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Cyclical Fluctuations in Dynamic Optimization Models

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Abstract

In this paper, we summarize the analytical results by Dockner and Feichtinger(1991) on the existence of cyclical fluctuations in continuous time dynamic optimization models with two state variables, and apply their analytical results to a particular economic dynamic model, that is, a modified Gaskins model of dynamic limit pricing.

Keywords: Cyclical fluctuations, Dynamic Optimization, Hopf Bifurcation, Dynamic Limit Pricing.

JEL classification: C61, D43

1.Introduction

The dynamic optimization model with only one state variable is very popular in dynamic economic analysis. It is well known, however, that usually such a model cannot produce cyclical fluctuations but it produces monotonic convergence to the equilibrium point. On the other hand, some economists studied the several economic models of dynamic optimization with two state variables that can produce cyclical fluctuations. Benhabib and Nishimura(1979), Benhabib and Rustichini(1990), and Asada and Semmler(2004) are examples of such works. However, all of the above-mentioned works are studies of the particular economic models rather than the systematic studies of the general mathematical principle. As far as we acknowledge, Dockner and Feichtinger(1991) provided the most systematic investigation of the general mathematical principle that produces cyclical fluctuations in continuous time dynamic optimization model with two state variables. In this paper, we reconsider on the contribution of Dockner and Feichtinger(1991) and the possibility of the economic application of their analytical results. It is worth noting that this paper is largely based on the analytical results which were presented in Asada(2013).

In section 2, we summarize the analytical results of Dockner and Feichtinger(1991) on the general theory of cyclical fluctuations in continuous time dynamic optimization models with two state variables. In section 3, we introduce the application of Dockner-Feichtinger theorem to the particular dynamic economic model, that is, the modified Gaskins limit pricing model that was presented by Asada(2013) for the first time. Section 4 is devoted to the concluding remarks. In appendix, we take up two useful theorems on the existence of the closed orbits in the general n-dimensional and four-dimensional system of linear and nonlinear differential equations, which is not necessarily restricted to the dynamic optimization model.

2. Dynamic Optimization Problem with Two State Variables

Let us consider the following continuous time dynamic optimization problem with two state variables.

$$\text{Maximize } \int_0^\infty F(k_1,k_2,u_1,u_2,\cdots,u_n)e^{-rt}dt \tag{1}$$

subject to

$$\dot{k}_1 = f(k_1,k_2,u_1,u_2,\cdots,u_n),\ \ \dot{k}_2 = g(k_1,k_2,u_1,u_2,\cdots,u_n;\varepsilon), \tag{2}$$

$$k_1(0) = k_{10} = \text{given},\ \ k_2(0) = k_{20} = \text{given},\ \ r = \text{constant} > 0, \tag{3}$$

where k_i is i'th state variable, u_j is j'th control variable, r is the rate of discount, and ε is a parameter. We assume that functions F, f, and g are at least twice continuously differentiable.

We can solve this type of dynamic optimization problem by using Pontryagin's maximum principle.[1]

Current value Hamiltonian of this system becomes as follows.

$$\begin{aligned} H &= F(k_1,k_2,u_1,u_2,\cdots,u_n) + \mu_1 f(k_1,k_2,u_1,u_2,\cdots,u_n) \\ &\quad + \mu_2 g(k_1,k_2,u_1,u_2,\cdots,u_n;\varepsilon), \end{aligned} \tag{4}$$

where μ_1 and μ_2 are two co-state variables. Then, the necessary conditions for optimality are given by the following set of equations.

