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Control of the Inflation Rate through Central Banks in an Equilibrium Model

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1 Introduction

In a continuous time stochastic equilibrium model, we describe a mechanism in which the central bank affects the inflation rates through ordinary banks, and studies under what conditions the change in the inflation rates alters the equilibrium consumption, investment and production. Every transaction is executed through bank accounts that bear a nominal interest rate and the bank accounts are called money in this paper. The central bank lends money to the ordinary banks by depositing the money into the reserve which is set in the ordinary banks. In this setting, the equilibrium processes are indeterminate. Since the real interest rates (nominal interest rates minus inflation rates) are not determined uniquely, we can study the effects of monetary policy on the real interest rates and therefore the consumption, investment and production. Because the real interest rates do not equal to the productivity of capital, the divergent capital paths can be called equilibrium unlike the existing dynamic stochastic equilibrium models.

2 The economy

The model we want to consider in the subsequent paper is based on the typical dynamic stochastic general equilibrium models in continuous time with production (see e.g. Duffie (2001), Dana and Jeanblanc-Picque (2003), Espino and Hintermaiser (2009)) except that in this paper the banking system is included in the setting.

The mathematical preliminaries are presented first. We work on the infinite interval of time $[0,\infty)$ and on a probability space $(\Omega,\mathcal{F},P)$. We assume that a Brownian motion $(B_t = (B_t^1, B_t^2);\ t \geq 0)$ is constructed on the probability space and set a filtration by $\mathcal{F}_t = \sigma(B_s;\ s \leq t)$ that is assumed to include the P-null sets. $(\mathcal{F}_t,\ t \geq 0)$ describes the flow of information available to every agent at t.

Although the analysis in this paper is executed in continuous time, in order to make the economic meaning clear, let us first illustrate our model in discrete time. The economy consisits of four entities; consumers, firms, ordinary banks and the central bank. The following steps show the procedure in which transactions among the consumers and firms occur through the bank accounts that bear an interest rate r_t, and accompanying behavior of the central bank.

We start from the following state of the ordinary banks 's balance sheet. The right hand of balance sheet (credit side) means that consumers have the deposit the quantity of which is b_0 and firms $p_0 f(0, k_{-1}, l)$ at time 0. The left hand (debit side) stands for the asset whose contents are lending to the consumers and firms by issuing the 'deposit bond' the quantity of which is $\theta_0^c + \theta_0^f$, θ_0^c to the consumers and θ_0^f the firms, at the price of X_0 at time 0. The central bank lends money to the banks, the quantity of which, called Φ_0, is deposited to the central bank as the reserve.

Ordinary Banks' Balance Sheet (B/S) (1)

reserve (to the central bank) Φ_0	consumers b_0
lend $X_0(\theta_0^c + \theta_0^f)$	firms $p_0 f(0, k_{-1}, l)$
	borrow (from the central bank) Φ_0

Second, the banks pay the interest rate r_0 which accrues on the deposits, and receive the interest rate Δ_0 paid by the borrowers (consumers and firms).

Ordinary Banks' Balance Sheet (B/S) (2)

reserve (to the central bank) Φ_0	consumers $b_0 + r_0 b_0 - \Delta_0 \theta_0^c$
lend $X_0(\theta_0^c + \theta_0^f)$	firms $p_0 f(0, k_{-1}, l) + r_0 p_0 f(0, k_{-1}, l) - \Delta_0 \theta_0^f$
	borrow (from the central bank) $\Phi_0 + r_0 \Phi_0$
profit $\Delta_0(\theta_0^c + \theta_0^f)$	equity capital
loss -$\{r_0 b_0 + r_0 p_0 f(0, k_{-1}, l) + r_0 \Phi_0\}$	$\Delta_0(\theta_0^c + \theta_0^f) - \{r_0 b_0 + r_0 p_0 f(0, k_{-1}, l) + r_0 \Phi_0\}$

Put

$$\Phi_0^B := \Delta_0(\theta_0^c + \theta_0^f) - \{r_0 b_0 + r_0 p_0 f(0, k_{-1}, l) + r_0 \Phi_0\}$$

which is paid as the wage to workers in the banks.

