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Chaos and multi-fractal nature in a Boussinesq magnetoconvection

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Abstract. In order to find multi-fractal extreme events from the viewpoint of the predictability and the nonlinear feedback mechanism, we investigate numerically a two-dimensional magnetoconvection near the Hopf bifurcation point. We find multi-fractal nature of intermittent chaotic magnetic fluctuations and a non-Gaussian distribution of the magnetic fluctuations which suggests the unpredictability of the extreme events. We show that a power law for a return interval distribution of extreme events can describe the multi-fractal nature of intermittent chaos.

Keywords: Chaos, Multi-fractal, Magnetoconvection, Intermittency, Hopf-bifurcation

Introduction

We can easily find a self-similarity for chaotic fluctuations in natural phenomena and for market fluctuations [1]. Price changes in finance, especially, are well known as a typical representative of self-similarity [2]. It is believed that all price-changes are distributed in a pattern that conforms to the standard bell curve (the Gauss distribution). The width of the bell shape (the standard deviation) depicts how far price-changes diverge from the mean. It is also well known that prices quoted in any financial market often change with heart-stopping swiftness and that they have never been predicted by any financial models yet. It is impossible to find a rigorous quantitative description of major financial upheavals, which are precipitous price jumps and intermittent extreme events. Events in the extremes are considered extremely rare as in the occurrence of the huge Tsunami (the Great East Japan Earthquake, March 11^{th}, 2011) as well as in the economic recession (the Lehman Brothers Shock, 2008).

It is important to understand statistical properties of extreme events, which show generally intermittent and multi-fractal properties [3]. The consequences of extreme events are very serious [4]. Therefore, by means of simple physical model, we investigate numerically some properties of multi-fractal extreme events from the viewpoint of the predictability and the feedback mechanism. In the present paper, we study numerically multi-fractal extreme events observed in a nonlinear magnetohydrodynamic (MHD) system of the Solar convection zone modeled by a 2-dimensional Boussinesq magnetoconvection [5,6], which is related to the fully developed fluid turbulence and intermittency. The magnetoconvection system can be reduced to the fifth-order system of nonlinear ordinary differential equations near the Hopf bifurcation point of the order parameter [6]. By use of the center manifold theory, a set of normal form equations was derived from the fifth-order system in order to understand numerical solutions qualitatively [6]. The normal form equations revealed that one attractor of the fifth-order system could be described by both the Duffing and Van der Pol equations [7]. The normal form equation generates the Melnikov function to analyze numerical solutions in codimension-two bifurcation [7].

In general, intermittency is a universal phenomenon due to the spatio-temporal inhomogeneity of stochastic fluctuations in the MHD turbulence, solar wind turbulence, a human heartbeat [8] and a financial dynamics [2]. A drastic transition to turbulence via spatio-temporal intermittency is one of the great interesting problems related to anomalous transport of drift wave turbulence [9] and Rayleigh-Bénard convection but not fully understood yet.

The problem of Rayleigh-Bénard convection interacting with a magnetic field, which is called magnetoconvection [5], has been investigated by many authors. In general, it is difficult to investigate analytically and numerically both spatial and temporal intermittency for fully developed turbulence. However, a strong external magnetic field enables us to freeze coherently a spatial pattern of magnetoconvection and to handle transition to chaos via a temporal intermittency near a critical Rayleigh number. In fact, a temporal intermittency with spatial coherent roll has been found in a two-dimensional Boussinesq magnetoconvection near a critical Rayleigh number [10]. The large deviation of magnetic fluctuations in drift wave turbulence has been considered as an anomalous transport, which is of interest to magnetically confined plasmas because it is the key issue in drift wave turbulence [9]. For this reason, the statistical properties of the anomalous transport have been intensely studied by many authors. However, it has been difficult to explain the occurrence of extremely large magnetic and thermal fluctuations by estimating the linear MHD turbulence theory.