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[1] See Chiang(1992), Dixit(1990) and Gandolfo(2009) as for the expositions of Pontryagin's maximum principle.

$$\dot{k}_i = \frac{\partial H}{\partial \mu_i} \quad (i = 1,2) \tag{5a}$$

$$\dot{\mu}_i = r\mu_i - \frac{\partial H}{\partial k_i} \quad (i = 1,2) \tag{5b}$$

$$\underset{(u_1,u_2,\cdots,u_n)}{Max} \; H \tag{5c}$$

$$\lim_{t\to\infty} k_i \mu_i e^{-rt} = 0 \quad (i = 1,2) \tag{5d}$$

We suppose that the condition (5c) is expressed by the following set of the first order conditions, assuming that the second order conditions are also satisfied.

$$\frac{\partial H}{\partial u_j} = 0 \quad (j = 1,2,\cdots,n) \tag{6}$$

Furthermore, we assume that we can obtain the unique set of the solution of equation (6) with respect to u_j as follows.

$$u_j = u_j(k_1,k_2,\mu_1,\mu_2) \quad (j = 1,2,\cdots,n) \tag{7}$$

Substituting equation (7) into equations (5a) and (5b), we obtain the following four-dimensional linear or nonlinear system of differential equations.

$$\dot{k}_1 = G_1(k_1,k_2,\mu_1,\mu_2;\varepsilon) \tag{8a}$$

$$\dot{k}_2 = G_2(k_1,k_2,\mu_1,\mu_2;\varepsilon) \tag{8b}$$

$$\dot{\mu}_1 = G_3(k_1,k_2,\mu_1,\mu_2;r,\varepsilon) \tag{8c}$$

$$\dot{\mu}_2 = G_4(k_1,k_2,\mu_1,\mu_2;r,\varepsilon) \tag{8d}$$

Let us assume that the system (8) has a meaningful equilibrium solution $(k_1{*},k_2{*},\mu_1{*},\mu_2{*})$ that ensures $\dot{k}_1 = \dot{k}_2 = \dot{\mu}_1 = \dot{\mu}_2 = 0$. In this case, the characteristic equation of this system *at the equilibrium point* becomes as follows.[2]

$$\Delta(\lambda) \equiv |\lambda I - J| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{9}$$

$$a_1 = -traceJ, \quad a_2 = M_2, \quad a_3 = -M_3, \quad a_4 = \det J, \tag{10}$$

where J is the (4×4) Jacobian matrix of the dynamic system (8) *at the equilibrium*

[2] See Asada, Chiarella, Flaschel and Franke(2010) Mathematical Appendix.

point, and M_j is the sum of all principal j'th order minors of $J(j = 2,3)$.

Dockner and Feichtinger(1991) studied the mathematical structure of this particular Jacobian matrix J that is attached to this type of dynamic optimization problem. They showed that the following properties are satisfied in case of this particular Jacobian matrix.

$$traceJ = 2r, \quad -M_3 + rK = 0, \quad K \equiv M_2 - r^2. \tag{11}$$

Substituting the relationships (11) into a set of equalities (10), we have the following expressions.

$$a_1 = -traceJ = -2r < 0, \quad a_2 = r^2 + K, \quad a_3 = -rK, \quad a_4 = \det J. \tag{12}$$

Dockner and Feichtinger(1991) proved that the following set of conditions (DF) is equivalent to the condition (H2) in **Theorem A1** in **Appendix**.[3]

$$\det J > (K/2)^2, \quad (K/2)^2 + r^2(K/2) - \det J = 0. \tag{DF}$$

In fact, they succeeded to provide the following complete mathematical characterization of the solution of the particular characteristic equation (9).

Theorem 1. (Dockner-Feichtinger Theorem, cf. Dockner and Feichtinger(1991) and Feichtinger, Novak and Wirl(1994))

The characteristic equation $\Delta \equiv |\lambda I - J| = 0$ of the particular Jacobian matrix of the system (8) has the following properties (1) – (4).