Ordinary Banks' Balance Sheet (B/S) (3)

reserve (to the central bank) Φ_0	consumers
lend $X_0(\theta_0^c + \theta_0^f)$	$b_0 + r_0 b_0 - \Delta_0\theta_0^c + \delta_0 a_0 + w_0 l + r_0\Phi_0 + \Phi_0^B$
	firms 0
	borrow (from the central bank) Φ_0

where

$$\frac{p_0 f(0, k_{-1}, l) + r_0 p_0 f(0, k_{-1}, l) - \Delta_0\theta_0^f - w_0 l}{a_0} =: \delta_0$$

which means the dividend per share at time 0.

Ordinary Banks' Balance Sheet (B/S) (4)

reserve (to the central bank) Φ_1	consumers
lend $X_1(\theta_0^c + \theta_0^f)$	$b_0 + r_0 b_0 - \Delta_0\theta_0^c + \delta_0 a_0 + w_0 l + r_0\Phi_0 + \Phi_0^B$
$+X_1(\theta_1^c - \theta_0^c) + X_1(\theta_1^f - \theta_0^f)$	$+X_1(\theta_1^c - \theta_0^c)$
	firms $X_1(\theta_1^f - \theta_0^f)$
	borrow (from the central bank) Φ_1
	equity capital (latent gain)
	$(X_1 - X_0)(\theta_0^c + \theta_0^f)$

Next, consumers select c_1, a_1 and b_1 under the budget constraint

$$p_1 c_1 + S_1(a_1 - a_0) + b_1 = b_0 + r_0 b_0 - \Delta_0\theta_0^c + \delta_0 a_0 + w_0 l + r_0\Phi_0 + \Phi_0^B + X_1(\theta_1^c - \theta_0^c) \tag{B}$$

where p_1 and S_1 represent the price of the good and the stock price at time 1 respectively. Thus,

Ordinary Banks' Balance Sheet (B/S) (5)

reserve (to the central bank) Φ_1	consumers b_1
lend $X_1(\theta_1^c + \theta_1^f)$	firms $p_1 c_1 + S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f)$
	borrow (from the central bank) Φ_1
	equity capital (latent gain)
	$(X_1 - X_0)(\theta_0^c + \theta_0^f)$

The firms conduct the investment by making use of $S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f)$. So letting $p_1(k_1 - k_0)$ stand for the quantity of the investment, it holds

$$p_1(k_1 - k_0) = S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f).$$

If we assume that the whole products are absorbed by consumption and investment, we see

$$p_1 f(1, k_0, l) = p_1 c_1 + S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f).$$

Ordinary Banks' Balance Sheet (B/S) (6)

reserve (to the central bank) Φ_1	consumers b_1
lend $X_1(\theta_1^c + \theta_1^f)$	firms $p_1 f(1, k_0, l)$
	borrow (from the central bank) Φ_1
	equity capital (latent gain)
	$(X_1 - X_0)(\theta_0^c + \theta_0^f)$

The above phase (6) is back to (1) replacing time 1 with 0 except the equity capital part. The ensuing process following (6) proceeds similarly to (1)-(6). The equity capital as latent gains at time t is represented as, by writing in continuous time,

$$\int_0^t (\theta_u^c + \theta_u^f) dX_u.$$

Ordinary Banks' Balance Sheet (B/S) t

reserve (to the central bank) Φ_t	consumers b_t
lend $X_t(\theta_t^c + \theta_t^f)$	firms $p_t f(t, k_t, l) dt$
	borrow (from the central bank) Φ_t
	equity capital (latent gain)
	$\int_0^t (\theta_u^c + \theta_u^f) dX_u$

The central bank obtains the interest rate r_t that accrues on Φ_t and pays it to consumers (e.g. workers to the central bank) as the wage at time $t \geq 0$. Then its balance sheet is represented as follows.

The Central Banks' Balance Sheet (B/S) t

lend Φ_t	banks Φ_t
profit $r_t \Phi_t$	equity capital 0
loss $-r_t \Phi_t$	

2.1 Consumers' problem

The problem that homogenous consumers face is the following;

$$\max_{(a_t, b_t, \theta_t^c, c_t)_{t \geq 0}} E_P \left[\int_0^\infty e^{-\int_0^t \rho_u du} u(c_t) dt \right] \tag{2.1.1}$$

subject to

$$p_t c_t dt + S_t da_t + db_t - X_t d\theta_t^c$$
$$= r_t b_t dt - \Delta_t \theta_t^c dt + \delta_t a_t dt + w_t l dt + r_t \Phi_t dt + \Phi_t^B dt \quad t \geq 0, \quad (2.1.2)$$