Here we demonstrate the statistical properties of return interval distribution of intermittent extreme events of magnetic fluctuations in a 2-D magnetoconvection, which are related to a power-law like in the Weibull-class distribution [12]. In addition, by means of multi-fractal analysis, we discuss consistently the relation between scaling exponents. Statistical properties of intermittent chaotic magnetoconvection may serve as a keystone of deep understandings for nonlinear physical phenomena and financial dynamics.

Numerical Simulation

We consider a horizontally stratified fluid layer of characteristic depth, in a Cartesian coordinate system with the z axis pointing vertically upwards and the x axis horizontally rightwards in the two-dimensional case. It is convenient to measure lengths in terms of the layer depth, time in terms of the thermal relaxation time, temperature in terms of a vertical temperature difference and magnetic fields in terms of the imposed field. A particular configuration is specified by five dimensionless parameters for magnetoconvection in the normalized region ($0 < x < L$ and $0 < z < 1$) driven by a vertical temperature difference; these are the Rayleigh number R, the Chandrasekhar number Q, the viscous Prandtl number Pr, the magnetic Prandtl number ζ and the normalized cell width L, respectively. Henceforth all the perturbed quantities will be expressed in dimensionless forms. Then, a two-dimensional Boussinesq magnetoconvection can be described by the following three scalar fields, stream function $\phi(x,z,t)$, thermal fluctuation $\theta(x,z,t)$ and magnetic flux function $A(x,z,t)$;

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial}{\partial t}\nabla^2\phi + \{\phi,\nabla^2\phi\}\right) = R\frac{\partial\theta}{\partial x} + \nabla^4\phi + \varsigma Q\left(\frac{\partial}{\partial z}\nabla^2 A + \{A,\nabla^2 A\}\right), \tag{1}$$

$$\frac{\partial\theta}{\partial t} + \{\phi,\theta\} = \nabla^2\theta + \frac{\partial\phi}{\partial x}, \tag{2}$$

$$\frac{\partial A}{\partial t} + \{\phi,A\} = \varsigma\nabla^2 A + \frac{\partial\phi}{\partial z}, \tag{3}$$

where $\{f, g\}$ denotes the Poisson bracket. The third term in the right hand side of (1) denotes the Lorentz force and plays a nonlinear feedback role.

From the linear theory of the Boussinesq magnetoconvection [6], we can obtain the Rayleigh number for the onset of convection in the absence of the magnetic field $R_0 = \beta^6/\alpha^2$, where α ($= \pi/L$) denotes the horizontal wave number and $\beta^2 = \pi^2 + \alpha^2$. The Rayleigh number for the onset of overstability is written as $R^{(o)} = (Pr + \zeta)((1 + \zeta)/Pr + \zeta q/(1 + Pr))R_0$, where $q = \pi^2 Q/\beta^4$ and $R^{(e)} = (1 + q)R_0$. The Hopf bifurcation occurs linearly at $R^{(o)}$ when $R^{(o)} < R^{(e)}$ and $\zeta < 1$. We have integrated the PDEs numerically as an initial-value problem with small disturbances after the solutions of PDEs (1)-(3) were expanded by Fourier series resolved by 64×64 grid points [10]. All the numerical results of interest were almost independent of the choice of initial conditions if we ignored the initial transient period. We used the fourth-order Runge-Kutta scheme with appropriately chosen time-steps. Numerical simulations were carried out using the two-dimensional pseudo-spectral code based on the code of the three-dimensional drift wave turbulence [13]. The Rayleigh number r ($= R/R_0$) was mainly chosen as the order parameter of the PDEs (1)-(3). In order to observe a transition to chaos via a temporal intermittency with spatial coherent roll of magnetoconvection, we chose a set of parameters as follows: $Pr = 1$, $\zeta = 0.1$, $q = 250$ ($Q = 9870$) and $L = 1$ ($R_0 = 8\pi^4$). The branch of the periodic solution of the PDEs bifurcates from $R^{(o)} = 15R_0$. The periodic solution of the PDEs is stable for $15 < r < 69$ and finally transits from the periodic solution to chaos via temporal intermittency near the critical bifurcation point $r_c = 69$.