(1) The characteristic equation has two positive real roots and two negative real roots *if and only if*

$$K < 0, \ 0 < \det J \leqq (K/2)^2. \tag{13}$$

(2) The characteristic equation has a pair of complex roots with positive real part and a

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[3] Asada and Yoshida(2003) proved that the condition (DF) is equivalent to the following seemingly simpler set of conditions (AY).

$$K > 0, \quad (K/2)^2 + r^2(K/2) - \det J = 0. \tag{AY}$$

The proof is quite straightforward. First, let us suppose that a set of conditions (DF) is satisfied. Then, we have

$$\det J = (K/2)^2 + r^2(K/2) > (K/2)^2, \tag{\#}$$

which means that $K > 0$. This proves the causality (DF)⇒(AY). Next, let us suppose that a set of conditions (AY) is satisfied. Also in this case we have the relationship (#), which means that a set of conditions (DF) is satisfied. This proves the causality (AY)⇒(DF). Asada and Yoshida(2003) also showed that a set of conditions (AY) is equivalent to a set of conditions (A2) in **Appendix**.

pair of complex roots with negative real part *if and only if*

$$\det J > (K/2)^2, \quad \det J - (K/2)^2 - r^2(K/2) > 0. \tag{14}$$

(3) The characteristic equation has three roots with positive real parts and one negative real root *if and only if*

$$\det J < 0. \tag{15}$$

(4) The characteristic equation has a pair of complex roots with positive real part and a pair of pure imaginary roots *if and only if* a set of the conditions (DF) is satisfied.

A set of conditions (DF) (or equivalently, a set of conditions (AY) in footnote 3) ensures the existence of the closed orbits in the dynamic optimization problem with two state variables that is characterized by equations (1) – (3).

3. An Application to Economics : Modified Gaskins Model of Dynamic Limit Pricing

In this section, we shall apply the analytical results by Dockner and Feichtinger(1991) that was summarized in the previous section to a particular economic model, that is, the modified Gaskins model of dynamic limit pricing.

Original model of dynamic limit pricing by Gaskins(1971) is formulated as follows.

$$\text{Maximize } W = \int_0^\infty (p-c)(a-bp-x)e^{-rt}dt \tag{16}$$

subject to

$$\dot{x} = \alpha(p - \bar{p}), \quad x(0) = \text{given}, \tag{17}$$

where $a, b, c, r, \bar{p},$ and α are the parameters such that $a > 0$, $b > 0$, $0 < c < \bar{p}$, $r > 0$ and $\alpha > 0$.

The rationale of this formulation is as follows. Let us consider the partial equilibrium micro-dynamic model of an industry in which one dominant large firm and many small fringe firms exist, and let us suppose that the demand function for the product of this industry is expressed by the following simple linear function.

$$q = a - bp, \tag{18}$$

where q is the demand for the product of this industry, and p is the price of this product.[4] Dominant firm acts as the price leader subject to the threat of entry by the fringe firms, and the entry dynamics of the fringe firms are expressed by the dynamic

[4] Although Gaskins(1971) did not necessarily assumed that the demand function is linear, we adopt the simplified version with linear demand function, which is due to Dixit(1990) Chap. 10.

equation (17), where x is total output of the fringe firms. If the dominant firm's output level is $(q-x)$ corresponding to the selected price level p, the discount present value of the dominant firm's net cash flow that is to be maximized is expressed by equation (16). In this model, p is the control variable and x is the state variable for the dominant firm.

The current value Hamiltonian of this dynamic optimization problem becomes

$$H=(p-c)(a-bp-x)+\mu\alpha(p-\bar{p}), \tag{19}$$

where μ is the co-state variable. A set of necessary conditions for optimality becomes as follows.

$$\dot{x}=\frac{\partial H}{\partial \mu}, \tag{20a}$$

$$\dot{\mu}=r\mu-\frac{\partial H}{\partial x}, \tag{20b}$$

$$\lim_{t\to\infty} x\mu e^{-rt}=0. \tag{20c}$$

In this formulation, the initial condition $x(0)$ is pre-determined, but the initial condition $\mu(0)$ is not pre-determined. Gaskins(1971) showed that the equilibrium point of the two-dimensional system of dynamic equations (20a), (20b) becomes a saddle point, and only the convergent path satisfies the 'transversality condition' (20c). This means that the solution of this dynamic optimization problem produces only the monotonically convergent path, and the cyclical fluctuation does not occur.