$$a_0, b_0 > 0, \text{ and } \theta_0^c > 0 \text{ are given} \quad (2.1.3)$$

$$\exp\left(- \int_0^t \frac{\Delta_u}{X_u} du \right) \theta_t^c = o(e^{-vt}), \ a.e., \text{ for some constant } v > 0, \quad (2.1.4)$$

$$(a, b, \theta^c, c) \in (C^+)^4 \quad (2.1.5)$$

where c_t is consumption at period t, a_t is the quantity of stock selected up to t, δ_t means the dividend at t, b_t is the quantity of bank accounts which remains by t, θ_t^c is the existing quantity of deposit bond which is bought from banks by t, S_t stands for the price of stock at t, w_t represents the nominal wage recieved at t, Φ_t is the lending from the central bank. We assume that the labors are supplied inelastically with respect to the wage, the interpretation of which is, for example, that if consumers do not work, they will die.

The first constraint (2.1.2) comes from (B). (2.1.4) reflects the condition that the debt should not grow faster than the interest rate (the no-Ponzi condition), but this is stronger requirement.

2.2 Firms' problem

The firms' maximization problem is synonymous with the stock holders' one whose aim is to maximize the expected rate of revenue which is defined as the discount rate under which the expectation of the discounted integration of all the net profits equals the current value of firms' equity capital, $p_0 k_0 - X_0 \theta_0^f$.

Then the firms' problem is described as follows;

$$\max_{(a_t, \theta_t^f, k_t, l_t)_{t \geq 0}} \quad \{\phi_t\}_{t \geq 0} \quad \text{maximize by the order } \phi_t \geq \phi_t' \ \ a.e. \text{ for all } t, \ \phi, \phi' \in L^+ \quad (2.2.1)$$

subject to

$$p_0 k_0 - X_0 \theta_0^f$$
$$= E_P\left[\int_0^\infty e^{-\int_0^t \phi_u du} \left\{ \left[(1 + r_t) p_t f(t, k_t, l_t) - \Delta_t \theta_t^f - w_t l_t \right] dt - S_t da_t \right\} \right], \quad (2.2.2)$$

$$k_t = k_0 + \int_0^t p_u^{-1} S_u da_u + \int_0^t p_u^{-1} X_u d\theta_u^f \tag{2.2.3}$$

$$a_0, \theta_0^f > 0 \text{ and } k_0 > 0 \text{ are given} \tag{2.2.4}$$

$$\exp\left(-\int_0^t \frac{\Delta_u}{X_u} du\right)\theta_t^f = o(e^{-v't}), \ a.e., \text{ for some constant } v' > 0, \tag{2.2.5}$$

$$(a, \theta^f, k, l) \in (C^+)^4 \tag{2.2.6}$$

where k_t is capital stock accumulated up to t, a_t is the quantity of stock issued up to t, θ_t^f is the existing quantity of deposit bond which is bought from banks by t, S_t stands for the price of the stock at t, w_t represents the nominal wage at t. Firms' all profit is assumed to be distributed to stock holders as dividends which is defined as;

$$\delta_t := \frac{(1 + r_t)p_t f(t, k_t, l_t) - \Delta_t \theta_t^f - w_t l_t}{a_t}$$

$\delta_t = 0$ in the case of $a_t = 0$.

Firms' balance sheet is then described as follows;

Firms' B/S at t

capital stock $p_t k_t$	debt $X_t \theta^f$
	equity capital

2.3 Ordinary banks' problem

In this paper, we assume the reserve requirement system in which a quantity of money needs to be deposited to the central bank as the reserve that is determined by ['the reserve ratio' $\times$ 'all deposit owed by ordinary banks']. We write the reserve ratio as $0 < \varepsilon_t < 1$. The ordinary banks control θ_t^c and θ_t^f subject to

$$b_t + \Phi_t + p_t f(t, k_t, l)dt + \int_0^t \theta_u dX_u - X_t(\theta_t^c + \theta_t^f) \geq \varepsilon_t(b_t + \Phi_t + p_t f(t, k_t, l)dt)$$

namely

$$b_t + \Phi_t + \int_0^t \theta_u dX_u - X_t(\theta_t^c + \theta_t^f) \geq \varepsilon_t(b_t + \Phi_t) \quad t \geq 0 \tag{2.3.1}$$

so as to maximize

$$\Phi_t^B = \Delta_t(\theta_t^c + \theta_t^f) - \{r_t b_t + r_t p_t f(t, k_t, l) + r_t \Phi_t\} \quad t \geq 0. \tag{2.3.2}$$