Finally we observed temporal intermittent chaos of magnetic fluctuations near the normalized critical Rayleigh number (r=69.5) [10] and obtained the time-series data of magnetic fluctuations [11], which can be assumed to be a sequence of random variable;

$$x(t) = E_M(t) = \frac{1}{2L}\zeta Q \int_0^L \int_0^1 |\nabla A(x,z,t)|^2 \, \mathrm{d}x\,\mathrm{d}z, \tag{4}$$

$$X_t(s) := \frac{1}{s}\int_t^{t+s} \left(E_M(\tau) - \bar{E}_M\right) \mathrm{d}\tau,$$

where $\bar{E}_M = \langle E_M(t) \rangle$ and the bracket denotes the temporal average.

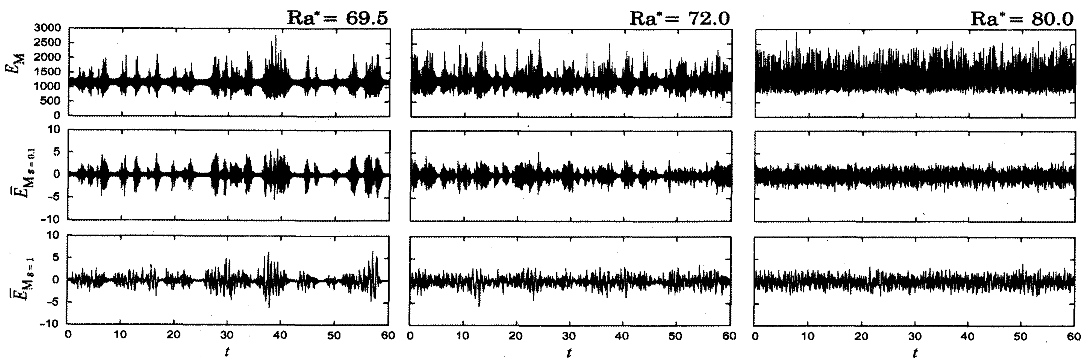

Fig. 1: A part of time-series data of intermittent magnetic fluctuations near the critical Hopf bifurcation point on different timescales s (s=0.1, s=1.0) for (a) r=69.5, (b) r=72 and (c) r=80, respectively (from left to right).

Typical intermittent time series data showed that the clustered behavior of large deviations from the average value of magnetic fluctuations results in a non-Gaussian PDFs with fat tails [11] which mean existence of strong correlations in variance fluctuations.

Figure 1 shows a part of time series data of intermittent chaotic magnetic fluctuations near the critical Rayleigh number [10]. We can clearly observe a temporal intermittency and its self-similarity in the two-dimensional magnetoconvection. In the next section, we show that the time series data of intermittent chaos has an autocorrelation function with a power law form. We can assume here that the time series data of Eq. (4) is a realization from a stationary stochastic process with long memory.

A non-Gaussian PDF with fat tails can be modeled by multiplicative stochastic process:

$$X_t(s) = \xi(t)\exp[\omega(t)], \qquad (5)$$

where ξ and ω are both Gaussian random variables with zero mean and variance σ^2 and λ^2, respectively. The PDF of magnetic fluctuations has fat tail depending on the variance λ^2 and is expressed by

$$P\left(X_t(s)\right) = \int F\left(\frac{X_t(s)}{\sigma}\right)\frac{1}{\sigma}G\left(\ln\sigma\right)\mathrm{d}(\ln\sigma), \qquad (6)$$

where $F(\xi)$ and $G(\omega)$ are both Gaussian random variables with zero mean and variance σ^2 and λ^2, respectively. Figure 2 shows standardized PDFs on scales s = 0.1(□), s=0.3(circle) and s=1.0(△) (from top to bottom) together with approximated PDFs (solid lines) by Eq. (6). Such a non-Gaussian distribution with a fat tail results from a series of randomly clustered bursts of fluctuations. We can clearly observe a logarithmic decay of λ^2 for r = 69.5, which means a slow convergence to a Gaussian in comparison with independent and identically distributed non-Gaussian process [11].

