_{ss}(z)\mathrm{Li}_s(1/z)\Delta_s(z)-P_s(z)\Delta_s(z) & = & \Delta_s(z)\left(\dfrac{c_{0(s)}}{z^{ns+s}}+\cdots\right) \\
\hline
 & D-\displaystyle\sum_{q=0}^{s}P_q(z)\Delta_q(z) & = & \beta c_{0(0)}+\dfrac{h_1}{z}+\cdots
\end{array}
$$

where

$$
\begin{aligned}
D &= (A_{10}(z)\Delta_0(z)+\cdots+A_{1s}(z)\Delta_s(z))\mathrm{Li}_1(1/z)+\cdots \\
&= (-1)^s\left\{A_{10}(z)\begin{vmatrix} {}_{11}(z) & \cdots & A_{s1}(z) \\ \vdots & & \vdots \\ A_{1s}(z) & \cdots & A_{ss}(z)\end{vmatrix}+\cdots+(-1)^s A_{1s}(z)\begin{vmatrix} A_{10}(z) & \cdots & A_{s0}(z) \\ \vdots & & \vdots \\ A_{1,s-1}(z) & \cdots & A_{s,s-1}(z)\end{vmatrix}\right\}+\cdots \\
&= (-1)^s\begin{vmatrix} A_{10}(z) & A_{10}(z) & \cdots & A_{s0}(z) \\ A_{11}(z) & A_{11}(z) & \cdots & A_{s1}(z) \\ \vdots & \vdots & & \vdots \\ A_{1,s-1}(z) & A_{1,s-1}(z) & \cdots & A_{s,s-1}(z) \\ A_{1s}(z) & A_{1s}(z) & \cdots & A_{ss}(z)\end{vmatrix}+\cdots \\
&= 0.
\end{aligned}
$$

Let us take $\alpha \in \overline{\mathbb{Q}}$ $(0<|\alpha|<1)$ such that

$$
|\alpha| \prod_{v|\infty, v\neq \mathrm{Id}} \max\{1,|\alpha|_v^{-s}\} < \frac{1}{b^{s(r_1+2r_2)}}\exp\{-s(sr_1+2sr_2-1)(\log s+2\log 2+1)\}. \tag{7}
$$

Here we set

$$
\ell = x_1\mathrm{Li}_1(\alpha)+\cdots+x_s\mathrm{Li}_s(\alpha)-x_0 \quad (x_i\in \mathfrak{o}_K,\ K=\mathbb{Q}(\alpha)),
$$

with $(x_1,\cdots,x_s,x_0)\neq(0,\cdots,0,0)$.

Putting $T_{kq}(z) = b^n d_n^s A_{kq}(z)$, $\Phi_q(z) = b^n d_n^s P_q(z)$, we have

$$\Delta := \det \begin{pmatrix} T_{10}(\alpha^{-1}) & T_{20}(\alpha^{-1}) & \cdots & T_{s0}(\alpha^{-1}) & \Phi_0(\alpha^{-1}) \\ T_{11}(\alpha^{-1}) & T_{21}(\alpha^{-1}) & \cdots & T_{s1}(\alpha^{-1}) & \Phi_1(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ T_{1,q-1}(\alpha^{-1}) & T_{2,q-1}(\alpha^{-1}) & \cdots & T_{s,q-1}(\alpha^{-1}) & \Phi_{q-1}(\alpha^{-1}) \\ x_1 & x_2 & \cdots & x_s & x_0 \\ T_{1,q+1}(\alpha^{-1}) & T_{2,q+1}(\alpha^{-1}) & \cdots & T_{s,q+1}(\alpha^{-1}) & \Phi_{q+1}(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ T_{1s}(\alpha^{-1}) & T_{2s}(\alpha^{-1}) & \cdots & T_{ss}(\alpha^{-1}) & \Phi_s(\alpha^{-1}) \end{pmatrix} \tag{8}$$

thus as we have seen before, we get $\Delta \neq 0$. Recall that $x_0, \cdots, x_s$ are algebraic integers in K.

By the linear algebra, we have

$$\Delta = \pm\det \begin{pmatrix} T_{10}(\alpha^{-1}) & T_{20}(\alpha^{-1}) & \cdots & T_{s0}(\alpha^{-1}) & s!b^n d_n^s H_{n0}(\alpha^{-1}) \\ T_{11}(\alpha^{-1}) & T_{21}(\alpha^{-1}) & \cdots & T_{s1}(\alpha^{-1}) & s!b^n d_n^s H_{n1}(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ T_{1,q-1}(\alpha^{-1}) & T_{2,q-1}(\alpha^{-1}) & \cdots & T_{s,q-1}(\alpha^{-1}) & s!b^n d_n^s H_{n,q-1}(\alpha^{-1}) \\ x_1 & x_2 & \cdots & x_s & \ell \\ T_{1,q+1}(\alpha^{-1}) & T_{2,q+1}(\alpha^{-1}) & \cdots & T_{s,q+1}(\alpha^{-1}) & s!b^n d_n^s H_{n,q+1}(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ T_{1s}(\alpha^{-1}) & T_{2s}(\alpha^{-1}) & \cdots & T_{ss}(\alpha^{-1}) & s!b^n d_n^s H_{ns}(\alpha^{-1}) \end{pmatrix}. \tag{9}$$