Thus we can see that the optimal solutions are always the corner's ones in which (2.3.1) holds binding. So we see in this paper

$$b_t + \Phi_t + \int_0^t \theta_u dX_u - X_t(\theta_t^c + \theta_t^f) = \varepsilon_t(b_t + \Phi_t) \quad t \geq 0. \tag{2.3.3}$$

Note (2.3.3) is equivalent to

$$-X_0(\theta_0^c + \theta_0^f) + (1 - \varepsilon_t)(b_t + \Phi_t) = \int_0^t X_u d(\theta_t^c + \theta_t^c) \quad t \geq 0. \tag{2.3.4}$$

3 Results

In this paper, we focus on the equilibrium which is defined as follows.

The definition of the equilibrium. An equilibrium of this economy is a set of stochastic processes

$$\{(a_t, b_t, \theta_t^c, \theta_t^f), (c_t, k_t, l_t), (p_t, S_t, X_t, r_t, \Delta_t, w_t), \varepsilon_t, \Phi_t\}_{t \geq 0}$$

such that

(1) given $\{p_t, S_t, X_t, r_t\}$, $\{a_t, b_t, \theta_t^c, c_t\}$ solves the consumers' problem; (2) given $\{p_t, S_t, X_t, r_t\}$, $\{a_t, \theta_t^f, k_t, l_t\}$ solves the firms' problem; (3) given $\{X_t, \Delta_t, r_t\}$, $\{\theta_t^c, \theta_t^f\}$ solves the banks' problem; (4) the good market clears;

$$c_t + \dot{k}_t = f(t, k_t, l_t)$$

(5) the stock market clears; (6) the deposit bond market clears; (7) the labor market clears; (8) Φ_t and ε_t are exogenous variables determined by the central bank.

We assume that the utility function is CRRA class.

Assumption 1.

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad c \geq 0$$

where $0 < \gamma < 1$.

We can see

$$-\frac{u''(c)c}{u'(c)} = \gamma, \quad \frac{u'''(c)c^2}{u'(c)} = \gamma(\gamma + 1).$$

The production function is assumed to be linear with respect to both variables.

Assumption 2.

$$f(t, x_t, l_t) = \alpha(t, B_t)x_t + \beta(t, B_t)l_t$$
$$0 < \inf_{(t,\omega)} \alpha(t, B_t), \quad 0 < \inf_{(t,\omega)} \beta(t, B_t)$$

where $\alpha(t, B_t), \beta(t, B_t) \in C^2([0, \infty) \times \mathbb{R}^2)$ (i.e., α, β is twice continuously differentiable on $[0, \infty) \times \mathbb{R}^2$.

Put

$$p_t := p_0 \exp\left(\int_0^t \pi_u du - \frac{1}{2} \int_0^t \{(\sigma_u^{p1})^2 + (\sigma_u^{p2})^2\} du - \int_0^t \{\sigma_u^{p1} dB_u^1 + \sigma_u^{p2} dB_u^2\} \right) \tag{3.1}$$

and

$$c_t := c_0 \exp\left(\int_0^t (1/\gamma) \Big[r_u - \pi_u - (\sigma_u^{p1}\lambda_u^1 + \sigma_u^{p2}\lambda_u^2) - \rho_u + \frac{1}{2}\{(\sigma_u^{p1} + \lambda_u^1)^2 + (\sigma_u^{p2} + \lambda_u^2)^2\} \Big] du \right.$$
$$\left. + \int_0^t \{((\sigma_u^{p1} + \lambda_u^1)/\gamma) dB_u^1 + ((\sigma_u^{p2} + \lambda_u^2)/\gamma) dB_u^2\} \right) \tag{3.2}$$

so that it holds that

$$p_0 = u'(c_0). \tag{3.3}$$

Put

$$k_t = \exp\Big(\int_0^t \alpha_u du \Big) \Big[k_0 + \int_0^t \exp\Big(- \int_0^u \alpha_\tau dtau \Big)(\beta_u l - c_u) du \Big]. \tag{3.4}$$

Set

$$S_t = \exp\left(\int_0^t \mu_u du - \frac{1}{2} \int_0^t \{(\sigma_u^1)^2 + (\sigma_u^2)^2\} du + \int_0^t \{\sigma_u^1 dB_u^1 + \sigma_u^2 dB_u^2\} \right) \tag{3.5}$$

for some $\mu, \sigma^1, \sigma^2 \in L$.