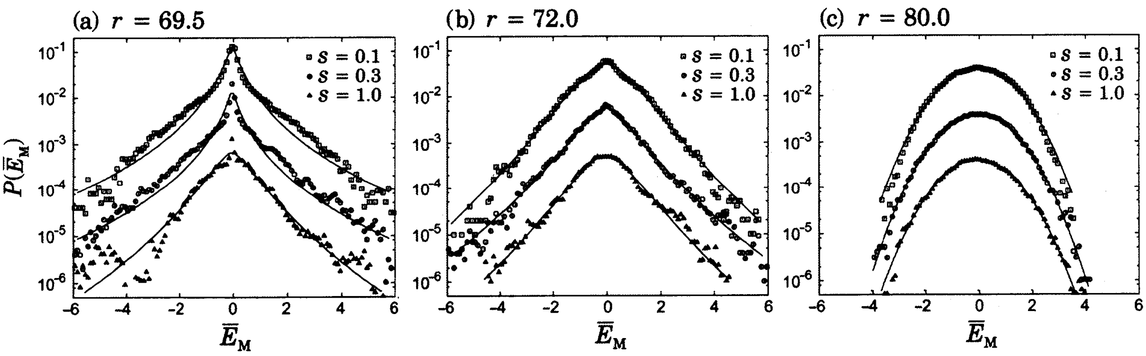

Fig. 2: Standardized probability density functions (PDF) of magnetic fluctuations on different timescales s approximated by Lognormal distribution for (a) r=69.5, (b) r=72 and (c) r=80, respectively (from left to right). The PDFs on scales s=0.1(□), s=0.3(circle) and s=1.0(△), respectively (from top to bottom). The PDFs for r = 80 show almost a Gaussian.

Return-Interval Analysis

If a random time series exhibits a long-range correlation, an autocorrelation function may be given by

$$C(\tau) = \langle x(t)x(t+\tau) \rangle \sim \tau^{-\gamma}, \qquad (7)$$

where τ is the time-lag and γ denotes the autocorrelation exponent ($0< \gamma <1$). Also, a structure function is given by

$$S_2 = \left\langle |x(t)x(t+\tau)|^2 \right\rangle \sim \tau^{2H}, \qquad (8)$$

where H is the Hurst exponent with the fixed moment of our interest.

Let us call an event extreme if a random variable is larger than some threshold value; $x(t) > q$ where q is some threshold value. The return interval τ is the time between successive occurrences of extreme events. Assuming that a time series of random data is sampled at discrete intervals, with respect to some threshold value q, we have a series of return intervals, as is shown in Fig. 3. At any instant $t = \tau$ if $x\ (t = \tau) > q$, it exceeds the threshold value q, then it is taken to be an extreme event. The recurrence interval distribution of earthquakes above some large magnitude may be a power law type. In fact, the Omori law for large earthquakes qualitatively explains the empirical seismic data [3].

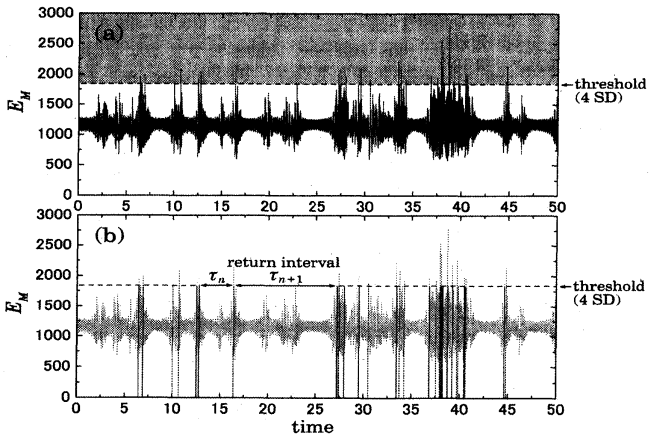

Fig. 3: The return intervals of magnetic fluctuations for the larger threshold q=4σ. This indicates that extreme events will be fewer when the threshold value is higher.