On the other hand, Asada(2013) presented the following modified version of Gaskins' dynamic limit pricing model.

$$\text{Maximize } W=\int_0^\infty (p-c)(a-bp-x)e^{-rt}dt \tag{21}$$

subject to

$$\dot{x}=\alpha(p^e-\bar{p}),\ \ x(0)=\text{given}, \tag{22}$$

$$\dot{p}^e=\beta(p-p^e),\ \ p^e(0)=\text{given}, \tag{23}$$

where $\beta>0$ is an additional parameter and p^e is an additional state variable that is the 'expected price' of the fringe firms. This means that the adaptive price expectation formation by the fringe firms is introduced in this modified version.[5] Also in this model, p is the control variable.

The current value Hamiltonian of this system becomes

[5] Judd and Petersen(1986) and Asada and Semmler(2004) introduced other types of modification of Gaskins model. In this paper, however, we do not consider their contributions.

$$H = (p-c)(a-bp-x) + \mu_1\alpha(p^e - \bar{p}) + \mu_2\beta(p-p^e), \tag{24}$$

where μ_1 and μ_2 are two co-state variables. Then, a set of necessary conditions of optimality becomes as follows.[6]

$$\dot{x} = \frac{\partial H}{\partial \mu_1} = \alpha(p^e - \bar{p}), \tag{25a}$$

$$\dot{p}^e = \frac{\partial H}{\partial \mu_2} = \beta(p - p^e), \tag{25b}$$

$$\dot{\mu}_1 = r\mu_1 - \frac{\partial H}{\partial x} = r\mu_1 + p - c, \tag{25c}$$

$$\dot{\mu}_2 = r\mu_2 - \frac{\partial H}{\partial p^e} = (r+\beta)\mu_2 - \mu_1\alpha, \tag{25d}$$

$$\frac{\partial H}{\partial p} = -2bp + a - x + bc + \mu_2\beta = 0, \tag{25e}$$

$$\lim_{t\to\infty} x\mu_1 e^{-rt} = 0,\ \lim_{t\to\infty} p^e\mu_2 e^{-rt} = 0. \tag{25f}$$

This system of equations can be reduced to the following four-dimensional linear system of differential equations together with the 'transversality conditions' (25f).

$$\dot{x} = \alpha(p^e - \bar{p}) = G_1(p^e;\alpha), \tag{26a}$$

$$\dot{p}^e = \beta\{\frac{1}{2b}(a - x + bc + \mu_2\beta) - p^e\} = G_2(x, p^e, \mu_2;\beta), \tag{26b}$$

$$\dot{\mu}_1 = r\mu_1 + \frac{1}{2b}(a - x + bc + \mu_2\beta) = G_3(x, \mu_1, \mu_2; r, \beta), \tag{26c}$$

$$\dot{\mu}_2 = (r+\beta)\mu_2 - \mu_1\alpha = G_4(\mu_1, \mu_2; r, \alpha, \beta). \tag{26d}$$

Asada(2013) showed that the equilibrium solution $(x^*, p^{e}{}^*, p^*, \mu_1{}^*, \mu_2{}^*)$ of the dynamic system (26) that ensures $\dot{x} = \dot{p}^e = \dot{\mu}_1 = \dot{\mu}_2 = 0$ is given by

$$x^* = (a - 2b\bar{p} + bc) - \frac{\bar{p}\alpha\beta}{r(r+\beta)}, \tag{27a}$$

$$p^{e}{}^* = p^* = \bar{p} > 0, \tag{27b}$$

$$\mu_1{}^* = \frac{-\bar{p}}{r} < 0, \tag{27c}$$

$$\mu_2{}^* = \frac{-\alpha\bar{p}}{r(r+\beta)} = \frac{\alpha\mu_1{}^*}{r+\beta} < 0. \tag{27d}$$

We can easily show that we have $x^* > 0$ for all $\beta > 0$ if the parameter $a > 0$ is sufficiently large and the parameter $\alpha > 0$ is sufficiently small (cf. Asada 2013, p. 218).