Put

$$X_t = \exp\left(\int_0^t \hat{r}_u du - \frac{1}{2}\int_0^t \{(\zeta_u^1)^2 + (\zeta_u^2)^2\} du + \int_0^t \{\zeta_u^1 dB_u^1 + \zeta_u^2 dB_u^2\} \right) \tag{3.6}$$

for some $\hat{r}, \zeta^1, \zeta^2 \in L$.

Select $a, \theta^f \in C^+$ so as to satisfy

$$k_t = k_0 + \int_0^t p_u^{-1} S_u da_u + \int_0^t p_u^{-1} X_u d\theta_u^f \tag{3.7}$$

and put

$$\delta_t = \frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\} p_t k_t - \Delta_t \theta_t^f}{a_t}. \tag{3.8}$$

Choose r_t so that it holds

$$\alpha_t = \frac{\hat{r}_t + (\Delta_t/X_t) - \pi_t}{1 + r_t}. \tag{3.9}$$

Solve $(\lambda_t^1, \lambda_t^2)$ as the solution of the following equation;

$$\begin{pmatrix} \sigma_t^1, & \sigma_t^2 \\ \zeta_t^1, & \zeta_t^2 \end{pmatrix} \begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} = \begin{pmatrix} \mu_t - r_t + (\delta_t/S_t) \\ \hat{r}_t - r_t + (\Delta_t/X_t) \end{pmatrix} \tag{3.10}$$

Determine the above parameters so that λ_t^1 and $\lambda_t^2 \in L$ satisfy the following Novikof condition;

$$E_P\left[\exp\left(\frac{1}{2}\int_0^t \{(\lambda_u^1)^2 + (\lambda_u^2)^2\} du \right) \right] < \infty, \quad t \geq 0. \tag{3.11}$$

Then, by the Girsanov theorem, $\hat{B}_t^i := B_t^i + \int_0^t \lambda_u^i du, i = 1, 2$, is a standard Brownian motion under the new measure Q defined by

$$dQ := \exp\left(-\frac{1}{2}\int_0^t \{(\lambda_u^1)^2 + (\lambda_u^2)^2\} du - \int_0^t \{\lambda_u^1 dB_u^1 + \lambda_u^2 dB_u^2\} \right) dP.$$

Under these settings we prove the following lemma.

Lemma 1. There exists a set of parameters $\{r, \hat{r}, \zeta^1, \zeta^2, \mu, \sigma^1, \sigma^2, \lambda^1, \lambda^2, \pi, \sigma^{p1}, \sigma^{p2}, \Delta, a, \theta^f\}$ which simultaneously satisfies (3.1) – (3.11).

(Proof) Set first the parameters so as to satisfy (3.1) – (3.6), (3.9) – (3.11). Determine θ^f by

$$p_t^{-1}S_t d\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t k_t - \Delta_t\theta_t^f}{\delta_t}\right) + p_t^{-1}X_t d\theta_t^f = dk_t,$$

which is equivalent to

$$\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t^{-1}S_t}{\delta_t} - 1\right)dk_t + d\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t}{\delta_t}\right)k_t$$
$$= \left(\frac{\Delta_t}{\delta_t} - p_t^{-1}X_t\right)d\theta_t^f + d\left(\frac{\Delta_t}{\delta_t}\right)\theta_t^f \qquad (3.12)$$

Lastly, a_t is determined by (3.8). □

In the subsequent lemmata, we prove that the parameters found in the lemma 1 consist of the equilibrium.

Let Φ_t, b_t and θ_t^c satisfy (2.3.4). Define Φ_t^B by (2.3.2).