The long-range correlation affects the return interval distribution of extreme events. For the given threshold q, a probability model [3] to find an extreme event at time $t = \tau$ may be given by

$$P_{ex} = a\tau^{-(2H-1)} = a\tau^{-(1-\gamma)}, \qquad (9)$$

where H $(1/2<H<1)$ is the Hurst exponent, γ the autocorrelation exponent, and a a normalization constant. Here we have used the well-known relation between H and γ : $\gamma=2-2H$. It is noted that the Hurst exponent $H=1/2$ means an independent and identically distribution and the Brownian motion. We have assumed the probability (9) to be some continuous function of τ with the aid of renewal process [3], which is related to the cumulative distribution function. Let $F(x)=P[Z<x]$ be the cumulative probability distribution of some random variable Z. The random variable Z is said to be of power law type when x is large enough,

$$1-F(x)=Cx^{-\mu}, \qquad (10)$$

where C is a positive normalization constant and the exponent is called the tail exponent. From Eqs. (9) and (10), we have a probability model for the return interval of extreme events:

$$P_{ex}=P[t>\tau], \qquad (11)$$

where t is some random variable. This model can be derived from the Weibull-type distribution [12].

It is expected that the extreme events will be fewer if the threshold is larger and hence the return intervals will be longer. Thus, larger threshold leads to larger average return intervals. Since the power law regime of extreme events varies with the threshold value, we have a question that the return interval distribution may be modified by changes in threshold value. Therefore, we investigate here the relation between the return interval distribution and the threshold which defines the extreme events for the intermittent time series data of magnetoconvection.

In order to study the relation between the return interval distribution and the threshold for various threshold values, we have plotted the return interval distribution for the data in a log-log plot as shown in Fig. 4, and measured each slope in a linear region of the log-log plot when there exists a linear region in it. Here we have adopted that the maximum threshold value is 4.5σ, where σ denotes the standard deviation of magnetic fluctuations. It is noted that the number of return intervals are not sufficient for reliable statistics beyond q = 4.5σ in our case. It seems that a value of the linear slope for larger threshold converges to a certain constant in Fig. 4. We can observe that the return interval distribution is closer to a straight line with slope ($2H$-1) in Eq. (9) as the threshold increases. It is shown numerically that the slope of log-log plot increases monotonically to reach a saturation value ($2H$-1) when the threshold value q is larger than 4σ. It is confirmed that Eq. (9) represents a return interval distribution in the limit when the threshold is large. When the threshold value is smaller than 4σ, we could not find any linear slopes nor a power law type distribution (9), as is seen in Fig. 4. Thus, it is shown that the return interval distribution of extreme events is almost independent of the threshold value q as long as q is enough large in our case, although the power law regime for extreme events varies with the threshold value in general case.

In fact, Fig. 5 shows the cumulative distribution of extreme events of the intermittent chaos for the large threshold q=4σ and the tail exponent μ=0.34 in Eq. (10). It is confirmed that the probability of return interval of extreme events indicates a power-law for the larger threshold value. This suggests that the Weibull-type distribution is a good representation for the return interval distribution of extreme events in our intermittent chaos.

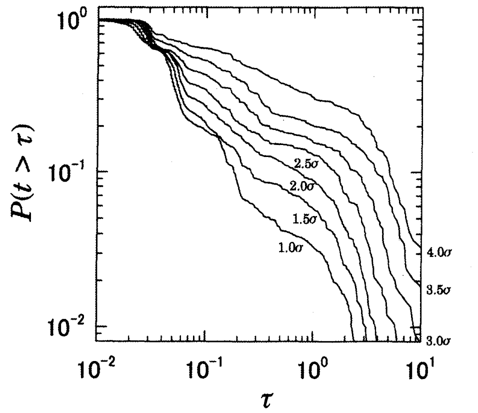

Fig. 4: Return interval distribution for the various values of threshold, $\sigma < q < 4.5\sigma$, where σ denotes the standard deviation of magnetic fluctuations. We can observe that the return interval distribution is closer to a straight line with the slope of the log-log plot when the threshold value is larger than 4σ.