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6 Equation (25e) is the first-order condition for the maximization of H with respect to p. Since $\partial^2 H / \partial p^2 = -2b < 0$, the second-order condition is satisfied.

We assume that these conditions are in fact satisfied. Then, the Jacobian matrix of the system (26) becomes

$$J = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\dfrac{\beta}{2b} & -\beta & 0 & \dfrac{\beta^2}{2b} \\ -\dfrac{1}{2b} & 0 & r & \dfrac{\beta}{2b} \\ 0 & 0 & -\alpha & r+\beta \end{bmatrix}. \tag{28}$$

In this case, the characteristic equation of this system is given by

$$\Delta(\lambda) \equiv |\lambda I - J| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{29}$$

where

$$a_1 = -traceJ = -2r < 0, \tag{30}$$

$$a_2 = M_2 = \begin{vmatrix} 0 & \alpha \\ -\dfrac{\beta}{2b} & -\beta \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ -\dfrac{1}{2b} & r \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & r+\beta \end{vmatrix} + \begin{vmatrix} -\beta & 0 \\ 0 & r \end{vmatrix}$$

$$+ \begin{vmatrix} -\beta & \dfrac{\beta^2}{2b} \\ 0 & r+\beta \end{vmatrix} + \begin{vmatrix} r & \dfrac{\beta}{2b} \\ -\alpha & r+\beta \end{vmatrix} = r^2 + \beta(\frac{\alpha}{b} - r - \beta), \tag{31}$$

$$a_3 = -M_3 = -(\text{sum of all principal third-order minors of } J), \tag{32}$$

$$a_4 = \det J = \frac{\alpha\beta r(r+\beta)}{2b} > 0. \tag{33}$$

Then, we obtain the following relationships by using the symbols that are introduced by Dockner and Feichtinger(1991).

$$K \equiv M_2 - r^2 = \beta(-\beta + \frac{\alpha}{b} - r) \equiv K(\beta), \tag{34}$$

$$\Omega(\beta) \equiv (K/2)^2 - \det J$$

$$= \frac{\beta}{2}[\frac{1}{2}\beta^3 + (r - \frac{\alpha}{b})\beta^2 + \{\frac{1}{2}(r - \frac{\alpha}{b})^2 - \frac{\alpha r}{b}\}\beta - \frac{\alpha r^2}{b}], \tag{35}$$

$$\Psi(\beta) \equiv (K/2)^2 + r^2(K/2) - \det J$$

$$= \beta[\beta^3 + \frac{1}{2}(r - \frac{\alpha}{b})\beta^2 + \frac{\alpha}{b}(\frac{\alpha}{b} - 4r)\beta - r^3]. \tag{36}$$

Let us select the parameter β as a bifurcation parameter. Then, it is clear that only the parameter value β_0 that satisfies the following relationships can satisfy the condition (AY) in the footnote 3, which ensures the existence of the closed orbits because of the Hopf Bifurcation.[7]

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[7] The system (26) is a system of linear differential equations, so that the Hopf

$$0 < \beta_0 < \frac{\alpha}{b} - r, \tag{37}$$

$$\beta_0^3 + \frac{1}{2}(r - \frac{\alpha}{b})\beta_0^2 + \frac{\alpha}{b}(\frac{\alpha}{b} - 4r)\beta_0 - r^3 = 0. \tag{38}$$

More accurately, Asada(2013) proved the following three propositions by applying Dockner-Feichtinger theorem (**Theorem 1**) to the system (29).

Proposition 1.

Suppose that $0 < r < \frac{\alpha}{b}$. Then, we have the following properties (1) – (2).

(1) The characteristic equation (29) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all sufficiently small values of $\beta > 0$. In this case, the equilibrium point becomes a complex roots type saddle point, and the *cyclical convergence* to the equilibrium point occurs.

(2) Equation (29) has two positive real roots and two negative real roots for all sufficiently large values of $\beta > 0$. In this case, the equilibrium point becomes a real roots type saddle point, and the *monotonic convergence* to the equilibrium point occurs.