Lemma 2. Let $\{(a_t, b_t, \theta_t^c, c_t)\}_{t\geq 0}$ meet the consumers' budget constraint (2.1.2) – (2.1.5). Then $c_t, t \geq 0$ satisfies

$$E_Q\left[\int_0^\infty R_t p_t c_t dt\right] \leq S_0 a_0 + b_0 - X_0\theta_0^c + E_Q\left[\int_0^\infty (R_t w_t l + r_t R_t \Phi_t dt + R_t \Phi_t^B)dt\right] \qquad (3.13)$$

where

$$R_t := \exp\left(-\int_0^t r_u du\right).$$

(Proof) From the condition in this claim, we see

$$p_t c_t dt + S_t da_t + db_t - X_t d\theta_t^c$$
$$= r_t b_t dt - \Delta_t \theta_t^c dt + \delta_t a_t dt + w_t l dt + r_t \Phi_t dt + \Phi_t^B \quad t \geq 0, \qquad (2.1.2)$$

Multiplying both sides by R_t and calculating a little yield that

$$\begin{aligned}
&R_tp_tc_tdt + d(R_tS_ta_t) + d(R_tb_t) - d(R_tX_t\theta_t^c) \\
&= a_td(R_tS_t) - \theta_t^f d(R_tX_t) - R_t\Delta_t\theta_t^c dt + R_t\delta_ta_tdt + R_tw_tldt + r_tR_t\Phi_tdt + R_t\Phi_t^B \\
&= \sigma_t^1R_tS_ta_tdB^1 + \sigma_t^2R_tS_ta_tdB^2 + R_tS_ta_t(\mu_t - r_t + (\delta_t/S_t))dt \\
&+ \zeta_t^1R_tX_t\theta_t^f dB^1 + \zeta_t^2R_tS_t\theta_t^f dB^2 + R_tX_t\theta_t^f(\hat{r}_t - r_t + (\Delta/X_t))dt + R_tw_tldt + r_tR_t\Phi_tdt + R_t\Phi_t^B \\
&= \sigma_t^1R_tS_ta_td\hat{B}^1 + \sigma_t^2R_tS_ta_td\hat{B}^2 + \zeta_t^1R_tX_t\theta_t^f d\hat{B}^1 + \zeta_t^2R_tS_t\theta_t^f d\hat{B}^2 \\
&\qquad + R_tw_tldt + r_tR_t\Phi_tdt + R_t\Phi_t^B.
\end{aligned}$$

Integrating both sides in time and taking expectations by the measure Q generate that

$$\begin{aligned}
&E_Q\left[\int_0^t R_up_uc_udu\right] + E_Q[R_tS_ta_t + R_tb_t - R_tX_t\theta_t^c] \\
&\qquad = S_0a_0 + b_0 - X_0\theta_0^c + E_Q\left[\int_0^t (R_uw_ul + r_uR_u\Phi_u + R_u\Phi_u^B)du\right].
\end{aligned}$$

Note

$$R_tX_t = \exp\left(-\int_0^t \frac{\Delta_u}{X_u}du - \frac{1}{2}\int_0^t \{(\zeta_u^1)^2 + (\zeta_u^2)^2\}du + \int_0^t \{\zeta_u^1d\hat{B}_u^1 + \zeta_u^2d\hat{B}_u^2\}\right).$$

So by the condition (2.2.5), we find that

$$\lim_{t\to\infty} E_Q[R_tX_t\theta_t^c] = 0.$$

That completes the proof. □

Lemma 3. If c and $k \in C^+$ satisfy

$$k_t = \exp\Big(\int_0^t \alpha_udu\Big)\Big[k_0 + \int_0^t \exp\Big(-\int_0^u \alpha_\tau d\tau\Big)(\beta_ul - c_u)du\Big] \tag{3.4}$$

and a and $\theta^f \in C^+$ are determined so that

$$k_t = k_0 + \int_0^t p_u^{-1}S_uda_u + \int_0^t p_u^{-1}X_ud\theta_u^f, \tag{3.7}$$

holds. Let b and $\theta^c \in C^+$ are chosen so that it holds

$$\Phi_t^B = \Delta_t(\theta_t^c + \theta_t^f) - \{r_tb_t + r_tp_tf(t, k_t, l) + r_t\Phi_t\} \quad t \geq 0. \tag{3.14}$$

and that the money market clears, namely,

$$b_t = X_0(\theta^c_0 + \theta^f_0) + \int_0^t X_u d(\theta^c_u + \theta^f_u) \quad t \geq 0. \tag{3.15}$$

Then it follows that

$$\begin{aligned} p_t c_t dt + &S_t da_t + db_t - X_t d\theta^c_t \\ &= r_t b_t dt - \Delta_t \theta^c_t dt + \delta_t a_t dt + w_t l dt + r_t \Phi_t dt + \Phi^B_t \quad t \geq 0. \end{aligned} \tag{2.1.2}$$