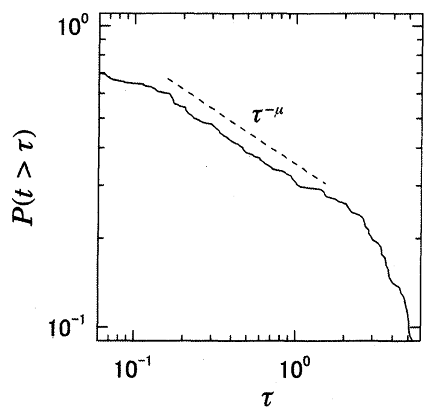

Fig. 5: Return interval distribution of extreme events for the larger threshold value 4σ, where σ denotes the standard deviation of magnetic fluctuations. We can clearly observe the tail exponent.

From Eqs. (9), (10) and (11), we obtain approximately the Hurst exponent H=0.67, the autocorrelation exponent γ=0.66, and the tail exponent μ=0.34. Since this Hurst exponent is larger than 0.5, our intermittent chaos of magnetic fluctuations does not describe the Brownian motion, and it has a certain long-range correlation. Our intermittent chaos of magnetic fluctuations may be described a non-Gaussian distribution [10] such as a multiplicative stochastic process [11].

Multi-Fractal Analysis

In this section, we study the multi-fractal properties of the intermittent chaos observed in the 2-D magnetoconvection. First, let us briefly review the multifractal formalism [3]. The simplest way to performing a multi-fractal analysis of a singular measure is to partition its support using boxes of size ε. Then the measure in each ε box can be characterized by a singular strength α according to its scaling behavior as follows:

$$\mu_i \sim \varepsilon^{\alpha_i}, \qquad (12)$$

where the index i denotes the box location. The number $N_\alpha(\varepsilon)$ of occurrences of a particular α defines the $f(\alpha)$ singularity spectrum:

$$N_\alpha(\varepsilon) \sim \varepsilon^{-f(\alpha)}. \qquad (13)$$

The partition function is given by

$$Z(q,\varepsilon) = \sum [\mu_i]^q \sim \varepsilon^{\tau(q)}, \qquad (14)$$

where q denotes the moment index that corresponds to the inverse of temperature in thermodynamics, instead of the threshold or the normalized Chandrasekhar number in the previous sections. Moreover, the generalized fractal dimension is also given by

$$D_q = \frac{\tau(q)}{q-1}. \qquad (15)$$

Equation (15) can be extracted from the power law behavior of the partition function (14). It is noted that the $\tau(q)$ in Eq. (14) is different from the time lag in Eq. (7). We obtain from the definition of $\tau(q)$ by the Legendre transforming the $f(\alpha)$ singularity spectrum,

$$\tau(q) = q\alpha - f(\alpha), \qquad (16)$$

and the relation containing the generalized Hurst exponent $h(q)$,

$$f(\alpha) = q[\alpha - h(q)] + 1, \qquad (17)$$

where

$$\alpha = h(q) + q\frac{dh(q)}{dq}.$$

For the moment $q = \pm\infty$, we can deduce from Eqs. (15) and (16),

$$D_{+\infty} = \alpha_{\min},\ D_{-\infty} = \alpha_{\max}.$$

We can compute the $f(\alpha)$ singularity spectrum from the scaling exponents $\tau(q)$ of the partition functions. The generalized Hurst exponent $h(q)$ is shown in Fig. 6, which gives

$$\alpha_{\max} = \lim_{q\to-\infty} h(q) = 1.8,$$

and

$$\alpha_{\min} = \lim_{q\to+\infty} h(q) = 0.4.$$

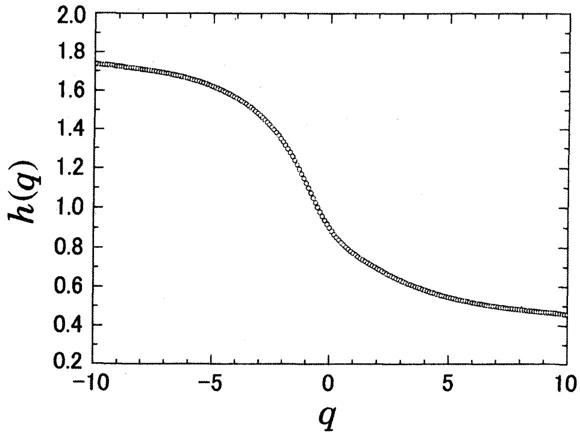

Fig. 6: The generalized Hurst exponent $h(q)$ for the moment q.