Proposition 2.

Suppose that $0 < r < \frac{\alpha}{b}$ and r is sufficiently small. Then, there exist the parameter values $\beta_j (j = 1,2,3,4)$ such that $0 < \beta_1 < \frac{\alpha / b - r}{2} < \beta_2 < \frac{\alpha}{b} - r < \beta_3 \leqq \beta_4 < \infty$ which satisfy the following properties (1) – (4).

(1) The characteristic equation (29) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all $\beta \in (0, \beta_1) \cup (\beta_2, \beta_3)$.

(2) Equation (29) has four roots with positive real parts for all $\beta \in (\beta_1, \beta_2)$.

(3) Equation (29) has a pair of complex roots with positive real part and a pair of pure imaginary roots at two points $\beta = \beta_1$ and $\beta = \beta_2$. These points are the degenerated type Hopf Bifurcation points, so that the *closed orbits* exist at these points.[8]

Bifurcation in this case is the degenerated type. In other words, the closed orbits exist only at the point $\beta = \beta_0$.

[8] It must be noted that a set of 'transversality conditions' (25f) is satisfied at the closed orbit.

(4) Equation (29) has two positive real roots and two negative real roots for all $\beta \in [\beta_4, \infty)$.

Proposition 3.

Suppose that $r \geqq \frac{\alpha}{b}$. Then, there exists a parameter value $\bar{\beta} \in (0, \infty)$ that satisfy the following properties (1) – (2).

(1) The characteristic equation (29) has a pair of complex roots with positive real part and a pair of complex roots with negative real part for all $\beta \in (0, \bar{\beta})$.

(2) Equation (29) has two positive real roots and two negative real roots for all $\beta \in [\bar{\beta}, \infty)$. The point $\beta = \bar{\beta}$ is *not* the Hop Bifurcation point.

4. Concluding Remarks

In this paper, we summarized the complete mathematical characterization by Dockner and Feichtinger(1991) on the existence of cyclical fluctuations in continuous time dynamic optimization models with two state variables, and then introduced an example of the application of the analytical results by Dockner and Feichtinge(1991) to dynamic economics, which was presented by Asada(2013) for the first time. Although mathematical proofs are omitted in this paper, the detailed mathematical proofs of three propositions in section 3 of this paper are contained in Asada(2013). The method of the proofs is the straightforward application of Dockner-Feichtinger theorem to the system (29).

Appendix : Two Useful Theorems

The following two theorems are quite useful for the investigation of the existence of cyclical fluctuations in the general n-dimensional or four-dimensional system of linear and nonlinear differential equations, which is not necessarily restricted to the dynamic optimization model that is studied in the text.

Theorem A1. (Hopf Bifurcation Theorem for an n-dimensional system, cf. Asada, Chiarella, Flaschel and Franke(2010) Mathematical Appendix and Gandolfo(2009) Chap. 24)

Let $\dot{x} = f(x; \varepsilon)$, $x \in R^n$, $\varepsilon \in R$ be an n-dimensional system of differential equations depending upon a parameter ε. Suppose that the following conditions (H1) –

(H3) are satisfied.

(H1) The system has a smooth curve of equilibria given by $f(x^*(\varepsilon);\varepsilon) = 0$.

(H2) The characteristic equation $|\lambda I - Df(x^*(\varepsilon_0);\varepsilon_0)| = 0$ has a pair of pure imaginary roots $\lambda(\varepsilon_0)$, $\bar{\lambda}(\varepsilon_0)$ and no other roots with zero real parts, where $Df(x^*(\varepsilon_0);\varepsilon_0)$ is the Jacobian matrix of the above system at $(x^*(\varepsilon_0),\varepsilon_0)$ with the parameter value ε_0.