(Proof) Firstly note that (3.4) is equivalent to

$$\dot{k}_t = f(t, k_t, l) - c_t.$$

Then by multiplying both sides by p_t

$$\begin{aligned} &p_t c_t dt + S_t da_t + X_t d\theta^f_t \\ &\quad = [p_t f(t, k_t, l) - \Delta_t \theta^f_t - w_t l]dt + \Delta_t \theta^f_t dt + w_t l dt \\ &\quad = [(1 + r_t) p_t f(t, k_t, l) - \Delta_t \theta^f_t - w_t l]dt - r_t p_t f(t, k_t, l)dt + \Delta_t \theta^f_t dt + w_t l dt \\ &\quad \text{From (3.14) and the definition of } \delta_t \\ &\quad = \delta_t a_t + r_t b_t - \Delta_t \theta^c_t dt + r_t \Phi_t dt + \Phi^B_t dt + w_t l dt. \end{aligned}$$

On the left hand side, we obtain from (3.15)

$$\begin{aligned} &p_t c_t dt + S_t da_t + X_t d\theta^f_t \\ &= p_t c_t dt + S_t da_t + X_t d(\theta^c_t + \theta^f_t) - X_t d\theta^c_t \\ &= p_t c_t dt + S_t da_t + db_t - X_t d\theta^c_t. \end{aligned}$$

Thus we find (2.1.2) holds. □

Lemma 4. The solution of the firms' problem

$$\max_{(a_t, \theta^f_t, k_t, l_t)_{t \geq 0}} \{\phi_t\}_{t \geq 0} \quad \text{maximize by the order } \phi_t \geq \phi'_t \ \ a.e. \text{ for all } t,\ \phi, \phi' \in L^+ \tag{2.2.1}$$

subject to (2.2.2)-(2.2.6) is

$$\left\{\hat{r}_t + \frac{\Delta_t}{X_t}\right\}_{t \geq 0}.$$

(Proof) Let

$$\hat{R}_t = \exp\left(- \int_0^t \left\{\hat{r}_u + \frac{\Delta_u}{X_u}\right\} du\right).$$

$$\begin{aligned}
&\big[(1+r_t)\hat{R}_t p_t f(t,k_t,l_t) - \Delta_t \hat{R}_t \theta_t^f - \hat{R}_t w_t l_t\big]dt - \hat{R}_t S_t da_t \\
&\quad = \big[\{\hat{r}_t + (\Delta_t/X_t)\}\hat{R}_t p_t k_t - (\Delta_t/X_t)\hat{R}_t X_t \theta_t^f\big]dt - \hat{R}_t S_t da_t \\
&\quad = -k_t d(\hat{R}_t p_t) - (\hat{R}_t p_t)dk_t + \hat{R}_t X_t d\theta_t^f - (\Delta_t/X_t)\hat{R}_t X_t \theta_t^f dt \\
&\quad = -d(\hat{R}_t p_t k_t) + d(\hat{R}_t X_t \theta_t^f) + \sigma_t^{p1}\hat{R}_t p_t k_t dB_t^1 + \sigma_t^{p2}\hat{R}_t p_t k_t dB_t^2 + \zeta_t^1 \hat{R}_t X_t \theta_t^f dB_t^1 + \zeta_t^2 \hat{R}_t X_t \theta_t^f dB_t^2.
\end{aligned}$$

Thus it follows by integrating and taking expectations of both sides

$$\begin{aligned}
&p_0 k_0 - X_0 \theta_0^f \\
&\quad = E_P\left[\int_0^t \exp\left(-\int_0^u \Big\{\hat{r}_\tau + \frac{\Delta_\tau}{X_\tau}\Big\}d\tau\right)\Big\{\big[(1+r_u)p_u f(t,k_u,l_u) - \Delta_u\theta_u^f - w_u l_t\big]du - S_u da_u\Big\}\right] \\
&\qquad + E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f].
\end{aligned}$$

Because for any (a,θ^f,k,l) that satisfies $\lim_{\to\infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0$ it holds that

$$\begin{aligned}
&p_0 k_0 - X_0 \theta_0^f \\
&\quad = E_P\left[\int_0^\infty \exp\left(-\int_0^t \Big\{\hat{r}_u + \frac{\Delta_u}{X_u}\Big\}du\right)\Big\{\big[(1+r_t)p_t f(t,k_t,l_t) - \Delta_t\theta_t^f - w_t l_t\big]dt - S_t da_t\Big\}\right],
\end{aligned}$$

it suffices to prove that $\lim_{\to\infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0$ follows for the selected (a,θ^f,k,l).