Since $\Delta \in \mathfrak{o}_K$, we have $\prod_{v|\infty} |\Delta|_v^{n_v} \geq 1$.

$$\begin{aligned} \therefore 1 \leq \prod_{v|\infty} |\Delta|_v^{n_v} &= |\Delta| \cdot \prod_{v|\infty, v\neq \mathrm{Id}} |\Delta|_v^{n_v} \\ &\leq \left\{ \sum_{j\neq q} |b^n d_n^s H_{nj}(\alpha^{-1})| \left(\max_{i,j} |T_{ij}(\alpha^{-1})| \right)^{s-1} \max_{1\leq\mu\leq s} |x_\mu| + |\ell| \left(\max_{i,j} |T_{ij}(\alpha^{-1})| \right)^s \right\} \\ &\times \prod_{v|\infty, v\neq \mathrm{Id}} \left\{ \sum_{j\neq q} |\Phi_j(\alpha^{-1})|_v \left(\max_{i,j} |T_{ij}(\alpha^{-1})|_v \right)^{s-1} \max_{1\leq\mu\leq s} |x_\mu|_v + |x_0|_v \left(\max_{i,j} |T_{ij}(\alpha^{-1})|_v \right)^s \right\}^{n_v}. \end{aligned}$$

Now we start to prove our theorem. Suppose $\max_\mu |x_\mu| \neq 0$ and we are going to show the linear combination $\ell \neq 0$.

For the first term , we have

$$\sum_{j\neq q}|b^n d_n^s H_{nj}(\alpha^{-1})|\left(\max_{i,j}|T_{ij}(\alpha^{-1})|\right)^{s-1}\max_{1\le\mu\le s}|x_\mu|+|\ell|\left(\max_{i,j}|T_{ij}(\alpha^{-1})|\right)^s$$

$$\le O(n^c)b^{ns}\exp\{ns^2\}|\alpha|^n\cdot\exp\{-ns\}\exp\{ns(s-1)(\log s+2\log 2)\}+|\ell|\left(\max_{i,j}|T_{ij}(\alpha^{-1})|\right)^s$$

$$=O(n^c)b^{ns}|\alpha|^n\exp\{ns(s-1)(\log s+\log 2+1)\}+|\ell|\left(\max_{i,j}|T_{ij}(\alpha^{-1})|\right)^s \quad \cdots\cdots(\mathrm{A}).$$

For the second term , we have

$$\prod_{v|\infty,v\neq\mathrm{Id}}\left\{\sum_{j\neq q}|\Phi_j(\alpha^{-1})|_v\left(\max_{i,j}|T_{ij}(\alpha^{-1})|_v\right)^{s-1}\max_{1\le\mu\le s}|x_\mu|_v+|x_0|_v\left(\max_{i,j}|T_{ij}(\alpha^{-1})|_v\right)^s\right\}^{n_v}$$

$$\le\prod_{i=2}^{r_1+2r_2}(b^n d_n^s)^s n^{cs}\max\{1,|\alpha_i^{-1}|\}^{ns}\exp\{ns^2(\log s+\log 2)\}\left(s\max_\mu|x_\mu^{(i)}|+|x_0^{(i)}|\right) \quad \cdots\cdots(\mathrm{B}).$$

Combining (A) and (B), we have

$$1\le\left\{O(n^c)b^{ns}|\alpha|^n\exp\{ns(s-1)(\log s+\log 2+1)\}+|\ell|\left(\max_{i,j}|T_{ij}(\alpha^{-1})|\right)^s\right\}$$
$$\times\prod_{i=2}^{r_1+2r_2}O(n^c)b^{ns}\max\{1,|\alpha_i^{-1}|^{ns}\}\exp\{ns^2(\log s+\log 2+1)\}$$

Suppose $\ell=0$. Then we have by taking $1/n$-th power,

$$1\le O(n^{cd})^{\frac{1}{n}}b^{s(r_1+2r_2)}\exp\{s(sr_1+2sr_2-1)(\log s+\log 2+1)\}\times|\alpha|\prod_{i=2}^{r_1+2r_2}\max\{1,|\alpha_i|^{-s}\} \qquad (10)$$

By our assumption, we obtain the contradiction since the right-hand side <1 for sufficiently large n. Thus $\ell\neq 0$.

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