(H3) $\left.\dfrac{d\{\operatorname{Re}\lambda(\varepsilon)\}}{d\varepsilon}\right|_{\varepsilon=\varepsilon_0} \neq 0$, where $\operatorname{Re}\lambda(\varepsilon)$ is the real part of $\lambda(\varepsilon)$.

Then, there exists a continuous function $\varepsilon(\gamma)$ with $\varepsilon(0) = \varepsilon_0$, and for all sufficiently small values of $\gamma \neq 0$ there exists a continuous family of non-constant periodic solution $x(t,\gamma)$ for the above dynamic system.

Theorem A2. (Asada-Yoshida Theorem for a four-dimensional system, cf. Asada and Yoshida(2003) and Gandolfo(2009) chap. 24)

(1) Consider the characteristic equation

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0. \tag{A1}$$

Then, we have the following results (i) and (ii).

(i) The characteristic equation (A1) has a pair of pure imaginary roots and two roots with nonzero real parts *if and only if* either of the following set of conditions (A2) or (A3) is satisfied.

$$b_1 b_3 > 0,\ b_4 \neq 0,\ \Phi \equiv b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0. \tag{A2}$$

$$b_1 = b_3 = 0,\ b_4 < 0. \tag{A3}$$

(ii) The characteristic equation (A1) has a pair of pure imaginary roots and two roots with negative real parts *if and only if* the following set of conditions (A4) is satisfied.

$$b_1 > 0,\ b_3 > 0,\ b_4 > 0,\ \Phi \equiv b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0. \tag{A4}$$

(2) Consider the characteristic equation

$$\lambda^4 + b_1(\varepsilon)\lambda^3 + b_2(\varepsilon)\lambda^2 + b_3(\varepsilon)\lambda + b_4(\varepsilon) = 0, \tag{A5}$$

where it is assumed that the coefficients b_j $(j = 1,2,3,4)$ are the continuously differentiable functions of a parameter ε. Then, we have the following propositions (i) and (ii).

(i) Suppose that we have $b_1(\varepsilon_0)b_3(\varepsilon_0)>0$, $b_4(\varepsilon_0)\neq 0$, and

$\Phi(\varepsilon_0)\equiv b_1(\varepsilon_0)b_2(\varepsilon_0)b_3(\varepsilon_0)-b_1(\varepsilon_0)^2 b_4(\varepsilon_0)-b_3(\varepsilon_0)^2=0$ at $\varepsilon=\varepsilon_0$. Then, the condition (H3) in **Theorem A1** is equivalent to the following condition (A6).

$$\left.\frac{d\Phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\varepsilon_0}\neq 0. \tag{A6}$$

(ii) Suppose that we have $b_1(\varepsilon_0)=0$, $b_3(\varepsilon_0)=0$, and $b_4(\varepsilon_0)<0$ at the point $\varepsilon=\varepsilon_0$. Then the condition (H3) in **Theorem A1** is equivalent to the following condition (A7).

$$[b_2(\varepsilon_0)+\sqrt{b_2(\varepsilon_0)^2-4b_4(\varepsilon_0)}]b_1'(\varepsilon_0)-2b_3'(\varepsilon_0)\neq 0. \tag{A7}$$

Remarks on Theorem A2.

(1) The condition $\Phi=0$ is always satisfied if a set of conditions (A3) is satisfied.

(2) The inequality $b_2>0$ is always satisfied if a set of conditions (A4) is satisfied.

(3) We can derive **Theorem A2**(1)(ii) from Liu's theorem on the n-dimensional system that is due to Liu(1994) as a special case with n = 4, although other parts of **Theorem A2** cannot be derived from Liu's theorem.

Theorem A1 (Hopf Bifurcation Theorem) establishes the existence of the closed orbits and the cyclical fluctuations in a general nonlinear n-dimensional system of differential equations, which is not necessarily restricted to the dynamic optimization model. **Theorem A2** provides a complete mathematical characterization of the Hopf Bifurcation in a four-dimensional system of differential equations. Part (1) of **Theorem A2** is called 'Asada-Yoshida Theorem' by Gandolfo(2009) p. 483.

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