Recall

$$\alpha_t = \frac{\hat{r} + (\Delta_t/X_t) - \pi_t}{1+r_t}. \tag{3.9}$$

Because

$$\alpha_t < \hat{r} + (\Delta_t/X_t) - \pi_t,$$

we can see

$$\lim_{t\to\infty} E_p[\hat{R}_t p_t k_t] = 0.$$

From (2.2.5)

$$\lim_{t\to\infty} E_p[\hat{R}_t X_t \theta_t^f] = 0.$$

Then this lemma is proved. □

Lemma 5. If $c^* \in C^+$ satisfies

$$R_t p_t \exp\left(-\frac{1}{2}\int_0^t \{(\lambda_u^1)^2+(\lambda_u^2)^2\}du - \int_0^t \{\lambda_u^1 dB_u^1 + \lambda_u^2 dB_u^2\}\right) = \exp\left(-\int_0^t \rho_u du\right)u'(c_t^*),$$

c^* is the solution of the problem;

$$\max_{c \in C^+} E_p\left[\int_0^\infty e^{-\int_0^t \rho_u du} u(c_t)dt\right]$$

subject to

$$E_Q\left[\int_0^\infty R_t p_t c_t dt\right] \leq S_0 a_0 + b_0 - X_0\theta_0^c + E_Q\left[\int_0^\infty (R_t w_t l + r_t R_t \Phi_t dt + R_t \Phi_t^B)dt.\right]$$

The proof of lemma 5 is almost the same as the standard argument (see e.g. Dana and Jeanblanc-Picque (2003)), so we omitted here.

Note that under

$$p_t = p_0 \exp\left(\int_0^t \pi_u du - \frac{1}{2}\int_0^t \{(\sigma_u^{p1})^2 + (\sigma_u^{p2})^2\}du - \int_0^t \{\sigma_u^{p1} dB_u^1 + \sigma_u^{p2} dB_u^2\}\right), \tag{3.1}$$

we can deduce that

$$\begin{aligned} c_t^* &:= c_0 \exp\left(\int_0^t (1/\gamma)\Big[r_u - \pi_u - (\sigma_u^{p1}\lambda_u^1 + \sigma_u^{p2}\lambda_u^2) - \rho_u + \frac{1}{2}\{(\sigma_u^{p1} + \lambda_u^1)^2 + (\sigma_u^{p2} + \lambda_u^2)^2\}\Big]du \right. \\ &\left. + \int_0^t \{((\sigma_u^{p1} + \lambda_u^1)/\gamma)dB_u^1 + ((\sigma_u^{p2} + \lambda_u^2)/\gamma)dB_u^2\}\right). \end{aligned} \tag{3.2}$$

4 Policy Implications

We consider the case in which the central bank attempts to ease the quantity of money by reducing the reserve rate, ε_t, or increasing the lending, Φ_t. The change in these parameters leads to the change of the quantity θ_t^c and θ_t^f from (2.3.4). By (3.2) and (3.4), we can see that the increase of the inflation rate π_t means the increase of the capital path k_t. Thus recalling (3.12)

$$\begin{aligned} &\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t^{-1}S_t}{\delta_t} - 1\right)dk_t + d\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t}{\delta_t}\right)k_t \\ &= \left(\frac{\Delta_t}{\delta_t} - p_t^{-1}X_t\right)d\theta_t^f + d\left(\frac{\Delta_t}{\delta_t}\right)\theta_t^f, \end{aligned} \tag{3.12}$$

the increase of θ_t^f accompanies the increase of the inflation rate π_t provided

$$\begin{aligned} \operatorname{sign}\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t^{-1}S_t}{\delta_t} - 1\right)dt &= \operatorname{sign}d\left(\frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\}p_t}{\delta_t}\right) \\ &= \operatorname{sign}\left(\frac{\Delta_t}{\delta_t} - p_t^{-1}X_t\right)dt = \operatorname{sign}d\left(\frac{\Delta_t}{\delta_t}\right). \end{aligned}$$

The case of tightening the money can also be studied in the similar manner so we omitted the discussion.

References

[1] Dana, R. A. and M. Jeanblanc-Picque (2003), *Financial Markets in Continuous Time*, Springer.

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[3] Espino, E. and T. Hintermaiser (2009), "Asset Trading Volume in a Production Economy," *Economic Theory*, 39, 231-258.