We can also estimate the relation between the Hurst and the generalized Hurst exponents from Fig. 6 at the moment $q = 2$,

$$H = h(2) = 0.67 \pm 0.1.$$

This result means that our intermittent chaos cannot be described by the Gaussian distribution.

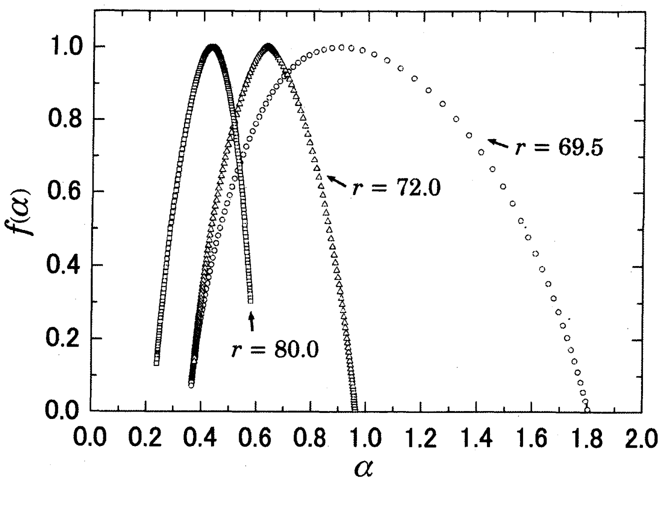

Fig. 7: Singularity spectrum for intermittent chaos of magnetic fluctuations.

The $f(\alpha)$ singularity spectrum is shown in Fig. 7. We can clearly observe the typical multifractal singularity spectrum, which stands below a tangential line to the diagonal at $\alpha(q = 1)$.

Conclusion

We have investigated numerically two-dimensional Boussinesq magnetoconvection near the Hopf bifurcation point in order to find multi-fractal extreme events from the viewpoint of the predictability and the nonlinear feedback mechanism. We have plotted the return interval distribution of extreme events for the data in a log-log plot and measured each slope in a linear region of the log-log plot when there exists a linear region in it. It has been observed that the slope of log-log plot increases monotonically to reach a saturation value ($2H$-1) when the threshold value is large enough. We have shown that the return interval distribution of extreme events is almost independent of the threshold value as long as it is enough large in our case. We have observed that the return interval distribution of extreme events is closer to a straight line with slope ($2H$-1) in Eq. (9) as the threshold increases. We have obtained the Hurst exponent H=0.67 and the tail exponent μ=0.34. Since this Hurst exponent is larger than 0.5, our intermittent magnetic fluctuations have a certain long-range correlation and cannot be explained by the Gaussian distribution.

Our intermittent chaos of magnetic fluctuations can be described by a non-Gaussian distribution with strong correlation [10] such as a multiplicative stochastic process [11] and with the multi-fractal properties [3]. We have shown that the multi-fractal nature of intermittent chaos near the critical bifurcation point is strongly related to a power law for a return interval distribution of extreme events and that it is consistent with the Kolmogorov-Obukov law for fully developed turbulence theory [3]. Here we demonstrated the statistical properties of return interval distribution of intermittent extreme events of magnetic fluctuations in the 2-D magnetoconvection, which are related to a power-law like in the Weibull-class distribution [12]. In addition, by means of multi-fractal analysis, we discussed consistently the relation between scaling exponents. Statistical properties of intermittent chaotic magnetoconvection may serve as a keystone of deep understandings for nonlinear physical phenomena and financial dynamics.

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